Teacher Support Materials 2009

Maths GCE

Paper Reference MFP3

Question 1

1 The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = f(x, y)$$

where $$f(x, y) = \sqrt{x^2 + y + 1}$$

and $$y(3) = 2$$

(a) Use the Euler formula

$$y_{r+1} = y_r + h\,f(x_r, y_r)$$

with $h = 0.1$, to obtain an approximation to $y(3.1)$, giving your answer to four decimal places. *(3 marks)*

(b) Use the formula

$$y_{r+1} = y_{r-1} + 2h\,f(x_r, y_r)$$

with your answer to part (a), to obtain an approximation to $y(3.2)$, giving your answer to three decimal places. *(3 marks)*

Student Response

		Leave blank
1.	(a) $y(3.1) = y(3) + 0.1 \times \sqrt{3^2+2+1}$	
	$= 2 + 0.1 \times \sqrt{12}$ ✓	3
	≈ 2.3464 ✓	
	(b). $y(3.2) = y(3) + 2\times0.1\times\sqrt{3^2+2+1}$	
	$= 2 + 2\times0.1\times\sqrt{12}$ M0	0
	≈ 2.6928	
		(3)

Commentary

The exemplar illustrates a common error in part (b).
The candidate presented a correct solution to part (a). Correct values were substituted into the given Euler formula, evaluated correctly and the final answer given to the required degree of accuracy as requested in the question.
In part (b) the candidate applied the given formula incorrectly in line1 by using 2×0.1×f(3,2) instead of 2×0.1×f(3.1,y(3.1)) in the final term on the right-hand-side.

Mark scheme

1(a)	$y(3.1) = y(3) + 0.1\sqrt{3^2+2+1}$	M1A1		
	$= 2 + 0.1\times\sqrt{12} = 2.3464(10..)$ $= 2.3464$	A1	3	Condone > 4dp if correct
(b)	$y(3.2) = y(3) + 2(0.1)[\mathrm{f}(3.1, y(3.1))]$	M1		
	$\ldots = 2 + 2(0.1)[\sqrt{(3.1^2+2.3464+1)}]$	A1F		ft on candidate's answer to **(a)**
	$\ldots = 2 + 0.2\times 3.599499.. = 2.719(89..)$ $= 2.720$	A1	3	CAO Must be 2.720
	Total		**6**	

Question 2

2 By using an integrating factor, find the solution of the differential equation

$$\frac{dy}{dx} - y\tan x = 2\sin x$$

given that $y = 2$ when $x = 0$. *(9 marks)*

Student response

2. $\frac{dy}{dx} - y\tan x = 2\sin x$ Integrating factor $= e^{\int \tan x\, dx}$ M1

$= e^{\ln|\sec x|}$ A0

$= |\sec x|$ ✓ A1F

$\frac{dy}{dx}(\sec x) - y\tan x \sec x = 2\sin x \sec x$

$\frac{d}{dy}(y\sec x) = 2\tan x$ ✓ M1

$y\sec x = \int 2\tan x\, dx$ ✓ A1F

$y\sec x = 2\ln|\sec x| + c$ ✗ M0

$\frac{y}{\cos x} = 2\ln|\sec x| + c$

$y = (2\ln|\sec x| + c)(\cos x)$ ✗ A0

$y = 2$ when $x = 0$

$2 = (2\ln 1 + c)(1)$ ✓ m1

$2 = c$

GENERAL SOLUTION

$y = (2\ln|\sec x| + 2)(\cos x)$ A0

5

(5)

Commentary

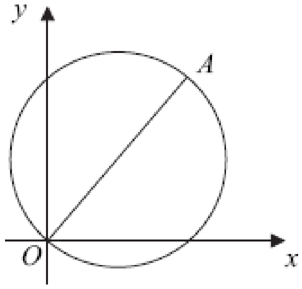
The exemplar shows the most common error that was made by candidates answering this question.
The candidate omitted the negative sign from the coefficient of *y* when writing down the integrating factor. The candidate should have used $e^{\int -\tan x \, dx}$ which would lead to cos*x*, not sec*x*, for the integrating factor; however 2 marks were awarded for a correct follow through integration and simplification of $e^{\ln \sec x}$. Although line 4 contained a slip, it is clear from line 5 that the candidate had integrated both sides of $\frac{d}{dx}(y \sec x) = 2\tan x$ with respect to *x* so both marks were awarded on follow through. Even though the candidate's work was correct on follow through the original sign error led to an integral which is stated in the formulae booklet so work had been significantly eased and the only other available mark to the candidate was for use of the boundary condition, *y*=2 when *x*=0 to find the constant of integration, *c*, which the candidate scored in line 12 of the solution.

Mark Scheme

2	IF is $e^{\int -\tan x \, dx}$	M1		Award even if negative sign missing
	$= e^{\ln(\cos x)\ (+c)}$	A1		OE Condone missing *c*
	$= (k)\cos x$	A1F		ft earlier sign error
	$\cos x \frac{dy}{dx} - y\tan x \cos x = 2\sin x \cos x$			
	$\frac{d}{dx}(y\cos x) = 2\sin x \cos x$	M1		LHS as $\frac{d}{dx}(y \times \text{IF})$ PI
	$y\cos x = \int 2\sin x \cos x \, dx \, dx$	A1F		ft on c's IF provided no exp or logs
	$y\cos x = \int \sin 2x \, dx$	m1		Double angle or substitution OE for integrating $2\sin x \cos x$
	$y\cos x = -\frac{1}{2}\cos 2x (+c)$	A1		ACF
	$2 = -\frac{1}{2} + c$	m1		Boundary condition used to find *c*
	$c = \frac{5}{2}$			
	$y\cos x = -\frac{1}{2}\cos 2x + \frac{5}{2}$	A1	9	ACF eg $y\cos x - 2 + \sin^2 x$ Apply ISW after ACF
	Total		**9**	

Question 3

3 The diagram shows a sketch of a circle which passes through the origin O.



The equation of the circle is $(x-3)^2+(y-4)^2=25$ and OA is a diameter.

(a) Find the cartesian coordinates of the point A. *(2 marks)*

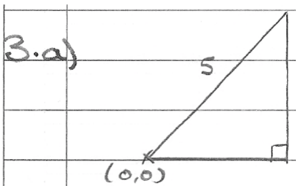
(b) Using O as the pole and the positive x-axis as the initial line, the polar coordinates of A are (k, α).

(i) Find the value of k and the value of $\tan\alpha$. *(2 marks)*

(ii) Find the polar equation of the circle $(x-3)^2+(y-4)^2=25$, giving your answer in the form $r=p\cos\theta+q\sin\theta$. *(4 marks)*

Student Response

3.a)



$\therefore A(x,y) = (5,5)$ ✗ 0

b) i) $R = 5$ ✗ $\tan\alpha = (y/x)$

~~tan~~ $\tan\alpha = (5/5)$

$\therefore$ ~~tanα = 1~~ $\tan\alpha = 1$ ✗

OR

Equilatoral triangle, therefore angle at origin must be $\pi/4$. $\alpha = \pi/4$ 0

$\Rightarrow \tan\alpha = \tan(\pi/4) = 1$

ii) Using $x = r\cos\theta$, $y = r\sin\theta$ and $x^2+y^2=r^2$.

$(x-3)^2 + (y-4)^2 = 25$ ✓

$x^2 - 6x + 9 + y^2 - 8y + 16 = 25$

$r^2 - 6x + 8y = 0$

$r^2 - 6(r\cos\theta) - 8(r\sin\theta) = 0$

$r^2 - 6r\cos\theta - 8r\sin\theta = 0$

Question number

Leave blank

$r^2 - 6r\cos\theta - 8r\sin\theta = 0$

~~r(r − 6cosθ~~

$r^2 - r(6\cos\theta + 8\sin\theta) = 0$

$r^2 = r(6\cos\theta + 8\sin\theta)$

$r = 6\cos\theta + 8\sin\theta$ ✓ 4

where $p = 6$ and $q = 8$.

Commentary

The exemplar illustrates the common wrong assumptions in parts (a) and (b)(i).
In part (a) the candidate wrote down the wrong coordinates for the point *A* without showing any method. Since the coordinates are the same it was clear that the candidate had incorrectly assumed that the point *A* lay on the line *y*=*x* and that angle *AOx* = 45º. In part (b)(i) the candidate gave the common wrong answer, 5, for *k* so equated *k* to the radius rather than the diameter of the circle. The candidate stated the incorrect value, 1, for $\tan\alpha$ which confirmed the candidate's incorrect assumption that angle *AOx* = 45º. In part (b)(ii) the candidate showed good examination technique by listing the three results for converting between cartesian and polar coordinates. The candidate correctly expanded the brackets in the given cartesian equation of the circle, applied the three conversions and obtained the correct polar equation of the circle in the form requested in the question.

Mark Scheme

3(a)	Centre of circle is $M(3, 4)$	B1		PI
	$A(6, 8)$	B1	2	
(b)(i)	$k = OA = 10$	B1		
	$\tan\alpha = \frac{y_A}{x_A} = \frac{4}{3}$	B1	2	SC " $r = 10$ and $\tan\theta = \frac{8}{6}$ " = B1 only
(b)(ii)	$x^2 + y^2 - 6x - 8y + 25 = 25$	B1		If polar form before expansion award the B1 for correct expansions of both $(r\cos\theta - m)^2$ and $(r\sin\theta - n)^2$ where $(m,n) = (3,4)$ or $(m,n) = (4,3)$
	$r^2 - 6r\cos\theta - 8r\sin\theta = 0$	M1M1		1st M1 for use of any one of $x^2 + y^2 = r^2$, $x = r\cos\theta$, $y = r\sin\theta$ 2nd M1 for use of these to convert the form $x^2 + y^2 + ax + by = 0$ correctly to the form $r^2 + ar\cos\theta + br\sin\theta = 0$
	$\{r = 0$, origin$\}$ Circle: $r = 6\cos\theta + 8\sin\theta$	A1	4	NMS Mark as 4 or 0
	ALTn			
	Circle has eqn $r = OA\cos(\alpha - \theta)$	(M2)		
	$r = OA\cos\alpha\cos\theta + OA\sin\alpha\sin\theta$	(m1)		OE
	Circle: $r = 6\cos\theta + 8\sin\theta$	(A1)		
	Total		**8**	

Question 4

4 Evaluate the improper integral

$$\int_1^\infty \left(\frac{1}{x} - \frac{4}{4x+1}\right) \mathrm{d}x$$

showing the limiting process used and giving your answer in the form $\ln k$, where k is a constant to be found. *(5 marks)*

Student Response

4. $\int_1^\infty \left(\frac{1}{x} - \frac{4}{4x+1}\right) dx$

$= \lim_{a\to\infty} \int_1^a \left(\frac{1}{x} - \frac{4}{4x+1}\right) dx$

$= \lim_{a\to\infty} \left[\ln|x| - \ln|4x+1|\right]_1^a$

$= \lim_{a\to\infty} \left[\ln \frac{x}{4x+1}\right]_1^a$

$= \lim_{a\to\infty} \left(\ln \frac{a}{4a+1}\right) - \ln\left(\frac{1}{5}\right)$

$= \lim_{a\to\infty} \left(\ln \frac{1}{4+\frac{1}{a}}\right) + \ln 5$

$= \ln\left(\frac{1}{4}\right) + \ln 5$ 5

$= \ln 5 - \ln 4$ (5)

$= \ln \frac{5}{4}$

Commentary

The exemplar illustrates an excellent solution to the worse answered question on the paper. The candidate recognised that the integral was improper because the interval of integration was infinite. In line 2 the infinite upper limit was replaced by *a* and the 'limit as $a\to\infty$' indicated. In line 3 the integral of $\frac{1}{x} - \frac{4}{4x+1}$ was found correctly and in line 4 a law of logarithms was used to write the difference of the two logarithms as a single logarithm. In line 5 the limits *a* and 1 were considered correctly. In line 6, which is the stage many candidates omitted, the candidate had realised that $\frac{a}{4a+1}$ required a further rearrangement to $\frac{1}{4+\frac{1}{a}}$ before the limit as $a\to\infty$ could be taken. In line 7 the candidate correctly found the limit as $a\to\infty$ of $\ln\left(\frac{1}{4+\frac{1}{a}}\right)$ to be $\ln\left(\frac{1}{4}\right)$ and subsequently evaluated the improper integral to the form requested in the question. Although line 8 could have been omitted in the candidate's solution, all other steps were required to show the process used.

Mark Scheme

4	$\int\left(\frac{1}{x}-\frac{4}{4x+1}\right)\mathrm{d}x=\ln x-\ln(4x+1)\{+c\}$	B1		OE
	$\mathrm{I}=\lim_{a\to\infty}\int_1^a\left(\frac{1}{x}-\frac{4}{4x+1}\right)\mathrm{d}x$	M1		∞ replaced by a (OE) and $\lim_{a\to\infty}$
	$=\lim_{a\to\infty}\left[\ln x-\ln(4x+1)\right]_1^a$			
	$=\lim_{a\to\infty}\left[\ln\left(\frac{a}{4a+1}\right)-\ln\frac{1}{5}\right]$	m1		$\ln a-\ln(4a+1)=\ln\left(\frac{a}{4a+1}\right)$ **and** previous M1 scored
	$=\lim_{a\to\infty}\left[\ln\left(\frac{1}{4+\frac{1}{a}}\right)-\ln\frac{1}{5}\right]$	m1		$\ln\left(\frac{a}{4a+1}\right)=\ln\left(\frac{1}{4+\frac{1}{a}}\right)$ **and** previous M1m1 scored
	$=\ln\frac{1}{4}-\ln\frac{1}{5}=\ln\frac{5}{4}$	A1	5	CSO
	Total		**5**	

Question 5

5 It is given that y satisfies the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 8\sin x + 4\cos x$$

(a) Find the value of the constant k for which $y = k\sin x$ is a particular integral of the given differential equation. *(3 marks)*

(b) Solve the differential equation, expressing y in terms of x, given that $y = 1$ and $\frac{dy}{dx} = 4$ when $x = 0$. *(8 marks)*

Student Response

Leave blank

5) a) PI: $y = (A\cos x +) k\sin x$

$\frac{dy}{dx} = k\cos x$

$\frac{d^2y}{dx^2} = -k\sin x$

$\rightarrow -k\sin x + 2k\cos x + 5k\sin x = 8\sin x + 4\cos x$

$\sin x \Rightarrow \quad -k + 5k = 8$

$4k = 8$

$k = 2$

$\cos x \Rightarrow \quad 2k = 4 \qquad \therefore k = 2$

$k = 2$

$y_{PI} = 2\sin x$

3

b) CF: $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$

$m^2 + 2m + 5 = 0$

$m = \frac{-2 \pm \sqrt{4 - 4\times5}}{2}$ M1

A1

$m = -1 \pm 2i$

$y_{CF} = e^{-1}(A\cos 2x + B\sin 2x)$ A0

GS: $y_{CF} + y_{PI}$

$y = e^{-1}(A\cos 2x + B\sin 2x) + 2\sin x \qquad y = 1 \quad x = 0$ B1F

$1 = e^{-1}(A)$ B1F

$A = e$

$\frac{dy}{dx} = e^{-1}(-2A\sin 2x + 2B\cos 2x) + 2\cos x$ M0

$4 = e^{-1}(2B) + 2$ A0

$e^{-1}2B = 2$

$B = e$ A0

$\therefore y = \cos 2x + \sin 2x + 2\sin x$

4

(7)

Commentary

In part (a) the candidate's initial thoughts were to apply the general particular integral for the case where f(x) is of the form $p\sin x+q\cos x$ but with the brackets around the '$A\cos x+$' and the subsequent working, benefit of doubt was given and the '$A\cos x+$' was ignored. It would have been clearer if the candidate had crossed out the '$A\cos x+$'. The candidate showed good examination technique by equating both the coefficients of $\sin x$ and the coefficients of $\cos x$ to confirm that $k=2$ was the solution in both cases. In part (b) the candidate formed and solved the auxiliary equation correctly but then gave the incorrect complementary function, the power of e should have been $-x$ not -1. Follow through credit was given for adding the complementary function to the particular integral to give the general solution and also for applying the boundary condition $y=1$ when $x=0$ correctly on follow through but since the candidate's differentiation of the general solution did not require the product rule, no further marks were available.

Mark Scheme

5(a)	$-k\sin x+2k\cos x+5k\sin x=8\sin x+4\cos x$	M1 A1		Differentiation and subst. into DE
	$k=2$	A1	3	
(b)	Auxl eqn $m^2+2m+5=0$			
	$m=\dfrac{-2\pm\sqrt{4-20}}{2}$	M1		Formula or completing sq. PI
	$m=-1\pm 2\text{i}$	A1		
	CF: $\{y_C\}=\text{e}^{-x}(A\sin 2x+B\cos 2x)$	A1F		ft provided m is not real
	GS $\{y\}=\text{e}^{-x}(A\sin 2x+B\cos 2x)+k\sin x$	B1F		ft on CF + PI; must have 2 arb consts
	When $x=0$, $y=1\Rightarrow B=1$	B1F		
	$\dfrac{\text{d}y}{\text{d}x}=-\text{e}^{-x}(A\sin 2x+B\cos 2x)$ $+\text{e}^{-x}(2A\cos 2x-2B\sin 2x)+k\cos x$	M1		Product rule
	When $x=0$, $\dfrac{\text{d}y}{\text{d}x}=4\Rightarrow 4=-B+2A+k$	A1		PI
	$\Rightarrow A=\dfrac{3}{2}$			
	$y=\text{e}^{-x}\left(\dfrac{3}{2}\sin 2x+\cos 2x\right)+2\sin x$	A1	8	CSO
	Total		**11**	

Question 6

6 The function f is defined by

$$f(x) = (9 + \tan x)^{\frac{1}{2}}$$

(a) (i) Find $f''(x)$. *(4 marks)*

(ii) By using Maclaurin's theorem, show that, for small values of x,

$$(9 + \tan x)^{\frac{1}{2}} \approx 3 + \frac{x}{6} - \frac{x^2}{216}$$

(3 marks)

(b) Find

$$\lim_{x \to 0} \left[\frac{f(x) - 3}{\sin 3x} \right]$$

(3 marks)

Student Response

Leave blank

6a i $f(x) = (9+\tan x)^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}\times\sec^2 x(9+\tan x)^{-\frac{1}{2}}$

$f''(x) = \frac{1}{2}\sec^2 x(9+\tan x)^{-\frac{3}{2}}(\frac{-1}{2})\sec^2 x + [\frac{1}{2}(\sec x)(\sec x\tan x)\times 2](9+\tan x)^{-\frac{1}{2}}$

$= \frac{-1}{4}\sec^4 x(9+\tan x)^{-\frac{3}{2}} + \sec^2 x\tan x(9+\tan x)^{-\frac{1}{2}}$

4

ii $f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2} + \ldots$

$= (\sqrt{9}) + \frac{1}{2}\frac{1\times x}{2\sqrt{9}} + \frac{1}{2}x^2(\frac{-1}{4}(9)^{-\frac{3}{2}} + 0) + \ldots$

$= 3 + \frac{x}{6} - \frac{x^2}{216} + \ldots$

3

b. $\lim_{x\to 0}\left[\frac{\frac{x}{6} - \frac{x^2}{216} + \ldots}{3x - \frac{27x^3}{6} + \ldots}\right] = \lim_{x\to 0}\left[\frac{x(\frac{1}{6} - \frac{x}{216} + \ldots)}{x(3 - \frac{27}{6}x + \ldots)}\right]$

M1 M0 x's still in

$\lim_{x\to 0} = \frac{(\frac{1}{6})}{3} = \frac{1}{18}$

1

8

Commentary

The exemplar illustrates the common error resulting in the loss of the final 2 marks for part (b).
In part (a)(i) the candidate displayed excellent skills in the methods of differentiation and in the use of brackets when applying the product rule and chain rule to find f″(*x*). The candidate's squared brackets in line 4 hold the correct differentiation of $\frac{1}{2}\sec^2 x$ which many other candidates could not find correctly. In (a)(ii) the candidate started by writing down the relevant terms in the general Maclaurin's theorem then substituted the correct unsimplified values using the answers from (a)(i) before completing the solution to obtain the printed answer. In part (b) the candidate recognised the need to use the series expansion from (a)(ii) and also the series expansion for sin3*x* but before the limit had been taken the candidate had not explicitly reduced the numerator and denominator to a constant term in each so in effect the limit taken would result in 0/0. If the candidate had written $\dfrac{\frac{1}{6}-\frac{x}{216}}{3-\frac{27}{6}x}$ where the examiner has written an inverted V in the final line, 2 further marks would have been awarded.

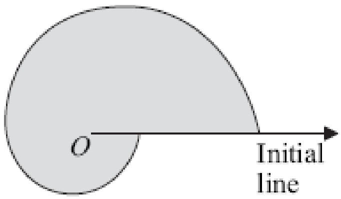
Mark Scheme

6(a)(i)	$f(x)=(9+\tan x)^{\frac{1}{2}}$			
	so $f'(x)=\frac{1}{2}(9+\tan x)^{-\frac{1}{2}}\sec^2 x$	M1 A1		Chain rule
	$f''(x)=-\frac{1}{4}(9+\tan x)^{-\frac{3}{2}}\sec^4 x$	M1		Product rule, OE
	$+\frac{1}{2}(9+\tan x)^{-\frac{1}{2}}(2\sec^2 x\tan x)$	A1	4	ACF
(a)(ii)	$f(0)=3$	B1		
	$f'(0)=\frac{1}{2}(9)^{-\frac{1}{2}}=\frac{1}{6}$; $f''(0)=-\frac{1}{4}(9)^{-\frac{3}{2}}=-\frac{1}{108}$	M1		Both attempted and at least one correct ft on c's f ′(*x*) and f ″(*x*)
	$f(x)\approx f(0)+x\,f'(0)+\frac{1}{2}x^2 f''(0)$			
	$(9+\tan x)^{\frac{1}{2}}\approx 3+\frac{x}{6}-\frac{x^2}{216}$	A1	3	CSO AG
(b)	$\frac{f(x)-3}{\sin 3x}\approx\dfrac{\frac{x}{6}-\frac{x^2}{216}\ldots}{3x-\frac{(3x)^3}{3!}\ldots}$	M1		Using series expns.
	$\approx\dfrac{\frac{1}{6}-\frac{x}{216}\ldots}{3-\ldots}$	m1		Dividing numerator and denominator by *x* to get constant term in each
	$\lim_{x\to 0}\left[\frac{f(x)-3}{\sin 3x}\right]=\frac{1}{18}$	A1	3	
	Total		**10**	

Question 7

7 The diagram shows the curve C_1 with polar equation

$$r = 1 + 6e^{-\frac{\theta}{\pi}}, \quad 0 \leqslant \theta \leqslant 2\pi$$



(a) Find, in terms of π and e, the area of the shaded region bounded by C_1 and the initial line. *(5 marks)*

(b) The polar equation of a curve C_2 is

$$r = e^{\frac{\theta}{\pi}}, \quad 0 \leqslant \theta \leqslant 2\pi$$

Sketch the curve C_2 and state the polar coordinates of the end-points of this curve. *(4 marks)*

(c) The curves C_1 and C_2 intersect at the point P. Find the polar coordinates of P. *(5 marks)*

Student Response

$e^{-x/\pi} = e^{-\frac{1}{\pi}x}$ $\quad -\frac{1}{\pi}x \quad = -\pi e^{-\frac{1}{\pi}x}$

$e^{-x/3} = e^{-1/3 x}$ $\quad -3e^{-1/3 x}$

Question number

Leave blank

7. $r = 1 + 6e^{-\theta/\pi}, \quad 0 \le \theta \le 2\pi$

a) Area bound by polar curve $= \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta$ ✓

$r^2 = (1+6e^{-\theta/\pi})^2 = 1 + 12e^{-\theta/\pi} + 6e^{-2\theta/\pi}$ ✗ B0

Area $= \frac{1}{2}\int_0^{2\pi} (1 + 12e^{-\theta/\pi} + 6e^{-2\theta/\pi}) \, d\theta$ M1 B1

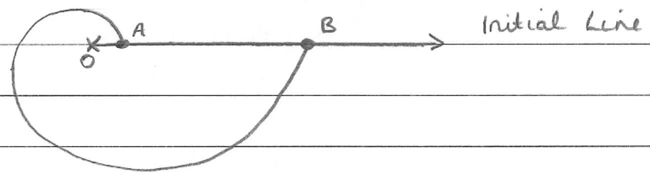
$= \frac{1}{2}\left[\theta - 12\pi e^{-\theta/\pi} - 6\pi e^{-2\theta/\pi}\right]_0^{2\pi}$ m1

$= \frac{1}{2}\left[2\pi - 12\pi e^{-2\pi/\pi} - 6\pi e^{-2\times 2\pi/\pi} + 12\pi + 6\pi\right]$

$= \frac{1}{2}\left[20\pi - 12\pi e^{-2} - 6\pi e^{-4}\right]$

$= 10\pi - 6\pi e^{-2} - 3\pi e^{-4}$ ✗ A0 3

b) $r = e^{\theta/\pi}, \quad 0 \le \theta \le 2\pi.$



B1 B1

$A(r,\theta) = (1, 0)$ ✓ and $B(r,\theta) = (7.38, 0)$ B1
$= (e^2, 0)$ ✗ B0 3

Leave blank

c) $C_1: r = 1 + 6e^{-\theta/\pi}$

$C_2: r = e^{\theta/\pi}$

At point of intersection:

$1 + 6e^{-\theta/\pi} = e^{\theta/\pi}$ ✓

Multiply through by $e^{\theta/\pi}$:

$e^{\theta/\pi} + 6 = e^{2\theta/\pi}$

$e^{2\theta/\pi} - e^{\theta/\pi} - 6 = 0$ ✓

$(e^{\theta/\pi} - 3)(e^{\theta/\pi} + 2) = 0$ ✓

$e^{\theta/\pi} = 3$ $\qquad$ $e^{\theta/\pi} = -2$

$\theta/\pi = \ln 3$ ✓ $\qquad$ Not possible because

$\theta = \pi \ln 3$ $\qquad$ you can't have ln of a ✓

$(= 3.451$ (3d.p.)) $\qquad$ negative number.

Sub into C_2:

$r = e^{\theta/\pi}$

$r = e^{\pi \ln 3/\pi} = e^{\ln 3} = 3$ ✓

So: Point of Intersection, $P = (r, \theta)$ ✓

$= (3, \pi \ln 3)$

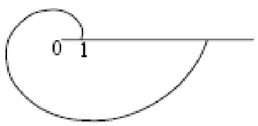
$[= (3, 3.451$ (3d.p.)$]$

5

(11)

Commentary

In part (a) the candidate displayed good examination technique by quoting the general formula for the area in polar coordinates referred to on page 8 in the booklet of formulae. The candidate made an error in squaring $6e^{-\frac{\theta}{\pi}}$ and so lost the mark for finding r^2 but still gained the method mark for integration, even though the final term is incorrect on follow through, as two of the three terms have been integrated correctly. The candidate also lost the final accuracy mark. In part (b) the candidate had drawn a correct sketch but the θ-coordinate for *B*, one of the end points, was incorrect since the 0 should be 2π. Examiners expected candidates to give answers in exact forms rather than use decimal approximations so $(e^2, 2\pi)$ would have been given B1 but $(7.389, 2\pi)$ would not have scored the B1. In part (c) the candidate found the coordinates of the point of intersection by first equating the *r*s and forming and solving a quadratic equation in $e^{\frac{\theta}{\pi}}$. Credit was awarded for rejecting the negative solution although it would have been sufficient if the candidate had just stated 'not possible since $e^{\frac{\theta}{\pi}} > 0$'. The candidate gave the correct exact coordinates for *P* but then in squared brackets gave a 3 d.p. approximation. If the $\pi \ln 3$ had not been seen, the final mark would not have been awarded for (3, 3.451).

7(a)	Area = $\frac{1}{2}\int\left(1+6e^{-\frac{\theta}{\pi}}\right)^2\ \mathrm{d}\theta$	M1		Use of $\frac{1}{2}\int r^2\,\mathrm{d}\theta$
	$=\frac{1}{2}\int_0^{2\pi}\left(1+12e^{-\frac{\theta}{\pi}}+36e^{-\frac{2\theta}{\pi}}\right)\mathrm{d}\theta$	B1		Correct expansion of $\left(1+6e^{-\frac{\theta}{\pi}}\right)^2$
		B1		Correct limits
	$=\frac{1}{2}\left[\theta-12\pi\,e^{-\frac{\theta}{\pi}}-18\pi\,e^{-\frac{2\theta}{\pi}}\right]_0^{2\pi}$	m1		Correct integration of at least two of the three terms $1,\ p\,e^{-\frac{\theta}{\pi}},\ q\,e^{-\frac{2\theta}{\pi}}$
	$=\pi\,(16-6e^{-2}-9e^{-4})$	A1	5	ACF
(b)		B1		Going the correct way round the pole
		B1		Increasing in distance from the pole
	End-points $(1, 0)$ and $(e^2, 2\pi)$	B2,1,0	4	Correct end-points B1 for each pair or for 1 and e^2 shown on graph in correct positions
(c)	$e^{\frac{\theta}{\pi}}=1+6\,e^{-\frac{\theta}{\pi}}$	M1		Elimination of r or θ $[r=1+\frac{6}{r}]$
	$\left(e^{\frac{\theta}{\pi}}\right)^2-e^{\frac{\theta}{\pi}}-6=0$	m1		Forming quadratic in $e^{\frac{\theta}{\pi}}$ or in $e^{-\frac{\theta}{\pi}}$ or in r. $[r^2-r-6=0]$
	$\left(e^{\frac{\theta}{\pi}}-3\right)\left(e^{\frac{\theta}{\pi}}+2\right)=0$	m1		OE
	$e^{\frac{\theta}{\pi}}>0$ so $e^{\frac{\theta}{\pi}}=3$	E1		Rejection of negative 'solution' PI $[r=3]$
	Polar coordinates of P are $(3,\ \pi\ln 3)$	A1	5	
	Total		**14**	

Question 8

8 (a) Given that $x = t^2$, where $t \geqslant 0$, and that y is a function of x, show that:

(i) $2\sqrt{x}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}$; *(3 marks)*

(ii) $4x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}^2y}{\mathrm{d}t^2}$. *(3 marks)*

(b) Hence show that the substitution $x = t^2$, where $t \geqslant 0$, transforms the differential equation

$$4x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2(1 + 2\sqrt{x})\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = 0$$

into

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 2\frac{\mathrm{d}y}{\mathrm{d}t} - 3y = 0$$ *(2 marks)*

(c) Hence find the general solution of the differential equation

$$4x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2(1 + 2\sqrt{x})\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = 0$$

giving your answer in the form $y = \mathrm{g}(x)$. *(4 marks)*

Student Response

8. $x = t^2$

$\frac{dx}{dt} = 2t$

$\frac{dy}{dx} \times \frac{dx}{dt} = \frac{dy}{dt}$

$\frac{dy}{dx} \times 2t = \frac{dy}{dt}$

$\sqrt{x} = t \quad \therefore \quad 2t = 2\sqrt{x}$

$\frac{dy}{dx} \times 2\sqrt{x} = \frac{dy}{dt}$

3

ii $\frac{d}{dx}\left(\frac{dy}{dx} 2\sqrt{x}\right) = \frac{d}{dx}\left(\frac{dy}{dt}\right)$

$\frac{d^2y}{dx^2} 2\sqrt{x} + \frac{dy}{dx} x^{-1/2} = \frac{d^2y}{dt^2} \times \frac{dt}{dx}$

$\frac{d^2y}{dx^2} 2\sqrt{x} + \frac{dy}{dx} x^{-1/2} = \frac{d^2y}{dt^2} \times \frac{1}{2\sqrt{x}}$

$\frac{d^2y}{dx^2} 4x + \frac{2\sqrt{x}}{\sqrt{x}} \frac{dy}{dx} = \frac{d^2y}{dt^2}$

3

$\frac{d^2y}{dx^2} 4x + 2\frac{dy}{dx} = \frac{d^2y}{dt^2}$

b $4x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4\sqrt{x}\frac{dy}{dx} - 3y = 0$

$\frac{d^2y}{dt^2} + 2 \times 2\sqrt{x}\frac{dy}{dx} - 3y = 0$

2

$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = 0$

c	$\lambda^2 + 2\lambda - 3 = 0$
	$(\lambda + 3)(\lambda - 1) = 0$
	$\lambda = -3$ and $\lambda = 1$
	$\therefore\ y = Ae^{-3t} + Be^{t}$
	$= Ae^{-3\sqrt{x}} + Be^{\sqrt{x}}$

4

12

Commentary

The exemplar illustrates a correct solution with all necessary steps clearly shown in obtaining the printed answers.

In part (a)(i) the candidate picked up an easy mark in line 2 for getting $\frac{dx}{dt} = 2t$, a further mark in line 3 for quoting a relevant chain rule and the final mark for a correct completion to the printed result stage. In part (a)(ii) the candidate differentiated the result shown in (a)(i) with respect to *x* and by line 2 had finished all the required differentiation and scored the 2 method marks. The remaining three lines showed clearly the completion to the printed result. The candidate started part (b) by multiplying out the brackets in the first differential equation and then clearly showed how the results from parts (a) were used to obtain the second differential equation. In part (c) the candidate wrote down and solved the auxiliary equation and in line 4 stated the solution for *y* in terms of *t*. A closer examination of this line suggests that the candidate may have initially made the common mistake of writing $y = Ae^{-3x} + Be^{x}$ but the final line confirmed that any error had been corrected by use of the substitution $t = \sqrt{x}$ and the candidate was awarded full marks for the correct solution.

Mark Scheme

8(a)(i)	$\frac{dx}{dt} = 2t$	B1		PI or for $\frac{dt}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$
	$\frac{dx}{dt}\frac{dy}{dx} = \frac{dy}{dt}$	M1		OE Chain rule $\frac{dy}{dx} = \ldots$ or $\frac{dy}{dt} = \ldots$
	$2t\frac{dy}{dx} = \frac{dy}{dt}$ so $2\sqrt{x}\frac{dy}{dx} = \frac{dy}{dt}$	A1	3	AG
(a)(ii)	$\frac{d}{dx}\left(2\sqrt{x}\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dt}\right) = \frac{dt}{dx}\frac{d}{dt}\left(\frac{dy}{dt}\right)$	M1		$\frac{d}{dx}(f(t)) = \frac{dt}{dx}\frac{d}{dt}(f(t))$ OE eg $\frac{d}{dt}(g(x)) = \frac{dx}{dt}\frac{d}{dx}(g(x))$
	$2\sqrt{x}\frac{d^2y}{dx^2} + x^{-\frac{1}{2}}\frac{dy}{dx} = \frac{1}{2t}\frac{d^2y}{dt^2}$	M1		Product rule OE
	$4t\sqrt{x}\frac{d^2y}{dx^2} + 2tx^{-\frac{1}{2}}\frac{dy}{dx} = \frac{d^2y}{dt^2}$			
	$\Rightarrow 4x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = \frac{d^2y}{dt^2}$	A1	3	AG Completion
(b)	$4x\frac{d^2y}{dx^2} + 2(1+2\sqrt{x})\frac{dy}{dx} - 3y = 0$			
	$(4x\frac{d^2y}{dx^2} + 2\frac{dy}{dx}) + 2(2\sqrt{x}\frac{dy}{dx}) - 3y = 0$			
	$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = 0$	M1 A1	2	Use of either (a)(i) or (a)(ii) AG Completion
(c)	$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = 0$ (*)			
	Auxl. Eqn. $m^2 + 2m - 3 = 0$			
	$(m+3)(m-1) = 0$	M1		PI
	$m = -3$ and 1	A1		PI
	GS of (*) $\{y\} = Ae^{-3t} + Be^{t}$	M1		$Ae^{-3x} + Be^{x}$ scores M0 here
	$\Rightarrow y = Ae^{-3\sqrt{x}} + Be^{\sqrt{x}}$	A1	4	
	Total		**12**	