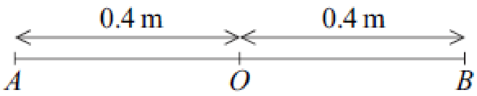
Teacher Support Materials 2009

Maths GCE

Paper Reference MM05

Question 1

1 A particle moves with simple harmonic motion along a straight line AOB, where $AO = OB = 0.4$ metres, as shown in the diagram.



The maximum speed of the particle is $1.2\,\text{m s}^{-1}$.

(a) Find the period of the motion. *(4 marks)*

(b) Find the magnitude of the maximum acceleration of the particle. *(2 marks)*

(c) The point C lies on OB. The speed of the particle as it passes through C is $0.9\,\text{m s}^{-1}$. Find the distance OC. *(3 marks)*

Student Response

1 (a) $V_{Max} = a\omega$

$1.2 = 0.4\omega$

$\omega = \dfrac{1.2}{0.4} = 3$

$T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{3}$ seconds

Commentary

This was a popular question usually yielding high marks; in part (c) some chose long methods involving times, and errors were more frequent here than with other methods. An exemplary 'good parctice' response, showing concise working.

Mark Scheme

1(a)	ωa = max speed			
	$0.4\omega = 1.2$	M1		
	$\omega = 3$	A1		
	$T = \frac{2\pi}{\omega} = \frac{2\pi}{3}$	M1A1	4	Accept $\frac{2\pi}{3}$ or 2.09
	($T = 2.09$ sec)			
(b)	$\lvert\text{Maximum acceleration}\rvert = \omega^2 a = 9 \times 0.4$	M1		
	$= 3.6\,\text{ms}^{-2}$	A1	2	
(c)	$\dot{x}^2 = \omega^2(a^2 - x^2)$	M1		
	$0.9^2 = 3^2(0.4^2 - x^2)$	A1		
	$OC = x = 0.265\,\text{m}$	A1	3	accept 0.264
	Total		**9**	

Question 2

2 A simple pendulum consists of a particle, of mass m, attached to one end of a light inextensible string, of length l. The other end of the string is attached to a fixed point. The pendulum is set into motion in a vertical plane. At time t, the angle between the string and the downward vertical is θ.

(a) Using a small angle approximation, show that the motion of the pendulum can be modelled by the differential equation

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\frac{g\theta}{l}$$

(4 marks)

(b) The period of the motion is 2.4 seconds. Find the value of l. *(2 marks)*

(c) During this motion, the pendulum is instantaneously at rest when the string is at an angle of 0.15 radians to the vertical. Find the maximum speed of the particle during the motion. *(3 marks)*

Student Response

(b) $T = 2\pi\sqrt{\frac{l}{g}}$ if $T = 2.4$ and $g = 9.8$

$2.4 = 2\pi\sqrt{\frac{l}{9.8}}$ $\therefore \frac{2.4^2}{2\pi^2} = \frac{l}{9.8}$

$\therefore \left(\frac{5.76}{4\pi^2}\right) \times 9.8 = l = 1.43\text{m}$

Leave blank

2

(c) Therefore max displacement = $s = l\theta$ $\therefore s = 1.43 \times 0.15^c$

$s = 0.2145\text{m}$

Using $T = \frac{2\pi}{\omega}$ $\therefore 2.4 = \frac{2\pi}{\omega}$ $\therefore \omega = \frac{2\pi}{2.4} = \frac{5}{6}\pi$

using $v^2 = \omega^2(a^2 - x^2)$ max speed when $x = 0$

$\therefore v^2 = \left(\frac{5}{6}\pi\right)^2 \times (0.2145)^2 = 0.315$ (3d.p)

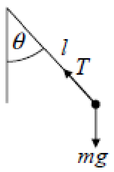
$\therefore v = 0.562\ \text{ms}^{-1}$ (3d.p) max speed

3

Commentary

A very sound response to parts (b) and (c), with an especially well explained and clearly set out solutions. The required proof in part (a) was well known and there were many concise solutions. In parts (b) and (c) some were unable to quote appropriate formulae, and there was some confusion between linear and angular quantities.

Mark Scheme

2(a)	 tangentially: $mg\sin\theta = -ml\ddot{\theta}$ $\sin\theta \approx \theta$ for small θ $\ddot{\theta} = -\frac{g\theta}{l}$	M1A1 m1 A1	4	AG
(b)	$T = 2\pi\sqrt{\frac{l}{g}} \Rightarrow 2.4 = 2\pi\sqrt{\frac{l}{9.8}}$ $l = 1.43\text{m}$	M1 A1	2	
(c)	Max speed = ωa $= \sqrt{\frac{9.8}{1.43}} \times 1.43 \times 0.15$ $= 0.561\text{ms}^{-1}$	M1A1 A1	3	accept 0.562
	Total		9	

Question 3

3 An astronaut on a spacewalk is travelling in a straight line. In order to increase his speed, he fires his rocket pack, which ejects burnt fuel backwards at a constant rate of $10\,\mathrm{kg\,s^{-1}}$ and at a constant speed of $30\,\mathrm{m\,s^{-1}}$ relative to the astronaut. Initially, the total mass of the astronaut with the rocket pack and fuel is 200 kg.

You should assume that gravitational forces can be ignored.

When the rocket pack has been fired for t seconds, the speed of the astronaut is $v\,\mathrm{m\,s^{-1}}$.

(a) Show that, while the rocket pack is being fired,

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{30}{20 - t}$$ *(6 marks)*

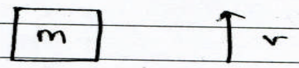
(b) The initial speed of the astronaut is $2\,\mathrm{m\,s^{-1}}$. Show that the speed of the astronaut at time t is given by

$$v = 30\ln\left(\frac{20}{20 - t}\right) + 2$$ *(4 marks)*

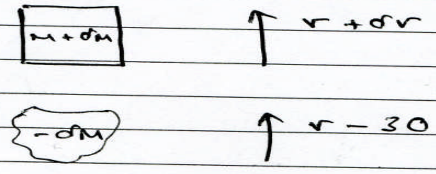
(c) The astronaut fires the rocket pack until his speed is $6\,\mathrm{m\,s^{-1}}$. Calculate the time for which the rocket pack is fired. *(3 marks)*

Student Response

3 At time 't'



At time '$t + \delta t$'



Using the impulse-momentum principle:

$0\,\delta t = (m+\delta m)(v+\delta v) + (-\delta m)(v-30) - mv$

$0\delta t = mv + m\delta v + v\delta m + \delta m\delta v - v\delta m + 30\delta m - mv$

$0\delta t = m\delta v + 30\delta m$

Dividing by δt, and letting $\delta t \to 0$

$0 = m\frac{dv}{dt} + 30\frac{dm}{dt}$

But, $m = 200 - 10t$

$\frac{dm}{dt} = -10$

b) $\frac{dv}{dt} = \frac{30}{20-t}$

$\int_2^v \frac{dv}{30} = \int_0^t \frac{dt}{20-t}$

$\left[\frac{v}{30}\right]_2^v = -\int_0^t \frac{-dt}{20-t}$

$\frac{v}{30} - \frac{2}{30} = -\left[\ln(20-t)\right]_0^t$

$\frac{v}{30} - \frac{2}{30} = -\left[\ln(20-t) - \ln(20)\right]$

$\frac{v}{30} - \frac{2}{30} = -\ln\left(\frac{20-t}{20}\right)$

$\frac{v}{30} - \frac{2}{30} = \ln\left(\frac{20}{20-t}\right)$

$v - 2 = 30\ln\left(\frac{20}{20-t}\right)$

$v = 30\ln\left(\frac{20}{20-t}\right) + 2$

Commentary

In part (a) there were some excellent solutions, but others revealed a lack of understanding, including an impulse in their momentum equation. Part (b) was done well, with more concise solutions from those who used limits as opposed to a constant of integration. Part (c) proved a good source of marks. This student's solution clear use of the momentum principle in part (a). In part (b), the use of limits brings a rapid and concise solution.

Mark Scheme

3(a)	$mv=(m+\delta m)(v+\delta v)-\delta m(v-V)$	M1A1		
	$m=200-10t,\ \frac{dm}{dt}=-10,\ V=30$	B1 B1		m the others
	$mv=mv+\delta mv+m\delta v-\delta mv+\delta mV$			
	$\frac{m dv}{dt}+V\frac{dm}{dt}=0$	m1		
	$(200-10t)\frac{dv}{dt}=300$			
	$\frac{dv}{dt}=\frac{30}{20-t}$	A1	6	CAO; AG
(b)	$\int_2^v dv=\int_0^t \frac{30}{20-t}dt$	M1		ignore limits
	$[v]_2^v=[-30\ln(20-t)]_0^t$	A1		
	$v-2=-30\ln(20-t)+30\ln 20$	m1		limits used or constant evaluated
	$v=30\ln\left(\frac{20}{20-t}\right)+2$	A1	4	AG
(c)	$v=6\Rightarrow 4=30\ln\left(\frac{20}{20-t}\right)$	M1		
	$(20-t)e^{\frac{2}{15}}=20$	M1		
	$t=20\left(1-e^{-\frac{2}{15}}\right)$			
	$t=2.50$ sec	A1	3	
	Total		**13**	

Question 4

4 A particle P, of mass m, is suspended from a fixed point O by a light elastic string, of natural length a and modulus of elasticity $5mn^2a$, where n is a positive constant. When the particle hangs in equilibrium vertically below O, the extension of the string is $\frac{g}{5n^2}$. At time $t=0$, the particle is projected vertically downwards from this equilibrium position with speed U.

During its subsequent motion, when P is moving with speed v, it experiences a resistance force of magnitude $2mnv$. The displacement of P below its equilibrium position at time t is x.

(a) Show that, whilst the string is taut, x satisfies the equation

$$\frac{d^2x}{dt^2}+2n\frac{dx}{dt}+5n^2x=0$$ *(5 marks)*

(b) Find x in terms of U and t. *(7 marks)*

(c) State whether the nature of the damping caused by the resistance force is light, critical or heavy. *(1 mark)*

(d) Find, in terms of U and π, the speed of P when it first returns to its equilibrium position. *(4 marks)*

Student Response

Question number

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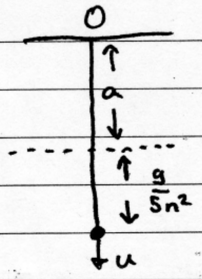
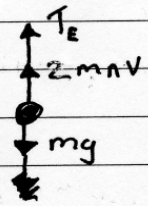
(4)



$\lambda = 5mn^2 a \quad e = \frac{g}{5n^2}$

$R = 2mnV$



(a) resolving with forces:

$-\frac{d^2x}{dt^2}$ as accel always towards centre.

~~[illegible]~~

$\therefore T_E = \frac{\lambda(x+e)}{l} = \frac{5mn^2ax + 5mn^2\left(\frac{g}{5n^2}\right)a}{a}$ ✓

$T_E = 5mn^2x + mg$ ✓

$\therefore -m\frac{d^2x}{dt^2} = T_E + 2mnV - mg$

$-m\frac{d^2x}{dt^2} = 5mn^2x + mg + 2mnV - mg$ ✓

~~[illegible]~~

$\therefore -m\frac{d^2x}{dt^2} = 5mn^2x + 2mnV$ ✓

$\therefore -\frac{d^2x}{dt^2} = 5n^2x + 2nV \quad \therefore \frac{d^2x}{dt^2} + 2nV + 5n^2x$

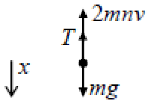
$V = \frac{dx}{dt} \quad \therefore \frac{d^2x}{dt^2} + 2n\frac{dx}{dt} + 5n^2x = 0$ ✓

6

Commentary

Part (a) was answered well, with occasional errors in finding the extension of the string. Part (b) was less successful, with some only being able to attempt solving the Auxiliary Equation and then stopping; there were various algebraic errors, the most frequent being the omission of ‘n’ at varying stages, but still a number of good solutions. The answer to part (c) was well known. Part (d) proved quite testing in choosing correct methods for solution, and again algebraic errors marred solutions.The solution to part (a) shows a clearly justified solution, taking into account all the forces and leading to the required differential equation.

Mark Scheme

Q	Solution	Marks	Total	Comments
4(a)				
		B1		T
	$m\ddot{x} = mg - \frac{5mn^2 a}{a}\left(x + \frac{g}{5n^2}\right) - 2mn\dot{x}$	M1 A1A1		all terms
	$\ddot{x} + 2n\dot{x} + 5n^2 x = 0$	A1	5	CAO; AG
(b)	AE: $p^2 + 2np + 5n^2 = 0$	M1		
	$(p+n)^2 + 4n^2 = 0$			
	$p = -n \pm 2n\mathrm{i}$	A1		
	$x = \mathrm{e}^{-nt}(A\cos 2nt + B\sin 2nt)$	A1		
	$t = 0,\ x = 0 \Rightarrow A = 0$	A1		
	$\dot{x} = -\mathrm{e}^{-nt}(B\sin 2nt) + \mathrm{e}^{-nt}(2nB\cos 2nt)$	M1		
	$t = 0,\ \dot{x} = U \Rightarrow$			
	$U = 2nB \quad B = \frac{U}{2n}$	A1		
	$x = \frac{U}{2n}\mathrm{e}^{-nt}\sin 2nt$	A1	7	
(c)	light	B1	1	
(d)	$x = 0 \therefore \sin 2nt = 0$	M1		
	$2nt = (0,)\ \pi, \ldots$			
	$t = \frac{\pi}{2n}$	A1		
	$\dot{x} = -\mathrm{e}^{-nt}\left(\frac{U}{2n}\sin 2nt\right) + \mathrm{e}^{-nt}(U\cos 2nt)$			
	$= -\mathrm{e}^{-\frac{\pi}{2}}(0) + \mathrm{e}^{-\frac{\pi}{2}}(U)(-1)$	M1		
	speed $= U\mathrm{e}^{-\frac{\pi}{2}}$	A1	4	
	Total		**17**	

Question 5

5 A particle P moves so that, at time t, its polar coordinates (r, θ) with respect to a fixed origin O are such that

$$r = \frac{2}{2 + \cos\theta}, \quad 0 \leqslant \theta < 2\pi$$

At all times during the motion the value of $r^2\dot{\theta} = 2$.

(a) Write down the value of $r\dot{\theta}$ in terms of θ. *(1 mark)*

(b) (i) Given that $\dot{r} = \sin\theta$, show that the speed of P is $\sqrt{5 + 4\cos\theta}$. *(3 marks)*

(ii) State the range of values of the speed of P. *(1 mark)*

(c) (i) Show that the transverse component of the acceleration of P is zero. *(2 marks)*

(ii) Hence find an expression for the acceleration of P in terms of r. *(4 marks)*

(iii) Deduce that the force acting on P is directed towards O at all times during the motion. *(2 marks)*

Student Response

ii) $a_r = \ddot{r} - r\dot{\theta}^2$

$a_r = \dfrac{d\sin\theta}{dt} - r\dot{\theta}^2$

$= \cos\theta \dfrac{d\theta}{dt} - (2+\cos\theta)\dfrac{d\theta}{dt}$

$= \dfrac{d\theta}{dt}\left(\cos\theta - 2 - \cos\theta\right) = -2\dfrac{d\theta}{dt}$

$\dfrac{d\theta}{dt} = \dfrac{2}{r^2}$ $\quad \therefore \quad a = \dfrac{-4}{r^2}$

iii) Transverse accelaration $= 0$ ∴ Force in radial direction only.
Radial accelaration is negative $\left(A_r = -\frac{4}{r^2}\right)$ ∴ As r is always positive the accelaration is opposite to r ∴ always towards O.
(Force = mass × accelaration) and mass constant.

Commentary

There were many concise solutions to part (a) but also many long winded ones, some giving answers in terms of differing variables. Part (b) was mostly done very well, although a minority thought the minimum value of the cosine function to be zero. In part (c)(i) those who could see the efficiency in differentiating the expression for $r^2 d\theta/dt$ were successful, but some worked with an alternative expression for the acceleration component and their solutions were lengthy and often contained errors. Those most successful in part (c)(ii) showed excellent skills in efficient substitution to obtain an expression in terms of *r*, but there were many meandering responses, and this request proved discriminating. Solutions to part (c)(iii) rarely considered all the necessary factors.In this solution, the candidate focuses on introducing the variable r in (c)(ii), leading to an efficient response; in part (c)(iii) the candidate provides all the necessary facts for the marks allowed.

Mark Scheme

5(a)	$r\dot{\theta} = \frac{2}{r} = 2 + \cos\theta$	B1	1	
(b)(i)	$\dot{r} = \sin\dot{\theta},\ r\dot{\theta} = 2 + \cos\theta$			
	$\text{speed}^2 = (\sin\theta)^2 + (2+\cos\theta)^2$	M1		
	$= \sin^2\theta + 4 + 4\cos\theta + \cos^2\theta$	A1		
	$\text{speed} = \sqrt{5 + 4\cos\theta}$	A1	3	AG
(ii)	$1 \leqslant \text{speed} \leqslant 3$	B1	1	
(c)(i)	$r^2\dot{\theta} = 2$ (constant)	M1		
	transverse component of acceleration			
	$= \frac{1}{r}\frac{\text{d}}{\text{d}t}\left(r^2\dot{\theta}\right) = \text{zero}$	A1	2	AG
(ii)	(radial) acceleration $= \ddot{r} - r\dot{\theta}^2$			
	$\ddot{r} = \cos\theta\dot{\theta}$	B1		
	(radial) acceleration			
	$= \cos\theta\dot{\theta} - (2+\cos\theta)\dot{\theta}$	M1A1		
	$= -2\dot{\theta}$			
	$= -\frac{4}{r^2}$	A1	4	
(iii)	$r^2 > 0 \therefore$ (radial) acceleration < 0			
	$\therefore$ force < 0 at all time	E1		
	direction radial, only component radial	E1	2	
	Total		**13**	

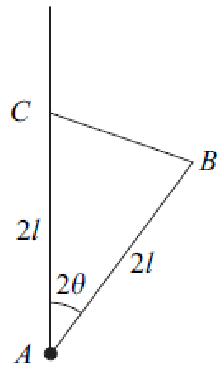
Question 6

6 A uniform rod AB is smoothly hinged at A, and is free to move in a vertical plane. A light spring connects B to a point C, vertically above A.

The rod is of length $2l$ and of mass $2m$.

The spring is of natural length l and modulus $2mg$, and the distance AC is $2l$.

The angle BAC is 2θ, as shown in the diagram, where $0 < \theta \leqslant \frac{\pi}{2}$.



(a) The gravitational potential energy is taken to be zero at the level of A. Show that V, the total potential energy of the system, is given by

$$V = mgl(12\sin^2\theta - 8\sin\theta + 3)$$ *(6 marks)*

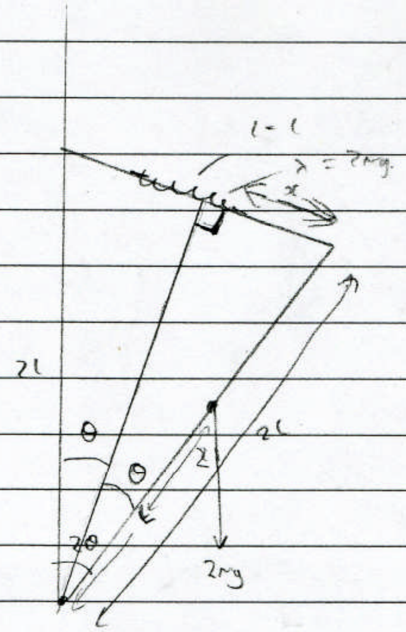
(b) Find the two values of θ for which the rod is in equilibrium. *(4 marks)*

(c) Determine, for each of these values, whether the rod is in stable or unstable equilibrium. *(4 marks)*

Student Response

6)

a)



$x = 2l\sin\theta$

$2x = 4l\sin\theta$

$Epe = \frac{\lambda x^2}{2l}$

$= \frac{2mg}{2l}(4l\sin\theta - l)^2$

$= \frac{2mgl^2(4\sin\theta-1)^2}{2l}$

$= mgl(16\sin^2\theta - 8\sin\theta + 1)$

gpe of $2m = mgh$

$= 2mg\,l\cos 2\theta$

$= 2mgl(1-2\sin^2\theta)$

$= mgl(2-4\sin^2\theta)$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= 1 - 2\sin^2\theta$

$V = mgl(16\sin^2\theta - 8\sin\theta + 1 + 2 - 4\sin^2\theta)$

$= mgl(12\sin^2\theta - 8\sin\theta + 3)$ QED.

b) $\frac{dV}{d\theta} = mgl(12 \times 2\sin\theta\cos\theta - 8\cos\theta)$

$= 8mgl\cos\theta(3\sin\theta - 1) = 0$

$\cos\theta = 0$, $3\sin\theta - 1 = 0$

$\theta = \frac{\pi}{2}$, $\theta = 0.340$

b) $\frac{d^2V}{d\theta^2} = -8mgl\sin\theta(3\sin\theta-1) + 8mgl\cos\theta(3\cos\theta)$

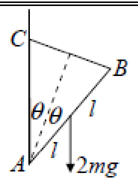
$V''\left(\frac{\pi}{2}\right) = -16mgl + 0 < 0$ ∴ unstable at $\theta = \frac{\pi}{2}$

$V''(0.340) = 21.3mgl > 0$ ∴ stable at $\theta = 0.340$

Commentary

Finding a correct expression for the extension of the spring in part (a) proved very challenging, and subsequent use of trigonometric identities was sometimes weak. Part (b) was a good source of marks for all candidates, and there was a pleasing improvement in the use of radians in solutions of trigonometric equations. Part (c) was mostly done well. Part (a) is answered well, with very clear and efficient use of trignometrical identities. Parts (b) and (c) show full and concise solutions.

Mark Scheme

6				
(a)	$V = 2mg(l\cos 2\theta) + \frac{2mg}{2l}(2\times 2l\sin\theta - l)^2$	M1 B1 A1		2 terms extension
	$V = 2mgl\cos 2\theta + mgl(16\sin^2\theta - 8\sin\theta + 1)$	A1		
	$V = 2mgl(1-2\sin^2\theta) + mgl(16\sin^2\theta - 8\sin\theta + 1)$	M1		identity
	$V = mgl(12\sin^2\theta - 8\sin\theta + 3)$	A1	6	AG
(b)	$\frac{dv}{d\theta} = mgl(24\sin\theta\cos\theta - 8\cos\theta)$	M1		
	$\frac{dv}{d\theta} = 0 \Rightarrow 8\cos\theta(3\sin\theta - 1) = 0$	m1		
	$\cos\theta = 0$ or $\sin\theta = \frac{1}{3}$	A1		
	$\theta = \frac{\pi}{2}$ or $\theta = \sin^{-1}\frac{1}{3} = 0.340$	A1	4	
(c)	$\frac{d^2v}{d\theta^2} = \frac{d}{d\theta}(mgl(12\sin 2\theta - 8\cos\theta))$			
	$= mgl(24\cos 2\theta + 8\sin\theta)$	M1A1		
	$\theta = \frac{\pi}{2} \Rightarrow mgl(-24+8) < 0$ unstable	A1		
	$\theta = \sin^{-1}\frac{1}{3} \Rightarrow mgl\left(24\left(1-\frac{2}{9}\right)+\frac{8}{3}\right) > 0$ stable	A1	4	
	Total		**14**	