Teacher Support Materials 2009

Maths GCE

Paper Reference MPC3

Question 1a

1 (a) The curve with equation

$$y = \frac{\cos x}{2x+1}, \qquad x > -\frac{1}{2}$$

intersects the line $y = \frac{1}{2}$ at the point where $x = \alpha$.

(i) Show that α lies between 0 and $\frac{\pi}{2}$. *(2 marks)*

(ii) Show that the equation $\frac{\cos x}{2x+1} = \frac{1}{2}$ can be rearranged into the form

$$x = \cos x - \frac{1}{2}$$ *(1 mark)*

(iii) Use the iteration $x_{n+1} = \cos x_n - \frac{1}{2}$ with $x_1 = 0$ to find x_3, giving your answer to three decimal places. *(2 marks)*

Student Response

Leave blank

1a. i) $y = \frac{\cos x}{2x+1}$

$f(0) = 1$

$f\left(\frac{\pi}{2}\right) = 0$

change in signs $\therefore$ root lies between 0 and $\frac{\pi}{2}$

$0 < x < \frac{\pi}{2}$

ii) $\frac{\cos x}{2x+1} = \frac{1}{2}$.

$\rightarrow \cos x = \frac{1}{2}(2x+1)$.

$\rightarrow \cos x = x + \frac{1}{2}$.

$\cos x - \frac{1}{2} = x$

$x = \cos x - \frac{1}{2}$ $\therefore$ as Required.

iii) $x_{n+1} = \cos x_n - \frac{1}{2}$.

$x_1 = 0 \rightarrow \cos 0 - \frac{1}{2}$

$x_2 = \frac{1}{2} = 0.5$

$\rightarrow \cos(0.5) - \frac{1}{2}$.

$x_3 = 0.37758$

$= 0.378$

1

1

2

Commentary

Questions are often set where candidates are required to test either side of zero to locate a root. In this question candidates were required to test either side of 0.5 and this caused many candidates to fail. They had been drilled in comparing to zero and the popular, but incorrect, response was as this candidate. After finding, correctly, the two values of 0 and 1, the candidate proceeded with the conclusion that there was a change of sign.

Mark scheme

Q	Solution	Marks	Total	Comments
1(a)(i)	$\mathrm{f}(x)=\dfrac{\cos x}{2x+1}-\dfrac{1}{2}$			OE
	$\mathrm{f}(0)=\dfrac{1}{2}$; $\mathrm{f}\left(\dfrac{\pi}{2}\right)=-\dfrac{1}{2}$	M1		$x=0$ LHS $=1$, $x=\dfrac{\pi}{2}$ LHS $=0$
	Change of sign $0<\alpha<\dfrac{\pi}{2}$	A1	2	Either side of $\dfrac{1}{2}$, $\therefore 0<\alpha<\dfrac{\pi}{2}$
(ii)	$\dfrac{\cos x}{2x+1}=\dfrac{1}{2}$			
	$\left.\begin{array}{l}2\cos x=2x+1\\2\cos x-1=2x\end{array}\right\}$ or, $\cos x=x+\dfrac{1}{2}$			Either line
	$x=\cos x-\dfrac{1}{2}$	B1	1	AG; or $\cos x-\dfrac{1}{2}=x$ All correct with no errors
(iii)	$x_1=0$			
	$x_2=0.5$	M1		Attempt at iteration (allow $x_2=-0.5$, $x_3=0.38, 0.4$)
	$x_3=0.378$	A1	2	CAO

Question 1b

(b) (i) Given that $y = \dfrac{\cos x}{2x+1}$, use the quotient rule to find an expression for $\dfrac{dy}{dx}$.

(3 marks)

(ii) Hence find the gradient of the normal to the curve $y = \dfrac{\cos x}{2x+1}$ at the point on the curve where $x = 0$.

(2 marks)

Student Response

$x_3 = 0.490$

b)i) $y = \dfrac{\cos x}{2x+1}$ $\qquad \dfrac{vu' - uv'}{v^2}$

$\dfrac{dy}{dx} = \dfrac{-(2x+1)\sin x - 2\cos x}{(2x+1)^2}$ $\qquad u = \cos x \quad v = 2x+1$ $\qquad u' = -\sin x \quad v' = 2$

$= \dfrac{-\sin x - 2\cos x}{2x+1}$ ISW

3

ii) $\text{grad} = \dfrac{dy}{dx}$ $\qquad$ CSO

$\dfrac{-\sin(0) - 2\cos(0)}{2(0)+1} = -2$ $\qquad$ grad for normal $= \dfrac{1}{2}$ $\quad (m_1 \times m_2 = -1)$

1

Commentary

Throughout the paper there were many instances where candidates' weak algebraic skills were seen. This answer to the question highlighted in this script was very common. The candidate correctly used the quotient rule, but then subsequently divided by (2x + 1). Although the candidate scored full marks in this first part of the question - as their subsequent incorrect working was ignored - they were penalised in part (ii), as full marks could only be obtained if the correct answer came from a completely correct solution.

Mark scheme

(b)(i)	$\frac{dy}{dx} = \frac{(2x+1)(-\sin x) - \cos x \times 2}{(2x+1)^2}$	M1		Attempt at quotient rule: $\frac{\pm(2x+1)\sin x \pm 2\cos x}{(2x+1)^2}$
		A1		Either term correct
		A1	3	All correct ISW
(ii)	$x = 0$			
	$\frac{dy}{dx} = -2$	m1		Correctly subst. $x = 0$ into their $\frac{dy}{dx}$
	$\therefore$ Gradient of normal $= \frac{1}{2}$	A1	2	CSO

Question 2c

(c)	The composite function fg is denoted by h.	
(i)	Find an expression for $\mathrm{h}(x)$.	*(1 mark)*
(ii)	Solve the equation $\mathrm{h}(x) = 3$.	*(3 marks)*

Student response

2ci $h(x) = \sqrt{2\left(\frac{1}{4x+1}\right) + 5}$

$= \sqrt{\frac{2}{4x+1} + 5}$ 1

2cii $3 = \sqrt{\frac{2}{4x+1} + 5}$

$\pm 9 = \frac{2}{4x+1} + 5$ 1

$\pm 9 - 5 = \frac{2}{4x+1}$

$\pm 9 \cancel{-5}(4x+1) = 2$

$4x + 1 = \frac{2}{\pm 9 - 5}$ 5

$4x = \frac{2}{\pm 9 - 5} - 1$

$x = -\frac{1}{8} \quad x = -\frac{2}{7}$

Commentary

Again, throughout the paper there were many instances where candidates' weak algebraic skills were seen. This shows the lack of understanding of the candidate in rearranging an equation in an attempt to find x. The candidate doesn't realise the order of the operations. There were many other, incorrect, versions in this part question.

Mark Scheme

2(c)(i)	$\mathrm{h}(x) = \mathrm{fg}(x)$ $= \sqrt{2\left(\frac{1}{4x+1}\right) + 5}$	B1	1	
(ii)	$\sqrt{2\left(\frac{1}{4x+1}\right) + 5} = 3$			
	$2\left(\frac{1}{4x+1}\right) + 5 = 9$	M1		one correct step from (c)(i), squaring
	$\frac{1}{4x+1} = 2$ $4x + 1 = \frac{1}{2}$ either or $16x + 4 = 2$	A1		
	$x = -\frac{1}{8}$ or equiv	A1	3	CSO

Question 3a

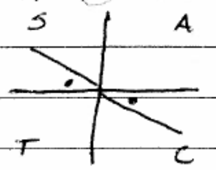
3 (a) Solve the equation $\tan x = -\frac{1}{3}$, giving all the values of x in the interval $0 < x < 2\pi$ in radians to two decimal places. *(3 marks)*

Student Response

③ a) $\tan x = -\frac{1}{3}$

$x = \tan^{-1}(-\frac{1}{3})$

$AAA = -0.3217505544^c$



$x = 3.46^c,\ x = 6.60^c$

Commentary

There were many completely correct responses to this question. However, there was a significant minority of questions who produced the same solution as this candidate. The candidate has correctly found the inverse tan, but is then unable to apply this solution. The candidate has used a CAST diagram, and if this method is used then the principal value should be found by ignoring the negative sign in the question. Also, the candidate hasn't realised that his second solution of 6.60 is outside the range.

Mark Scheme

3(a)	$\tan^{-1}\left(-\frac{1}{3}\right) = -0.32$	M1		Sight of ± 0.32 or 18.43
	$x = 2.82, 5.96$	A1 A1	3	a correct answer AWRT –1 for any extra in range, ignore extra answers not in range. [SC 161.57, 341.57 AWRT M1A1 (max 2/3)]

Question 4

4 (a) Sketch the graph of $y = |50 - x^2|$, indicating the coordinates of the point where the graph crosses the y-axis. *(3 marks)*

(b) Solve the equation $|50 - x^2| = 14$. *(3 marks)*

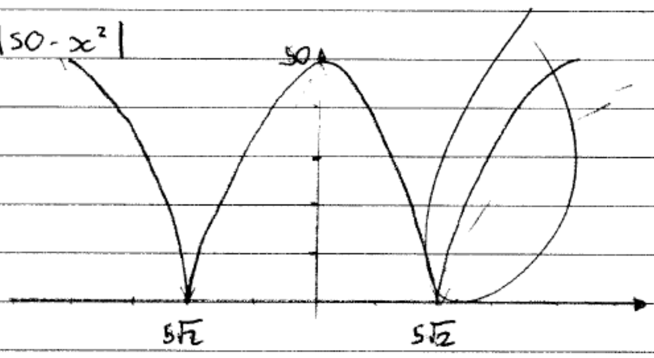
(c) Hence, or otherwise, solve the inequality $|50 - x^2| > 14$. *(2 marks)*

(d) Describe a sequence of two geometrical transformations that maps the graph of $y = x^2$ onto the graph of $y = 50 - x^2$. *(4 marks)*

Student Response

4

a) $y = |50 - x^2|$



2

b) $|50 - x^2| = 14$

$50 - x^2 = -14$ or $50 - x^2 = 14$

$50 + 14 = x^2$ $\quad$ $50 - 14 = x^2$

$\sqrt{64} = x$ $\quad$ $\sqrt{36} = x$

$x = 8$ $\quad$ $x = 6$

2

c) $x < 6$

$x > 8$

0

Leave blank

d) $y = x^2$

$\downarrow$ Reflection in the y-axis

$y = -x^2$

$\downarrow$ translation through $\begin{bmatrix} 0 \\ 50 \end{bmatrix}$

$y = 50 - x^2$

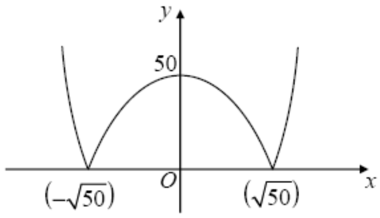
2

6

Commentary

There were few completely correct solutions for this question. The solution in this script was far more common than the correct answer. In part (a) the candidate has realised the essentials of a modulus graph, but has failed to sketch the graph correctly in the two extreme sections of the graph.
In part (b), the candidate has started the solution of the equation correctly and identified the two parts to the solution, but the candidate has then failed to handle the fact that the square root will also produce two solutions, giving four solutions in total. If the candidate had sketched the line y = 14 on their graph it would have been obvious that that there were four solutions in total.

Mark Scheme

Q	Solution	Marks	Total	Comments
4(a)		M1		Modulus graph, 3 section, condone shape inside + outside $\pm\sqrt{50}$
		A1		Cusps + curvature outside $\pm\sqrt{50}$
		A1	3	Value of y and shape inside $(\pm\sqrt{50})$
(b)	$\lvert 50-x^2 \rvert = 14$ $50-x^2=14 \quad x^2=36$ $50-x^2=-14 \quad x^2=64$	M1		Either
	$x=\pm 6, \pm 8$	A1 A1	 3	2 correct, from correct working All 4 correct, from correct working
(c)	$-6<x<6$ $x>8,\ x<-8$	B1 B1	 2	
(d)	Reflect in x-axis Translate $\begin{bmatrix} 0 \\ 50 \end{bmatrix}$ Reflect in $y = 25$ scores 4/4	M1,A1 E1, B1	 4	or $\left\{\begin{array}{l}\text{Reflect in } y=a \\ \text{Translate } \begin{bmatrix} 0 \\ 50-2a \end{bmatrix}\end{array}\right\}$ or $\left\{\begin{array}{l}\text{Translate } \begin{bmatrix} 0 \\ -50 \end{bmatrix} \\ \text{Reflect in } x-\text{axis}\end{array}\right\}$ or $\left\{\begin{array}{l}\text{Translate } \begin{bmatrix} 0 \\ 2a-50 \end{bmatrix} \\ \text{Reflect in } y=a\end{array}\right\}$
	Total		**12**	

Question 5b

(b) Solve the equation

$$2\ln x + \frac{15}{\ln x} = 11$$

giving your answers as exact values of x. *(5 marks)*

Student Response

b) $2\ln x + \dfrac{15}{\ln x} = 11$

$\dfrac{15}{\ln x} = 11 - 2\ln x$

$15 = 11\ln x - 2\ln x + \ln x$

$11\ln x - 2\ln x + \ln x = 15$

$\ln \dfrac{x^{11}}{x^2} + \ln x = 15$

$\ln x^9 + \ln x = 15$

$\ln x^{10} = 15$

$x^{10} = e^{15}$ $\quad x = \sqrt[10]{e^{15}}$

$= \pm 4.48$

1

Commentary

This part of the question was the least well answered on the whole paper. We condoned poor notation in the marking of the paper, but the question showed the severe weakness of candidates in algebraic manipulation when using logarithms. The candidate knows that logarithms may be combined, but is unsure as to how and when to apply any rules that they know.

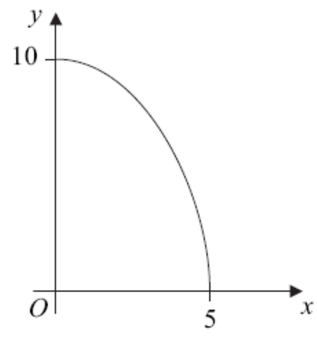
Mark Scheme

(b)	$2\ln x + \dfrac{15}{\ln x} = 11$			
	$2(\ln x)^2 - 11\ln x + 15 = 0$	M1		Forming quadratic equation in $\ln x$, condone poor notation
	$(2\ln x - 5)(\ln x - 3) = 0$	m1		Attempt at factorisation/formula
	$\ln x = \dfrac{5}{2}, 3$ condone $2\ln x = 5$	A1		
	$x = e^{\frac{5}{2}}, e^3$	A1,A1	5	[SC for substituting $x = e^{\frac{5}{2}}$ or equivalent into equation and verifying B1 $(\frac{1}{5})$]

(b)	$2\ln x + \dfrac{15}{\ln x} = 11$			
	$2(\ln x)^2 - 11\ln x + 15 = 0$	M1		Forming quadratic equation in $\ln x$, condone poor notation
	$(2\ln x - 5)(\ln x - 3) = 0$	m1		Attempt at factorisation/formula
	$\ln x = \dfrac{5}{2}, 3$ condone $2\ln x = 5$	A1		
	$x = e^{\frac{5}{2}}, e^3$	A1,A1	5	[SC for substituting $x = e^{\frac{5}{2}}$ or equivalent into equation and verifying B1 $(\frac{1}{5})$]

Question 6a

6 The diagram shows the curve with equation $y = \sqrt{100 - 4x^2}$, where $x \geqslant 0$.



(a) Calculate the volume of the solid generated when the region bounded by the curve shown above and the coordinate axes is rotated through 360° about the ***y*-axis**, giving your answer in terms of π. *(5 marks)*

Student Response

Leave blank

6) $y = \sqrt{100 - 4x^2}$

a)

$\int \pi x^2 \, dy$

$\cancel{y^2 = 100 - 4x^2}$

$\cancel{4x^2 = 100 - y^2}$

x^2

$\int \pi \left(\frac{10-y}{2}\right)^2 dy$

$y = \sqrt{100 - 4x^2}$

$y = 10 - 2x$

$2x = 10 - y$

$x = \frac{10-y}{2}$

$= \pi \int \left(\frac{10-y}{2}\right)^2 dy$ B1

$= \pi \int \frac{100 - y^2}{4} \, dy$ M1

$= \pi \left[\frac{100y - \frac{1}{3}y^3}{4y}\right]$ A0

$= \pi \left[\frac{100y}{4y} - \frac{1\,y^3}{12y}\right]$

$= \pi \left[25 - (y^3 \times 12y^{-1})\right]$

$= \pi \left[25 - 12y^2\right]$

$= 25\pi - 12y^2\pi$

2

Commentary

Again, algebra. The candidate realises that integration has to be wrt y. Then to rearrange the equation, the candidate has square rooted each term individually, rearranged, then squared each term individually. The candidate has then found the correct expression to integrate, even although the algebra has been very poor. This solution was common. Obviously it was heavily penalised.

Mark Scheme

6(a)	$V = \pi\int x^2\,\mathrm{d}y$	B1		PI
	$V = \frac{(\pi)}{4}\int \left(100 - y^2\right)\,\mathrm{d}y$	M1		$k\int\left(100 - y^2\right)\mathrm{d}y$ may be recovered Allow $\int\left(\text{their } x\right)^2 \mathrm{d}y$, expanded
	$= \frac{(\pi)}{4}\left[100y - \frac{y^3}{3}\right]_{(0)}^{(10)}$	A1		
	$= \frac{(\pi)}{4}\left[\frac{2000}{3}\right]$	m1		For F(10) − F(0)
	$= \frac{500\pi}{3}$	A1	5	OE CSO [SC: if rotated about x-axis $V = \pi\left[100x - \frac{4x^3}{3}\right]_0^5$ M1 $= \frac{1000}{3}\pi$ A1 max 2/5]

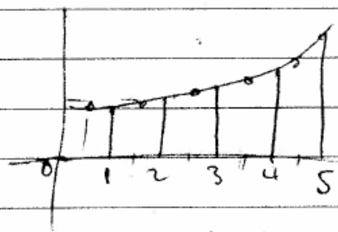
Question 6b

(b) Use the mid-ordinate rule with five strips of equal width to find an estimate for $\int_0^5 \sqrt{100 - 4x^2}\,\mathrm{d}x$, giving your answer to three significant figures. *(4 marks)*

Student Response

b) $\int_0^5 \sqrt{100 - 4x^2}$

$\frac{5}{5} = 1$ $h = 1$



x	y
0.5	~~9.49~~ 9.95
1.5	~~3.16~~ 9.54
2.5	8.66
3.5	7.14
4.5	4.36
	39.65

$\frac{1}{2}(39.65)$

19.8

2

Commentary

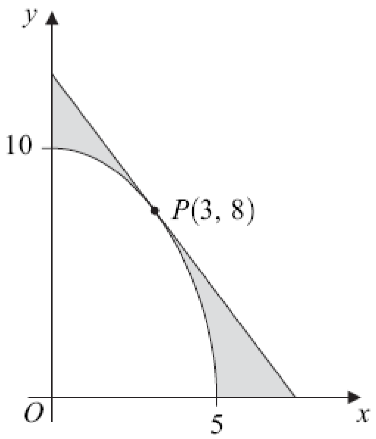
Candidates were asked to find an area numerically giving the answer accurate to three significant figures. Candidates **must** then work to a greater degree of accuracy. Although the candidate has used an incorrect formula, they would have still obtained an incorrect answer as when they would round 39.65 they would have obtained an answer of 39.7

Mark Scheme

(b)	x y 0.5 9.95(0) 1.5 9.539 2.5 8.66(0) 3.5 7.141 4.5 4.359 } or better	B1 M1 A1		Correct x 4 + correct y to 2sf All y correct
	$A = 1 \times \sum y = 39.6$	A1	4	(39.6 scores $\frac{4}{4}$)

Question 6d

(d) The shaded regions on the diagram below are bounded by the curve, the tangent at P and the coordinate axes.



Use your answers to part (b) and part (c)(ii) to find an approximate value for the **total** area of the shaded regions. Give your answer to three significant figures. *(5 marks)*

Student Response

Leave blank

d. $2y+3x=25$

$2y=25-3x$

$y=\frac{25-3x}{2}$

$\cancel{\frac{dy}{dx}=\frac{25-3x}{2}}$ u v

$\cancel{u=25-3x}$

$\cancel{\frac{du}{dx}=-3}$

$\cancel{v=2}$

$\cancel{\frac{dv}{dx}=0}$

$\cancel{\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}}$

$\cancel{=\frac{2(-3)-(25-3x)(0)}{2^2}}$

$\cancel{=\frac{-6}{4}}$

$\cancel{=-\frac{3}{2}}$

$y=\frac{25-3x}{2}$

$\int\frac{25-3x}{2}$

$\frac{1}{2}\int 25-3x$

$\frac{1}{2}\times(25x-3x^2$

$=\frac{25x-3x^2}{2}$ let $\cancel{x}=3$

0

6d $= \dfrac{25(3) - 3(3)^2}{2}$

$= \dfrac{75 - 27}{2}$

$= 24$

Using answer in b

$39.6 - 24$

$= 15.6$ Unit^2 area of shaded regions.

0

0

Commentary

Candidates had to find the total of shaded areas. To do this they needed to find the total area of the triangle. There were, in general, two methods used. This candidate used integration of the given equation of the line. However they had to clearly identify that they were finding the correct area. This candidate didn't know which limits to use as all that they have done is substitute x = 3 into their integral. There were many similar scripts.

Mark Scheme

6(d)	$x = 0 \quad y = \frac{25}{2}$ or equivalent	B1		
	$y = 0 \quad x = \frac{25}{3}$	B1		OE
	Area of $\Delta = \frac{1}{2} \times \frac{25}{2} \times \frac{25}{3}$	M1		for $\frac{1}{2}$(their y)$\times$(their x) or $\frac{1}{2}ab\sin C$
	Area = Area Δ − (b)	m1		PI $\Delta >$ (b)
	Required area = 12.5 AWRT	A1	5	Condone 12.4 AWRT
(d)	**Alternative**			
	Area $\Delta = \int_0^{\frac{25}{3}} \frac{1}{2}(25 - 3x)\,(dx)$	(B1) (B1)		
	$= \frac{1}{2}\left[25x - \frac{3x^2}{2}\right]_0^{\frac{25}{3}}$ $\frac{1}{2}\left[\frac{625}{3} - \frac{625}{6}\right]$	(M1)		For integration **and** $f(\frac{25}{3}) - f(0)$
	$= \frac{625}{12}$			

Question 7a

7 (a) Use integration by parts to find $\int (t-1)\ln t\,\mathrm{d}t$. *(4 marks)*

Student Response

7a) $\int (t-1)\ln t\,dt$

$u = \ln t \qquad v' = t - 1$

$u' = \frac{1}{t} \qquad v = \frac{t^2}{2} - t \qquad \frac{1}{2}\int t^2 - 2t$

$= \ln t\left(\frac{t^2}{2} - t\right) - \int \frac{t^2 - t}{2}\left(\frac{1}{t}\right)$

$= \ln t\left(\frac{t^2}{2} - t\right) - \frac{1}{2}\left(\frac{t^3}{3} - \cancel{\frac{t^3}{2}}\right)\ln t$

$= \ln t\left(\frac{t^2}{2} - t\right) - \left(\frac{t^3}{6} - \frac{t^2}{2}\right)\ln t$

$\ln t\left(\frac{t^2}{2} - t\right) - \frac{t^2}{2}\left(\frac{t}{3} - 1\right)\ln t + C$

$\left(\ln t\,(t)\left(\frac{t}{2} - 1\right) - \frac{t^2}{2}\left(\frac{t}{3} - 1\right)\ln t + C\right)$

2

Commentary

Many candidates knew the basic principles for integration by parts, and correctly integrated one term and differentiated the other. It is expected that to score the initial accuracy mark there should be no mistakes. However, the candidate then has to handle the second integration. This solution was all too common. Having substituted their, incorrect, terms into the parts formula, they then have no idea as to how to deal with the subsequent integration. The second integration was MPC2 work, and to a large extent was found wanting.

7(a)	$\int (t-1)\ \ln t\ \ dt$			
	$u=\ln t \quad \frac{dv}{dt}=t-1$	M1		Differentiate + integrate, correct direction
	$\frac{du}{dt}=\frac{1}{t} \quad v=\frac{t^2}{2}-t$	A1		All correct
	$\int=\left(\frac{t^2}{2}-t\right)\ln t-\int\left(\frac{t^2}{2}-t\right)\times\frac{1}{t}(\ dt)$			
	$=\left(\frac{t^2}{2}-t\right)\ln t-\int\left(\frac{t}{2}-1\right)(\ dt)$	A1		Condone missing brackets
	$=\left(\frac{t^2}{2}-t\right)\ln t-\frac{t^2}{4}+t(+c)$	A1	4	CAO

7(a)	**Alternative**			
	$\int (t-1)\ \ln t$	(M1)		$u=\ln t \quad v'=(t-1)$
		(A1)		$u'=\frac{1}{t} \quad v=\frac{(t-1)^2}{2}$
	$\int \quad =\frac{(t-1)^2}{2}\ln t\ -\int\frac{(t-1)^2}{t}\ \frac{1}{t}\,dt$			
	$\frac{(t-1)^2}{2}\ \ln t-\frac{1}{2}\quad\int\frac{t^2-2t+1}{t}\ \ dt$			
	$\frac{(t-1)^2}{2}\ \ \ln t\ -\frac{1}{2}\int t-2+\frac{1}{t}\ dt$	(A1)		
	$\frac{(t-1)^2}{2}\ \ \ln t\ -\frac{1}{2}\left[\frac{t^2}{2}-2t+\ln t\right]$	(A1)		
	$=\frac{t^2}{2}\ \ln t-t\ln t+\cancel{\frac{1}{2}\ln t}\ -\frac{t^2}{4}+t\ -\cancel{\frac{1}{2}\ln t}$			
	$=\left(\frac{t^2}{2}-t\right)\ \ln t-\frac{1}{4}\ t^2+t\ \ +c$		(4)	