A-LEVEL

Further Mathematics

F1

Mark scheme

Specimen

Version 1.0

Mark schemes are prepared by the Lead Assessment Writer and considered, together with the relevant questions, by a panel of subject teachers. This mark scheme includes any amendments made at the standardisation events which all associates participate in and is the scheme which was used by them in this examination. The standardisation process ensures that the mark scheme covers the students' responses to questions and that every associate understands and applies it in the same correct way. As preparation for standardisation each associate analyses a number of students' scripts. Alternative answers not already covered by the mark scheme are discussed and legislated for. If, after the standardisation process, associates encounter unusual answers which have not been raised they are required to refer these to the Lead Assessment Writer.

It must be stressed that a mark scheme is a working document, in many cases further developed and expanded on the basis of students' reactions to a particular paper. Assumptions about future mark schemes on the basis of one year's document should be avoided; whilst the guiding principles of assessment remain constant, details will change, depending on the content of a particular examination paper.

Further copies of this mark scheme are available from aqa.org.uk

Mark scheme instructions to examiners

General

The mark scheme for each question shows:

- the marks available for each part of the question
- the total marks available for the question
- marking instructions that indicate when marks should be awarded or withheld including the principle on which each mark is awarded. Information is included to help the examiner make his or her judgement and to delineate what is creditworthy from that not worthy of credit
- a typical solution. This response is one we expect to see frequently. However credit must be given on the basis of the marking instructions.

If a student uses a method which is not explicitly covered by the marking instructions the same principles of marking should be applied. Credit should be given to any valid methods. Examiners should seek advice from their senior examiner if in any doubt.

Key to mark types

M	mark is for method
dM	mark is dependent on one or more M marks and is for method
R	mark is for reasoning
A	mark is dependent on M or m marks and is for accuracy
B	mark is independent of M or m marks and is for method and accuracy
E	mark is for explanation
F	follow through from previous incorrect result

Key to mark scheme abbreviations

CAO	correct answer only
CSO	correct solution only
ft	follow through from previous incorrect result
'their'	Indicates that credit can be given from previous incorrect result
AWFW	anything which falls within
AWRT	anything which rounds to
ACF	any correct form
AG	answer given
SC	special case
OE	or equivalent
NMS	no method shown
PI	possibly implied
SCA	substantially correct approach
sf	significant figure(s)
dp	decimal place(s)

Examiners should consistently apply the following general marking principles

No Method Shown

Where the question specifically requires a particular method to be used, we must usually see evidence of use of this method for any marks to be awarded.

Where the answer can be reasonably obtained without showing working and it is very unlikely that the correct answer can be obtained by using an incorrect method, we must award **full marks**. However, the obvious penalty to candidates showing no working is that incorrect answers, however close, earn **no marks**.

Where a question asks the candidate to state or write down a result, no method need be shown for full marks.

Where the permitted calculator has functions which reasonably allow the solution of the question directly, the correct answer without working earns **full marks**, unless it is given to less than the degree of accuracy accepted in the mark scheme, when it gains **no marks**.

Otherwise we require evidence of a correct method for any marks to be awarded.

Diagrams

Diagrams that have working on them should be treated like normal responses. If a diagram has been written on but the correct response is within the answer space, the work within the answer space should be marked. Working on diagrams that contradicts work within the answer space is not to be considered as choice but as working, and is not, therefore, penalised.

Work erased or crossed out

Erased or crossed out work that is still legible and has not been replaced should be marked. Erased or crossed out work that has been replaced can be ignored.

Choice

When a choice of answers and/or methods is given and the student has not clearly indicated which answer they want to be marked, only the last complete attempt should be awarded marks.

Q	Marking Instructions	AO	Marks	Typical Solution
1	Circles correct answer	AO2.2a	B1	$\begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix}$
	Total		**1**	
2	Recalls correct definitions of cosh x and sinh x	AO1.2	B1	$\cosh^2 x - \sinh^2 x \equiv \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$ $\equiv \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4}$ $\equiv \frac{4}{4}$ $\equiv 1$ **AG**
	Demonstrates clearly that $\cosh^2 x - \sinh^2 x = 1$ **AG** Award only for completely correct argument including expansion and simplification	AO2.1	R1	
	Total		**2**	

Q	Marking Instructions	AO	Marks	Typical Solution
3(a)	Forms equation using the numerators from each side	AO1.1a	M1	$\frac{2}{(r+1)(r+2)(r+3)} \equiv \frac{A}{(r+1)(r+2)} + \frac{B}{(r+2)(r+3)}$
	Obtains the correct values of A and B	AO1.1b	A1	$\Rightarrow 2 \equiv A(r+3) + B(r+1)$ $\Rightarrow A = 1, B = -1$
(b)	Uses 'their' result from part **(a)** to write fraction as sum of differences	AO1.1a	M1	$\sum_{r=9}^{97} \frac{1}{(r+1)(r+2)(r+3)} = \frac{1}{2}\sum_{r=9}^{97} \frac{1}{(r+1)(r+2)} - \frac{1}{(r+2)(r+3)}$
	Clearly shows step of cancelling of terms	AO2.4	M1	$\sum_{r=9}^{97} \frac{1}{(r+1)(r+2)} - \frac{1}{(r+2)(r+3)}$ $= \frac{1}{10 \times 11}$
	Obtains correct two term difference Ft 'their' values for A and B provided that 'their' A = − 'their' B	AO1.1b	A1	$- \frac{1}{11 \times 12} + \frac{1}{11 \times 12}$ $- \frac{1}{12 \times 13} + \frac{1}{12 \times 13}$ ...
	States that solution gives $2 \times \sum_{r=9}^{97} \frac{1}{(r+1)(r+2)(r+3)}$ so divides their answer by 2 to obtain correct rational solution from fully correct working **AG** (If student merely divides by 2 without justification withhold this mark)	AO2.1	R1	$- \frac{1}{98 \times 99} + \frac{1}{98 \times 99}$ $- \frac{1}{99 \times 100}$ $= \frac{1}{10 \times 11} - \frac{1}{99 \times 100}$ $= \frac{89}{9900}$ $\therefore \sum_{r=9}^{97} \frac{1}{(r+1)(r+2)(r+3)}$ $= \frac{1}{2} \times \frac{89}{9900}$ $= \frac{89}{19800}$
	Total		**6**	

Q	Marking Instructions	AO	Marks	Typical Solution
4	Correctly states that the student's argument is invalid because the integrand is undefined in the range (0 to $\frac{\pi}{2}$)	AO2.3	E1	When $x = \frac{\pi}{4}$, $\cos x - \sin x = 0$
	Correctly justifies the reason why it is undefined and states a correct conclusion	AO2.4	E1	$\therefore$ the integrand is undefined at this point and the integral *is* improper
	Total		**2**	

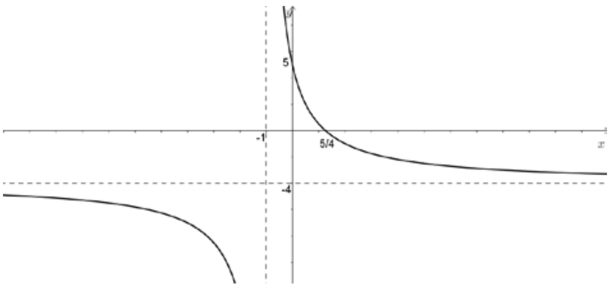
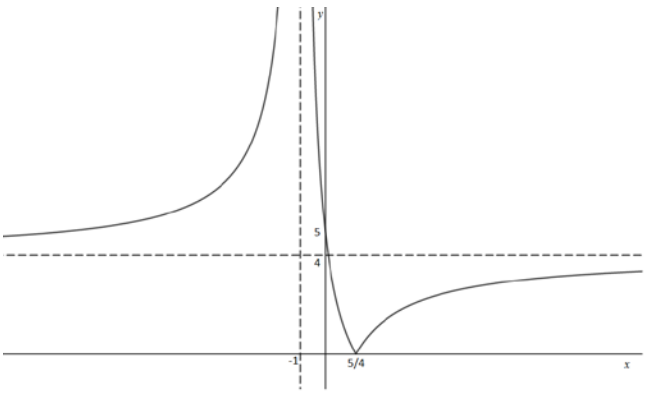
Q	Marking Instructions	AO	Marks	Typical Solution
5(a)	Makes a correct deduction about another root (PI)	AO2.2a	B1	$(z-(2-3i))(z-(2+3i))=z^2-4z+13$
	Finds quadratic factor by expanding brackets or using sum and product of roots	AO1.1a	M1	$\therefore$ $\mathrm{p}(z)=(z^2-4z+13)(z^2+cz+d)$ $(z^2-4z+13)(z^2+cz+d)\equiv z^4+3z^2+az+b$
	Finds a correct quadratic factor	AO1.1b	A1	
	Compares coefficients with quartic z^4+3z^2+az+b	AO1.1a	M1	$c-4=0$ $13-4c+d=3$ $\Rightarrow c=4,\ d=6$
	States the correct product of quadratic factors	AO1.1b	A1	$\therefore$ $\mathrm{p}(z)=\left(z^2-4z+13\right)\left(z^2+4z+6\right)$
ALT	Makes a correct deduction about another root	AO2.2a	B1	$(z-(2-3i))(z-(2+3i))=z^2-4z+13$ $\therefore$ $\mathrm{p}(z)=(z^2-4z+13)(z^2+cz+d)$
	Finds quadratic factor by expanding brackets or using sum and product of roots	AO1.1a	M1	$(z^2-4z+13)(z^2+cz+d)\equiv z^4+3z^2+az+b$
	Obtains a correct quadratic factor	AO1.1b	A1	$\alpha+\beta+\gamma+\delta=0 \Rightarrow \gamma+\delta=-4$ $\therefore c=4$
	Uses coefficients/roots to set up equations and find required coefficients	AO1.1a	M1	$\left(\sum\alpha\right)^2=\sum\alpha^2+2\sum\alpha\beta$ $0=\sum\alpha^2+2\times3$ $0=\left(2-3i\right)^2+\left(2+3i\right)^2+\gamma^2+\delta^2+6$
	States the correct product of quadratic factors	AO1.1b	A1	$\therefore\gamma^2+\delta^2=4$ $2\gamma\delta=\left(\gamma+\delta\right)^2-\gamma^2+\delta^2$ $\gamma\delta=\dfrac{16-4}{2}$ $\therefore d=6$ $p\left(z\right)=\left(z^2-4z+13\right)\left(z^2+4z+6\right)$

Q	Marking Instructions	AO	Marks	Typical Solution
(b)	States all four correct solutions FT 'their' two quadratic factors from part **(a)** provided both M1 marks have been awarded	AO1.1b	B1F	$z = 2 \pm 3\mathrm{i}, -2 \pm \sqrt{2}\mathrm{i}$
	Total		**6**	

Q	Marking Instructions	AO	Marks	Typical Solution
6(a)	Selects appropriate method for example by changing to reduced equation by dividing by tan x	AO3.1a	M1	$\frac{\mathrm{d}y}{\mathrm{d}x}+(\cot x)y=\sin x$
	Finds correct integrating factor	AO1.1b	B1	Integrating factor $=\mathrm{e}^{\int(\cot x)\mathrm{d}x}$ $=\mathrm{e}^{\ln(\sin x)}=\sin x$
ALT	Alt. finds an integrating factor by inspection, using original equation. (PI)	AO3.1a	M1	$\cos x\tan x\frac{\mathrm{d}y}{\mathrm{d}x}+y\cos x=\sin x\tan x\cos x$ $\sin x\frac{\mathrm{d}y}{\mathrm{d}x}+\left(\cos x\right)y=\sin^2 x$
	finds integrating factor = cos x	AO1.1b	B1	
	Multiples reduced or original equation by 'their' integrating factor and identifies LHS as differential of $y\times\sin x$ PI	AO1.1a	M1	$\sin x\frac{\mathrm{d}y}{\mathrm{d}x}+(\cos x)y=\sin^2 x$ $\frac{\mathrm{d}}{\mathrm{d}x}[y\sin x]=\sin^2 x$
	Uses appropriate integration method for RHS of 'their' equation	AO1.1a	M1	$y\sin x=\int\left\{\frac{1}{2}(1-\cos 2x)\right\}\mathrm{d}x$
	Integrates correctly to obtain correct solution	AO1.1b	A1	$y\sin x=\frac{1}{2}x-\frac{1}{4}\sin 2x+C$
(b)	Uses boundary condition after integration completed in either $y\sin x=\ldots$ or $y=\ldots$ form OE	AO1.1a	M1	$\sin\frac{\pi}{4}.\frac{1}{2\sqrt{2}}=\frac{1}{2}.\frac{\pi}{4}-\frac{1}{4}\sin\frac{\pi}{2}+C$ $C=\frac{1}{2}-\frac{\pi}{8}$
	States fully correct particular solution	AO1.1b	A1	$y\sin x=\frac{1}{2}x-\frac{1}{4}\sin 2x+\frac{1}{2}-\frac{\pi}{8}$
	Total		**7**	

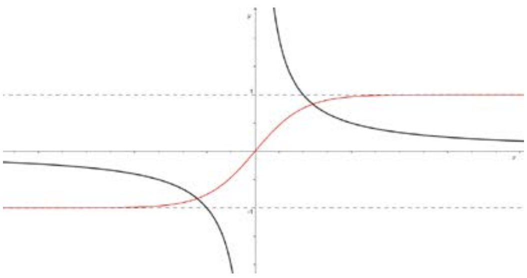
Q	Marking Instructions	AO	Marks	Typical Solution
7(a)	Uses an appropriate method for finding the values of k (for example expanding appropriate determinant)	AO1.1a	M1	$\begin{vmatrix} 1 & -1 & k \\ k & -3 & 5 \\ 1 & -2 & 3 \end{vmatrix} = 0$
	Obtains a quadratic equation in k	AO1.1a	M1	$\begin{vmatrix} -3 & 5 \\ -2 & 3 \end{vmatrix} + \begin{vmatrix} k & 5 \\ 1 & 3 \end{vmatrix} + k\begin{vmatrix} k & -3 \\ 1 & -2 \end{vmatrix} = 0$
	Obtains two correct values for k	AO1.1b	A1	$1 + 3k - 5 + k(-2k + 3) = 0$ $-2k^2 + 6k - 4 = 0$ $k^2 - 3k + 2 = 0$ $(k-2)(k-1) = 0$ $k = 2$ or 1
(b)	Selects an appropriate method to determine the appropriate geometrical configuration and substitutes 'their' first value of k	AO3.1a	M1	when $k = 1$ $x - y + z = 3$ $x - 3y + 5z = -1$ $x - 2y + 3z = -4$
	Eliminates one variable or uses row reduction	AO1.1a	M1	$-2y + 4z = -4$ $y - 2z = 7$
	Obtains a contradiction and makes correct deduction about the geometric configuration (must have correct value for k)	AO2.2a	R1	$y - 2z = 2$; $y - 2z = 7$ Hence equations are inconsistent and the three planes form a prism
	Substitutes 'their' 2$^{\text{nd}}$ value of k into selected method to determine the appropriate geometrical configuration	AO1.1a	M1	when $k = 2$ $x - y + 2z = 3$ $2x - 3y + 5z = -1$ $x - 2y + 3z = -4$
	Obtains a consistent set of equations and makes correct deduction about geometric configuration (must have correct value for k)	AO2.2a	R1	$R_2 - 2R_1 : -y + z = -7$ $R_3 - R_1 : -y + z = -7$ Hence equations are consistent and the three planes form a sheaf – they meet in line

Q	Marking Instructions	AO	Marks	Typical Solution
(c)	Deduces that the planes must meet in a line and hence that $k = 2$	AO2.2a	R1	$x - y + 2z = 3$ $2x - 3y + 5z = -1$ $x - 2y + 3z = -4$ $\Rightarrow -y + z = -7$ Let $z = \lambda$ Then $y = \lambda + 7$ and $x = 3 + y - 2z$ $= 3 + \lambda + 7 - 2\lambda$ $= -\lambda + 10$
	Selects method to find solution: For example, sets one variable = λ, substitutes and attempts to find other variables in terms of λ	AO1.1a	M1	**ALT** $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ 7 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$
	Fully states correct solution CAO	AO1.1b	A1	
	Total		**11**	

Q	Marking Instructions	AO	Marks	Typical Solution
8(a)	Indicates correct coordinate or intercept on x and y axes	AO1.1b	B1	$x=0$ $\Rightarrow y=5$ $\Rightarrow (0,5)$ $y=0$ $\Rightarrow 5-4x=0$ $\Rightarrow x=\frac{5}{4}$ $\Rightarrow (\frac{5}{4},0)$
	Indicates correct vertical asymptote or horizontal	AO1.1b	B1	$x=-1$ As $x\to\infty$, $y\to -4$ $y=-4$
	Sketches correct shape of curve	AO1.2	B1	
	Draws fully correct sketch including intercepts and both asymptotes marked	AO1.1b	B1	
(b)	Draws sketch fully correct including shape at x-intercept both asymptotes marked	AO1.1b	B1	
	Total		**5**	

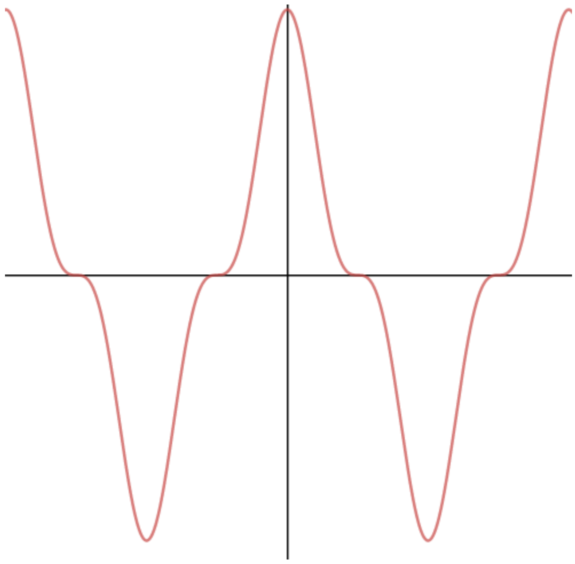
Q	Marking Instructions	AO	Marks	Typical Solution
9(a)	Uses an appropriate method for ensuring the line lies in the plane	AO3.1a	M1	Let $\lambda = x - p = \dfrac{y+2}{q} = 3 - z$, then $x = \lambda + p,\ y = q\lambda - 2,\ z = 3 - \lambda$
	Obtains equation(s) in p and q	AO1.1a	M1	sub into equation of plane $(\lambda + p) - (q\lambda - 2) - 2(3 - \lambda) + 3 = 0$ $\lambda(3 - q) + (p - 1) = 0$ this is true for all λ therefore p = 1 and q = 3
	Deduces the values of p and q	AO2.2a	A1	ALT vector equation of line is $\mathbf{r} = \begin{pmatrix} p \\ -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ q \\ -1 \end{pmatrix}$ therefore $\begin{pmatrix} p \\ -2 \\ 3 \end{pmatrix}$ lies on the plane $\begin{pmatrix} p \\ -2 \\ 3 \end{pmatrix} . \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} + 3 = 0$ And $\begin{pmatrix} 1 \\ q \\ -1 \end{pmatrix}$ is perpendicular to $\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$ therefore $\begin{pmatrix} 1 \\ q \\ -1 \end{pmatrix} . \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = 0$ $\Rightarrow q = 3$ and $p = 1$
(b)	States that to have a solution the coefficient of λ cannot be 0 OR dot product must ≠ 0	AO2.4	R1	$\begin{pmatrix} 1 \\ q \\ -1 \end{pmatrix} . \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \neq 0 \Rightarrow q \neq 3$
	Deduces the range of values for q	AO2.2a	R1	
	Deduces correct range of values for p	AO2.2a	R1	p can take any value

Q	Marking Instructions	AO	Marks	Typical Solution
(c)(i)	Finds the correct scalar product of the normal to the plane and the direction vector	AO1.1b	B1	$\mathbf{n} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$ $\mathbf{d} = \begin{pmatrix} 1 \\ q \\ -1 \end{pmatrix}$ $\mathbf{n.d} = 3-q$
	Correctly deduces the value of cos α	AO2.2a	R1	Let α be angle between the line and the normal to the plane $\sin\theta = \frac{1}{\sqrt{6}} \Rightarrow \cos\alpha = \frac{\pm 1}{\sqrt{6}}$
	Forms an equation connecting all relevant parts using $\mathbf{n.d} = \lvert\mathbf{n}\rvert\lvert\mathbf{d}\rvert\cos\theta$	AO3.1a	M1	$q-3 = \sqrt{6}\sqrt{q^2+2} \times \left(\frac{\pm 1}{\sqrt{6}}\right)$ $(3-q)^2 = q^2+2$
	Obtains correct value for q	AO1.1b	A1	$\Rightarrow 6q = 7$ giving $q = \frac{7}{6}$
(c)(ii)	Uses 'their' expressions for x and y and 'their' value for q and the equation of the plane to form an equation to find p	AO3.1a	M1	$x-p = \frac{y+2}{\frac{7}{6}} = 3-z$ $z = 0 \Rightarrow x = p+3, y = 1.5$
	Uses z = 0 to deduce expressions for x and y in terms of p and q	AO2.2a	R1	$\begin{pmatrix} p+3 \\ 1.5 \\ 0 \end{pmatrix} . \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = -3$ $\Rightarrow p+3-1.5 = -3$
	Obtains the correct value of p CAO	AO1.1b	A1	$\Rightarrow p = -4.5$
	Total		**13**	

Q	Marking Instructions	AO	Marks	Typical Solution
10(a)	Uses quotient or product rule to obtain correct derivative	AO1.1b	B1	$y=\dfrac{x}{\cosh x}\Rightarrow\dfrac{dy}{dx}=\dfrac{\cosh x - x\sinh x}{\cosh^2 x}$
	Clearly sets 'their' $\dfrac{dy}{dx}$ numerator equal to 0	AO2.4	R1	Stationary point $\Rightarrow\dfrac{dy}{dx}=0$ $\Rightarrow\dfrac{\cosh x - x\sinh x}{\cosh^2 x}=0$ $\Rightarrow \cosh x - x\sinh x = 0$
	Rearranges to complete a rigorous argument to show the required result. **AG**	AO2.1	R1	$\Rightarrow\dfrac{\sinh x}{\cosh x}=\dfrac{1}{x}$ $\Rightarrow \tanh x=\dfrac{1}{x}$ **AG**
(b)(i)	Sketches $\tanh x$ correctly including asymptotes	AO1.2	B1	
	Sketches $\dfrac{1}{x}$ correctly	AO1.2	B1	
(ii)	Deduces correct number of stationary points FT 'their' sketch in **(b)(i)**	AO2.2a	B1F	2 stationary points

Q	Marking Instructions	AO	Marks	Typical Solution
10(c)	Finds the second derivative	AO1.1a	M1	$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\cosh^2 x(\sinh x - x\cosh x - \sinh x)}{\cosh^4 x}$
	Obtains a correct expression for the second derivative	AO1.1b	A1	$-\frac{2\cosh x \sinh x(\cosh x - x\sinh x)}{\cosh^4 x}$
	Deduces that the second term is zero by using results from part **(a)**	AO2.2a	R1	second term is zero at stationary points $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\frac{x}{\cosh x}=-y$
	Completes a rigorous argument to show the required result. AG Mark awarded if they have a completely correct solution, which is clear, easy to follow and contains no slips	AO2.1	R1	$\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2}+y=0$ **AG**
	Total		**10**	

Q	Marking Instructions	AO	Marks	Typical Solution
11(a)	Commences proof by considering one side of the identity only: if considering LHS combines terms as a single fraction with a common denominator. If considering RHS writes coth θ as a fraction and introduces factor of (1 + cosh θ to both numerator and denominator) Note alternative valid approaches include commencing proof by considering LHS minus RHS or LHS divided by RHS	AO2.1	R1	$\dfrac{\sinh\theta}{1+\cosh\theta}+\dfrac{1+\cosh\theta}{\sinh\theta}$ $\equiv\dfrac{\sinh^2\theta+1+\cosh^2\theta+2\cosh\theta}{(1+\cosh\theta)\sinh\theta}$ $\equiv\dfrac{\cosh^2\theta+\cosh^2\theta+2\cosh\theta}{(1+\cosh\theta)\sinh\theta}$, $\because 1+\sinh^2\theta\equiv\cosh^2\theta$ $\equiv\dfrac{2\cosh\theta(1+\cosh\theta)}{(1+\cosh\theta)\sinh\theta}$ $\equiv\dfrac{2\cosh\theta}{\sinh\theta}$ $\equiv 2\coth\theta$ **AG**
	Explicitly states identity $\cosh^2\theta - \sinh^2\theta \equiv 1$ and uses it to eliminate (or introduce) $\sinh^2\theta$	AO2.4	R1	
	Factorises numerator and cancels correctly for ‘their’ fraction (if considering RHS rearranges ‘their’ numerator correctly into two factorised expressions)	AO1.1b	B1F	
	Completes rigorous proof to obtain result **AG** Only award if they have a completely correct argument, which is clear and contains no slips.	AO2.1	R1	
(b)	Uses result from part **(a)** to deduce that tanh $\theta = \dfrac{1}{2}$	AO2.2a	R1	2coth θ = 4 tanh $\theta = \dfrac{1}{2}$ $\theta = \tanh^{-1}\dfrac{1}{2} = \dfrac{1}{2}\ln 3$
	Uses natural log form and substitutes correct value to obtain correct exact form	AO1.1b	A1	
	Total		**6**	

<table>
<tr><th>Q</th><th>Marking Instructions</th><th>AO</th><th>Marks</th><th>Typical Solution</th></tr>
<tr><td rowspan="3">12</td><td>Correctly manipulates $y=\left[\mathrm{f}(x)\right]^n$ into a form that can be sketched.</td><td>AO3.1a</td><td>M1</td><td rowspan="3">$\cosh(\mathrm{i}x)=\dfrac{e^{\mathrm{i}x}+e^{-\mathrm{i}x}}{2}$
$=\dfrac{\cos(x)+\mathrm{i}\sin(x)+\cos(-x)+\mathrm{i}\sin(-x)}{2}$
$=\dfrac{\cos(x)+\mathrm{i}\sin(x)+\cos(x)-\mathrm{i}\sin(x)}{2}$
$=\cos(x)$
$\therefore\left[\mathrm{f}(x)\right]^n=\cos^n(x)$

Since the graph has the same shape/area above and below the x-axis over the given domain, the mean value of $y=\left[\mathrm{f}(x)\right]^n$ must be 0.</td></tr>
<tr><td>Sketches an appropriate section of the graph.

Sketch does not need to be accurate but should look symmetrical about y-axis and the area above and below the x-axis should look similar</td><td>AO1.1b</td><td>M1</td></tr>
<tr><td>Deduces, with fully correct reasoning, the correct mean value</td><td>AO2.2a</td><td>R1</td></tr>
<tr><td></td><td>Total</td><td></td><td>3</td><td></td></tr>
</table>

Q	Marking Instructions	AO	Marks	Typical Solution
13	Uses proof by induction and investigates formula for n = 1 and n=k (must see evidence of both n = 1 and n=k being considered)	AO3.1a	M1	Using induction method, Let P(n) be the statement $\mathbf{M}^n = \begin{bmatrix} 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \end{bmatrix}$
	Demonstrates that formula is true for n = 1	AO1.1b	A1	For n =1 $\begin{bmatrix} 3^0 & 3^0 & 3^0 \\ 3^0 & 3^0 & 3^0 \\ 3^0 & 3^0 & 3^0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \mathbf{M}^1$ ∴P(1) is true
	States assumption that formula true for n = k and uses $\mathbf{M}^{k+1} = \mathbf{M} \times \mathbf{M}^k$	AO2.1	R1	Assume P(k) is true $\mathbf{M}^{k+1} = \mathbf{M} \times \mathbf{M}^k$
	Deduces that formula is also true for n = k + 1 from correct working	AO2.2a	R1	$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 3^{k-1} & 3^{k-1} & 3^{k-1} \\ 3^{k-1} & 3^{k-1} & 3^{k-1} \\ 3^{k-1} & 3^{k-1} & 3^{k-1} \end{bmatrix}$ (since P(k) is true) $= \begin{bmatrix} (3^{k-1}+3^{k-1}+3^{k-1}) & (3^{k-1}+..) & (3^{k-1}+.. \\ (3^{k-1}+3^{k-1}+3^{k-1}) & (3^{k-1}+..) & (3^{k-1}+.. \\ (3^{k-1}+3^{k-1}+3^{k-1}) & (3^{k-1}+..) & (3^{k-1}+.. \end{bmatrix}$
	Completes a rigorous argument and explains how their argument proves the required result. **AG**	AO2.4	R1	But $3^{k-1} + 3^{k-1} + 3^{k-1} = 3 \times 3^{k-1}$ $= 3^k$ Hence $\mathbf{M}^{k+1} = \begin{bmatrix} 3^k & 3^k & 3^k \\ 3^k & 3^k & 3^k \\ 3^k & 3^k & 3^k \end{bmatrix}$ $\therefore \mathbf{M}^{k+1} = \begin{bmatrix} 3^{(k+1)-1} & 3^{(k+1)-1} & 3^{(k+1)-1} \\ 3^{(k+1)-1} & 3^{(k+1)-1} & 3^{(k+1)-1} \\ 3^{(k+1)-1} & 3^{(k+1)-1} & 3^{(k+1)-1} \end{bmatrix}$ ∴P(k +1) is true Since P(1) is true *and* P(k)⇒P(k + 1) , hence, by induction, P(n) is true for all $n \in N$ **AG**
	Total		**5**	

Q	Marking Instructions	AO	Marks	Typical Solution
14(a)	Models the motion of the particle by forming a second order differential equation. (must have correct terms but allow sign errors	AO3.3	M1	$M\dfrac{d^2x}{dt^2} = -4M\dfrac{dx}{dt} - 8Mx$ $\dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} + 8x = 0$
	Obtains correct differential equation	AO1.1b	A1	$\lambda^2 + 4\lambda + 8 = 0$ $\lambda = -2 \pm 2\mathrm{i}$
	Forms and solves auxiliary equation for 'their' D.E.	AO1.1a	M1	Complex roots ⇒ General solution is of the form: $x = Ae^{-2t}\cos(2t + B)$
	States a correct form of the general solution for 'their' auxiliary solution. (ft only if both M1 marks have been awarded)	AO1.1b	A1F	$\dot{x} = -2Ae^{-2t}\cos(2t + B) - 2Ae^{-2t}\sin(2t + B)$ $\dot{x}(0) = 0$ so, $-2A\cos(B) - 2A\sin(B) = 0$
	Uses initial conditions to find arbitrary constants for 'their' solution	AO1.1a	M1	$\Rightarrow \tan B = -1$ $\Rightarrow B = -\dfrac{\pi}{4}$ $x(0) = 1$
	Obtains correct value for one of 'their' constants (ft only if all M1 marks have been awarded)	AO1.1b	A1F	$\Rightarrow \dfrac{A}{\sqrt{2}} = 1$ $\Rightarrow A = \sqrt{2}$
	Obtains correct value for both of 'their' constants (ft only if all M1 marks have been awarded)	AO1.1b	A1F	$\therefore\ x = \sqrt{2}e^{-2t}\cos(2t - \frac{\pi}{4})$ ∴ the particle oscillates about O, with period π seconds and decreasing amplitude.
	Uses 'their' model to describe the motion of the particle either as a written description or shown on a clearly labelled graph.	AO3.4	A1F	

Q	Marking Instructions	AO	Marks	Typical Solution
(b)	Refines their DE model to account for altered resistive force by introducing a new coefficient for $\frac{\mathrm{d}x}{\mathrm{d}t}$	AO3.5c	B1	$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+\alpha\frac{\mathrm{d}x}{\mathrm{d}t}+8x=0$ Critical damping $\Rightarrow$ $\lambda^2+\alpha\lambda+8=0$ must have equal roots $\alpha^2=32$ $\alpha=4\sqrt{2}$ Resistive force should have magnitude $4\sqrt{2}Mv$
	Uses or states condition for critical damping	AO1.2	B1	
	Deduces value for coefficient of $\frac{\mathrm{d}x}{\mathrm{d}t}$	AO2.2a	R1	
	States resistive force	AO3.4	B1	
	Total		**12**	

Q	Marking Instructions	AO	Marks	Typical Solution
(a)	Forms a differential equation for the foxes.	AO3.3	B1	$\frac{\mathrm{d}y}{\mathrm{d}t} \propto x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = kx$
	Forms a differential equation for the rabbits.	AO3.3	B1	$\frac{\mathrm{d}y}{\mathrm{d}t} = 20, x = 200 \Rightarrow k = 0.1$
	Differentiates 'their' equation that contains y and obtains expression with at least two terms correct.	AO1.1a	M1	$\frac{\mathrm{d}y}{\mathrm{d}t} = 0.1x$ $\frac{\mathrm{d}x}{\mathrm{d}t} = 1.2x - 1.1y$
	Formulates a second order linear differential equation.	AO3.1a	M1	$y = \frac{1.2x}{1.1} - \frac{1}{1.1}\frac{\mathrm{d}x}{\mathrm{d}y}$
	Obtains roots of auxiliary equation for 'their' second order differential equation.	AO1.1a	M1	$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1.2}{1.1}\frac{\mathrm{d}x}{\mathrm{d}t} - \frac{1}{1.1}\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$
	States correct general solution. FT provided all M marks have been awarded	AO1.1b	A1F	$0.1x = \frac{1.2}{1.1}\frac{\mathrm{d}x}{\mathrm{d}t} - \frac{1}{1.1}\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$ $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 1.2\frac{\mathrm{d}x}{\mathrm{d}t} + 0.11x = 0$
	Uses initial population to find equation linking constants for their general solution.	AO3.4	M1	$\lambda^2 - 1.2\lambda + 0.11 = 0$ $\lambda = 0.1$ or 1.1 $x = A\mathrm{e}^{0.1t} + B\mathrm{e}^{1.1t}$
	Obtains initial rate of change for rabbits from 'their' DE.	AO3.4	M1	$t = 0, x = 80$ $A + B = 80$
	Differentiates and obtains a second equation for A and B from 'their' general solution.	AO1.1a	M1	$t = 0, x = 74$ $74 = 0.1A + 1.1B$ $74 = 0.1(80 - B) + 1.1B$ $B = 68, A = 12$
(a)(ii)	Substitutes 0.7 and obtains approximately 160. CAO	AO3.4	A1	$12e^{0.07} + 68e^{0.77} = 159.7$

Q	Marking Instructions	AO	Marks	Typical Solution
(b)	States a suitable refinement about the fact that an increased rabbit population will require more food supply or other valid refinement	AO3.5c	B1	Take account of the food available for the rabbits as this may limit population growth.
	Total		**11**	
	Total		**100**	