Teacher Support Materials 2008

Maths GCE

Paper Reference MFP3

Question 1

1 The function $y(x)$ satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x, y)$$

where $$\mathrm{f}(x, y) = \ln(x + y)$$

and $$y(2) = 3$$

Use the improved Euler formula

$$y_{r+1} = y_r + \tfrac{1}{2}(k_1 + k_2)$$

where $k_1 = h\mathrm{f}(x_r, y_r)$ and $k_2 = h\mathrm{f}(x_r + h, y_r + k_1)$ and $h = 0.1$, to obtain an approximation to $y(2.1)$, giving your answer to four decimal places. *(6 marks)*

Student Response

number

1.)

x	y	k_i	k_{ii}
2	3	0.160943 ✓	0.164865 ✗
2.1			

?? M1 A1 M0 A0

$y(2.1) = 3 + \frac{1}{2}(0.160943 + 0.164865)$ ∴ M0 A0

$= 3 + 0.162904431\ldots$

$= 3.016290$

$= 3.0163$ (4dp) A0

2

(2)

Commentary

Although this question was generally answered very well by candidates, the exemplar illustrates partial poor examination technique and also a common wrong value. In the exemplar the candidate stated the values of k_1 and k_2 without showing any method. The correct value for k_1 gained two marks but if the candidate had miscopied the value from the calculator display in this case, without showing the working, no marks could have been awarded for method. The candidate gave a wrong value for k_2. Although no method was shown, the value given was the same as that obtained by a significant number of other candidates who showed that they had used $k_2 = 0.1\ln(2.1 + 3.1)$, that is, the candidate has used $y_r + h$ instead of $y_r + k_1$ in finding k_2. No further marks could be awarded as all subsequent marks were dependent on gaining the first two method marks.

Mark scheme

1	$k_1 = 0.1 \times \ln(2+3)$ $= 0.1609(4379\ldots)$ (= *)	M1 A1		 PI
	$k_2 = 0.1 \times f(2.1,\ 3 + *\ldots)$ $\ldots = 0.1 \times \ln(2.1 + 3.16094\ldots)]$	M1		
	$\ldots. = 0.1660(31\ldots)$	A1		PI
	$y(2.1) = y(2) + \frac{1}{2}[k_1 + k_2]$ $= 3 + 0.5 \times 0.3269748\ldots.$	m1		Dep on previous two Ms and numerical values for k's
	$= 3.163487\ldots = 3.1635$ to 4dp	A1	6	Must be 3.1635
	Total		**6**	

Question 2

2 (a) Find the values of the constants a, b, c and d for which $a + bx + c\sin x + d\cos x$ is a particular integral of the differential equation

$$\frac{dy}{dx} - 3y = 10\sin x - 3x \qquad \textit{(4 marks)}$$

(b) Hence find the general solution of this differential equation. *(3 marks)*

Student response

Question number

Leave blank

2

a. $y = a + bx + c\sin x + d\cos x$

$\frac{dy}{dx} = b + c\cos x - d\sin x$ ✓

$\frac{d^2y}{dx^2} = -c\sin x - d\cos x$

$\frac{dy}{dx} - 3y = 10\sin x - 3x$

✓ M1

$= b + c\cos x - d\sin x - 3a - 3bx - 3c\sin x - 3d\cos x = 10\sin x - 3x$

$b - 3a = 0$ ✓ $\qquad a = \frac{1}{3}$ ✓

$-3b = -3$ ✓ $\qquad b = 1$ ✓

$c - 3d = 0$ ✓ ①

$-d - 3c = 10$ ✓ ② $\qquad$ M1 A1

① × 3 = $3c - 9d = 0$ ③

③ + ② = $-10d = 10$ $\qquad d = -1$ ✓

$c = -3$ ✓

A1

$a = \frac{1}{3} \quad b = 1 \quad c = -3 \quad d = -1$ ✓ $\qquad$ 4

b. (CF) $m - 3 = 0$ $\qquad$ M1

$m = 3$ ✓

$y = e^{3x}(A + Bx)$ ✗ $\qquad$ A0

(GS) = $y = e^{3x}(A + Bx) + \frac{1}{3} + x - 3\sin x - \cos x$ $\qquad$ B0 $\qquad$ 1

(5)

Commentary

In the exemplar the candidate gave a correct solution to part (a) by equating coefficients to form and then solve the four equations to find the correct values for the four unknowns a, b, c and d. A significant number of candidates, like the one in the exemplar, wasted time by finding an expression for $\frac{d^2y}{dx^2}$ which was not required in the solution to find the particular integral of the first order differential equation. In part (b) the exemplar illustrates a common error. The candidate correctly solved the auxiliary equation $m - 3 = 0$ but incorrectly took this to be a repeated root of an auxiliary equation to a second order differential equation and gave the general solution of the first order differential equation with two arbitrary constants instead of the required one.

Mark Scheme

2(a)	PI: $y_{PI} = a + bx + c\sin x + d\cos x$			
	$y'_{PI} = b + c\cos x - d\sin x$			
	$b + c\cos x - d\sin x - 3a - 3bx - 3c\sin x$ $-3d\cos x = 10\sin x - 3x$	M1		Substituting into DE
	$b-3a=0$; $-3b=-3$; $c-3d=0$; $-d-3c=10$	M1		Equating coefficients (at least 2 eqns)
	$a = \frac{1}{3}$; $b = 1$; $c = -3$; $d = -1$	A2,1	4	A1 for any two correct
	$y_{PI} = \frac{1}{3} + x - 3\sin x - \cos x$			
(b)	Aux. eqn. $m - 3 = 0$	M1		Altn. $\int y^{-1}\,dy = \int 3\,dx$ OE (M1)
	$(y_{CF} =)Ae^{3x}$	A1		Ae^{3x} OE
	$(y_{GS} =)Ae^{3x} + \frac{1}{3} + x - 3\sin x - \cos x$	B1F	3	(c's CF + c's PI) with 1 arbitrary constant
	Total		7	

Question 3

3	(a)	Show that $x^2 = 1 - 2y$ can be written in the form $x^2 + y^2 = (1 - y)^2$.	*(1 mark)*
	(b)	A curve has cartesian equation $x^2 = 1 - 2y$.	
		Find its polar equation in the form $r = \mathrm{f}(\theta)$, given that $r > 0$.	*(5 marks)*

Student Response

3. (a) $x^2 = 1-2y \Rightarrow x^2+y^2 = 1-2y+y^2 = (1-y)^2$ ✓ 1

(b) $\because x^2+y^2 = r^2$ ✓ and $r\sin\theta = y$ ✓

$\therefore x^2+y^2 = (1-y)^2 \Rightarrow r^2 = (1-r\sin\theta)^2$ ✓

$\Rightarrow r = \pm(1-r\sin\theta)$ ✓

If $r = 1-r\sin\theta$, $(1+\sin\theta)r = 1 \Rightarrow r = \frac{1}{1+\sin\theta}$ ✓

If $r = -(1-r\sin\theta)$, $r\sin\theta - 1 = r \Rightarrow (\sin\theta-1)r = 1 \Rightarrow r = \frac{1}{\sin\theta - 1}$

Since $-1 \le \sin\theta \le 1$, $1+\sin\theta \ge 0$ while $\sin\theta - 1 \le 0$ ✓

$\because r > 0 \quad \therefore r = \frac{1}{1+\sin\theta}$ ✓ 5

(6)

Commentary

Part (a) was generally answered correctly but it was unusual to see solutions for which the fifth mark was awarded in part (b). In the exemplar the candidate scored this final mark because, within this excellent solution, both square roots (the ±) had been considered and a full and accurate justification for eliminating the solution $r = \frac{1}{\sin\theta - 1}$ was given by the candidate.

Mark Scheme

3(a)	$x^2 + y^2 = 1 - 2y + y^2 \Rightarrow x^2 + y^2 = (1-y)^2$	B1	1	AG
(b)	$x^2 + y^2 = r^2$	M1		Or $x = r\cos\theta$
	$y = r\sin\theta$	M1		
	$x^2 = 1 - 2y$ so $x^2 + y^2 = (1-y)^2$ $\Rightarrow r^2 = (1 - r\sin\theta)^2$	A1		OE eg $r^2\cos^2\theta = 1 - 2r\sin\theta$ PI by the next line
	$r = 1 - r\sin\theta$ or $r = -(1 - r\sin\theta)$ $r(1+\sin\theta) = 1$ or $r(1-\sin\theta) = -1$	m1		Either
	$r > 0$ so $r = \frac{1}{1+\sin\theta}$	A1	5	CSO
	Total		**6**	

Question 4

4 (a) A differential equation is given by

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} = 3x^2$$

Show that the substitution

$$u = \frac{dy}{dx}$$

transforms this differential equation into

$$\frac{du}{dx} - \frac{1}{x}u = 3x$$ *(2 marks)*

(b) By using an integrating factor, find the general solution of

$$\frac{du}{dx} - \frac{1}{x}u = 3x$$

giving your answer in the form $u = f(x)$. *(6 marks)*

(c) Hence find the general solution of the differential equation

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} = 3x^2$$

giving your answer in the form $y = g(x)$. *(2 marks)*

Student Response

Leave blank

4a $x\frac{d^2y}{dx^2} - \frac{dy}{dx} = 3x^2$

$u = \frac{dy}{dx}$

$\frac{du}{dx} = \frac{d^2y}{dx^2}$

$x\frac{du}{dx} - u = 3x^2$

$\frac{du}{dx} - \frac{1}{x}u = 3x$

2

b $I = e^{\int P(x)\,dx}$

$-\int \frac{1}{x}\,dx = -\ln x = \ln x^{-1}$ M1 A1

$I = e^{\ln x^{-1}} = \frac{1}{x}$ A1

$\frac{1}{x}u = \int 3\,dx$ M1

$\frac{1}{x}u = 3x$ m0

$u = 3x^2$ A0

4

c $\frac{dy}{dx} = 3x^2$

$y = x^3$ M1 A0

1

(7)

Commentary

A significant number of candidates lost some marks because they forgot to include the constants of integration. The exemplar illustrates this error which resulted in the candidate giving a general solution of the first order differential equation in part (b) with no arbitrary constant and giving a general solution of the second order differential equation in part (c) also with no arbitrary constants. Candidates would have been well advised to check that in their general solution of a differential equation, the number of arbitrary constants was the same as the order of the differential equation.

4(a)	$u = \frac{dy}{dx} \Rightarrow \frac{du}{dx} = \frac{d^2y}{dx^2}$	M1		
	$x\frac{du}{dx} - u = 3x^2 \Rightarrow \frac{du}{dx} - \frac{1}{x}u = 3x$	A1	2	AG Substitution into LHS of DE and completion
(b)	IF is exp ($\int -\frac{1}{x}\,dx$)	M1		and with integration attempted
	$= e^{-\ln x}$	A1		
	$= x^{-1}$ or $\frac{1}{x}$	A1		or multiple of x^{-1}
	$\frac{d}{dx}\left[ux^{-1}\right] = 3$	M1		LHS as differential of $u \times$ IF . PI
	$\Rightarrow ux^{-1} = 3x + A$	m1		Must have an arbitrary constant (Dep. on previous M1 only)
	$u = 3x^2 + Ax$	A1	6	
(c)	$\frac{dy}{dx} = 3x^2 + Ax$	M1		Replaces u by $\frac{dy}{dx}$ and attempts to integrate
	$y = x^3 + \frac{Ax^2}{2} + B$	A1F	2	ft on cand's u but solution must have two arbitrary constants
	Total		**10**	

Question 5

5	(a)	Find $\int x^3 \ln x \, dx$.	*(3 marks)*
	(b)	Explain why $\int_0^e x^3 \ln x \, dx$ is an improper integral.	*(1 mark)*
	(c)	Evaluate $\int_0^e x^3 \ln x \, dx$, showing the limiting process used.	*(3 marks)*

Student Response

5a $\int x^3 \ln x \, dx$ $u = \ln x$ $\frac{du}{dx} = \frac{1}{x}$ $\frac{dv}{dx} = x^3$ $v = \frac{x^4}{4}$

$\int x^3 \ln x \, dx = \frac{x^4}{4} \ln x - \frac{1}{4}\int x^3 \, dx$ $uv - \int v \frac{du}{dx}$

$= \frac{x^4}{4} \ln x - \frac{x^4}{16} + c$. 3

b $x^3 \ln x$ is not defined at point 0 as $\ln 0$ does not exist 1

c $\int_0^e x^3 \ln x \, dx = \lim_{a \to 0} \int_a^e x^3 \ln x \, dx$

$= \lim_{a \to 0} \left[\frac{x^4}{4} \ln x - \frac{x^4}{16}\right]_a^e$

$= \lim_{a \to 0} \left[\frac{e^4}{4} - \frac{e^4}{16} - \frac{a^4}{4}\ln a + \frac{a^4}{16}\right]$ M1

as $a \to 0$ $\frac{-a^4}{4}\ln a$ + $\frac{a^4}{16} \to 0$ B1

$\therefore \int_0^e x^3 \ln x \, dx = \frac{e^4}{4} - \frac{e^4}{16}$ 2

A A0

6

Commentary

The candidate in the exemplar used integration by parts to find the correct expression for $\int x^3 \ln x \, dx$ and, in part (b), provided a correct explanation for why $\int_0^e x^3 \ln x \, dx$ is an improper integral. In part (c) the candidate showed excellent detail of the limiting process used, in particular the inclusion of $\int_0^e x^3 \ln x \, dx = \left\{\lim_{a\to 0}\int_a^e x^3 \ln x \, dx\right\}$ and the statement {as $a \to 0$, $-\frac{a^4}{4}\ln a \to 0$}. The candidate failed to score the final accuracy mark because the expression $\frac{e^4}{4} - \frac{e^4}{16}$ had not been simplified to $\frac{3e^4}{16}$.

Mark Scheme

5(a)	$\int x^3 \ln x \, dx = \frac{x^4}{4}\ln x - \int \frac{x^4}{4}\left(\frac{1}{x}\right) dx$	M1 A1		$... = kx^4 \ln x \pm \int f(x)$, with f(x) not involving the 'original' $\ln x$
	$...... = \frac{x^4}{4}\ln x - \frac{x^4}{16} + c$	A1	3	Condone absence of '$+ c$'
(b)	Integrand is not defined at $x = 0$	E1	1	OE
(c)	$\int_0^e x^3 \ln x \, dx = \left\{\lim_{a\to 0}\int_a^e x^3 \ln x \, dx\right\}$ $= \frac{3e^4}{16} - \lim_{a\to 0}\left[\frac{a^4}{4}\ln a - \frac{a^4}{16}\right]$	M1		F(e) – F(a)
	But $\lim_{a\to 0} a^4 \ln a = 0$	B1		Accept a general form eg $\lim_{x\to 0} x^k \ln x = 0$
	So $\int_0^e x^3 \ln x \, dx$ exists and $= \frac{3e^4}{16}$	A1	3	CSO
	Total		7	

Question 6

6 (a) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 10e^{-2x} - 9$$ *(10 marks)*

(b) Hence express y in terms of x, given that $y = 7$ when $x = 0$ and that $\frac{dy}{dx} \to 0$ as $x \to \infty$. *(4 marks)*

Student Response

Question number | Leave blank

(6)

(a) $m^2 - 2m - 3$ ✓ M1

$(m-3)(m+1)$ ✓

$m = 3 \quad m = -1$ ✓ A1

CF: $y = Ae^{3x} + Be^{-x}$ ✓ M1

PI $y = \lambda e^{-2x} + \beta$ ✓ M1

$y' = -2\lambda e^{-2x}$ ✓ A1

$y'' = 4\lambda e^{-2x}$ ✓ A1

$4\lambda e^{-2x} - 2(-2\lambda e^{-2x}) - 3(\lambda e^{-2x}) - 3\beta = 10e^{-2x} - 9$ ✓ M1

$5\lambda e^{-2x} - 3\beta = 10e^{-2x} - 9$ ✓

$3\beta = 9$ ✓ B1

$\beta = 3$

$5\lambda = 10$ ✓ A1

$\lambda = 2$ ✓ B1

$\therefore$ PI $= y = 2e^{-2x} + 3$

10

$\therefore$ general solution $=$ CF + PI ✓

$y = Ae^{3x} + Be^{-x} + 2e^{-2x} + 3$

(b)

$7 = A + B + 2 + 3$ ✓ B1

$2 = A + B$

$y' = 3Ae^{3x} - Be^{-x} - 4e^{-2x}$ ✓

~~$y' = e^{-x}(3Ae^{4x} - B - 4e^{-x})$~~

~~y~~ ~~$y' = 3A - B - 4$~~

$0 = 3A - B - 4$ ✗

$B = -0.5$ ✗

$A = 1$ ✗

$(A + B)2A - 2B = 4$

$2 + 2A - 2B = 4$

$2 = 2A - 2B \qquad 2(A - B) = 2$

$\therefore y = e^{3x} - 0.5e^{-x} + 2e^{-2x} + 3$ ✗ $\quad A - B = 1 \quad A + B = 2$

1

(11)

Commentary

The exemplar illustrates a typical answer to this mainly unstructured question. The candidate gave a full correct solution to find the general solution of the given second order differential equation in part (a). In part (b) the candidate correctly used the given boundary condition, $y = 7$ when $x = 0$, to get $2 = A + B$ but did not apply the limiting boundary condition $\frac{dy}{dx} \to 0$ as $x \to \infty$ correctly. The incorrect equation, $0 = 3A - B - 4$, was obtained by many candidates and effectively came from using the more familiar boundary condition $\frac{dy}{dx} = 0$ when $x = 0$.

Mark Scheme

Q	Solution	Marks	Total	Comments
6(a)	Aux eqn: $m^2 - 2m - 3 = 0$	M1		
	$m = -1,\ 3$	A1		PI
	CF $(y_C =)Ae^{3x} + Be^{-x}$	M1		
	Try $(y_{PI} =)a\,e^{-2x}\ (+b)$	M1		
	$\frac{dy}{dx} = -2ae^{-2x}$	A1		
	$\frac{d^2y}{dx^2} = 4ae^{-2x}$	A1		
	Substitute into DE gives $4ae^{-2x} + 4ae^{-2x} - 3ae^{-2x} - 3b = 10e^{-2x} - 9$	M1		
	$\Rightarrow a = 2$	A1		
	$b = 3$	B1		
	$(y_{GS} =)Ae^{3x} + Be^{-x} + 2e^{-2x} + 3$	B1F	10	(c's CF+c's PI) with 2 arbitrary constants
(b)	$x = 0,\ y = 7 \Rightarrow 7 = A + B + 2 + 3$	B1F		Only ft if exponentials in GS and two arbitrary constants remain
	$\frac{dy}{dx} = 3Ae^{3x} - Be^{-x} - 4e^{-2x}$			
	As $x \to \infty$, $e^{-kx} \to 0$, $\frac{dy}{dx} \to 0$ so $A = 0$	B1		
	When $A = 0$, $5 = 0 + B + 3 \Rightarrow B = 2$	B1F		Must be using 'A' = 0
	$y = 2e^{-x} + 2e^{-2x} + 3$	A1	4	CSO
	Total		**14**	

Question 7

7 (a) Write down the expansion of $\sin 2x$ in ascending powers of x up to and including the term in x^3. *(1 mark)*

(b) (i) Given that $y = \sqrt{3 + e^x}$, find the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when $x = 0$. *(5 marks)*

(ii) Using Maclaurin's theorem, show that, for small values of x,

$$\sqrt{3 + e^x} \approx 2 + \frac{1}{4}x + \frac{7}{64}x^2$$ *(2 marks)*

(c) Find

$$\lim_{x \to 0} \left[\frac{\sqrt{3 + e^x} - 2}{\sin 2x} \right]$$ *(3 marks)*

Student Response

Leave blank

(7)

(a) $\sin 2x = 2x - \frac{(2x)^3}{3!}$

$\sin 2x = 2x - \frac{8x^3}{6}$ ✓ B1

$\sin 2x = 2x - \frac{4}{3}x^3$

1

(b)

(i) $y = (3 + e^x)^{1/2}$

$y' = \frac{1}{2}e^x (3+e^x)^{-1/2}$ ✓ M1 A1

$u = \frac{1}{2}e^x$ $\quad v = (3+e^x)^{-1/2}$

$u' = \frac{1}{2}e^x$ $\quad v' = -\frac{1}{2}e^x(3+e^x)^{-3/2}$

$y'' = -\frac{1}{4}e^{2x}(3+e^x)^{-3/2} + \frac{1}{2}e^x(3+e^x)^{-1/2}$ ✓ M1 A1

$F'(0) = \frac{1}{4}$ ✓

$F''(0) = \frac{7}{32}$ ✓ A1

5

(ii) $F(0) = 2$ ✓

$\therefore \sqrt{3+e^x} = 2 + \frac{1}{4}x + \frac{7}{32 \times 2}x^2$ ✓ M1

$\Rightarrow = 2 + \frac{1}{4}x + \frac{7}{64}x^2$ ✓ A1

2

(c) $\lim_{x \to 0} \left[\frac{2 + \frac{1}{4}x + \frac{7}{64}x^2 - 2}{2x - \frac{4}{3}x^3} \right]$ ✓ M1

$\lim_{x \to 0} \left[\frac{\frac{1}{4}x}{2x} \right] = \frac{1}{8}$ M0 A0

1

9

Commentary

In part (a) the candidate in the exemplar quoted the correct expansion of $\sin 2x$ and, in particular, had replaced $3!$ by 6. In part (b)(i) the candidate showed good skills in applying the chain rule and product rule for differentiating the function. In (b)(ii) the candidate clearly stated the remaining value, $f(0)=2$, which is required and applied Maclaurin's theorem correctly. In part (c) the candidate had used previously found expansions but did not divide the denominator and numerator by x to get a constant term in each before applying the limit as x tends to zero.

Mark Scheme

7(a)	$\sin 2x \approx 2x - \frac{(2x)^3}{3!} + .. = 2x - \frac{4}{3}x^3 + ..$	B1	1	
(b)(i)	$\frac{dy}{dx} = \frac{1}{2}(3+e^x)^{-\frac{1}{2}}(e^x)$	M1 A1		Chain rule
	$\frac{d^2y}{dx^2} = \frac{1}{2}e^x(3+e^x)^{-\frac{1}{2}} - \frac{1}{4}(3+e^x)^{-\frac{3}{2}}(e^{2x})$	M1 A1		Product rule OE OE
	$y'(0) = \frac{1}{4}$; $y''(0) = \frac{1}{4} - \frac{1}{32} = \frac{7}{32}$	A1	5	CSO
(ii)	$y(0) = 2$; $y'(0) = \frac{1}{4}$; $y''(0) = \frac{1}{4} - \frac{1}{32} = \frac{7}{32}$ McC. Thm: $y(0) + x\,y'(0) + \frac{x^2}{2}\,y''(0)$ $\sqrt{3+e^x} \approx 2 + \frac{1}{4}x + \frac{7}{64}x^2$	M1 A1	2	CSO; AG
(c)	$\left[\frac{\sqrt{3+e^x}-2}{\sin 2x}\right] = \left[\frac{2+\frac{1}{4}x+\frac{7}{64}x^2-2}{2x-\frac{4}{3}x^3}\right]$	M1		
	$= \left[\frac{\frac{1}{4}+\frac{7}{64}x+...}{2-\frac{4}{3}x^2+..}\right]$	m1		Dividing numerator and denominator by x to get constant term in each
	$\lim_{x\to 0}\left[\frac{\sqrt{3+e^x}-2}{\sin 2x}\right] = \frac{\frac{1}{4}}{2} = \frac{1}{8}$	A1F	3	Ft on cand's answer to (a) provided of the form $ax+bx^3$
	Total		**11**	

Question 8

8	The polar equation of a curve C is

$$r = 5 + 2\cos\theta, \qquad -\pi \leqslant \theta \leqslant \pi$$

(a) Verify that the points A and B, with **polar coordinates** $(7, 0)$ and $(3, \pi)$ respectively, lie on the curve C. *(2 marks)*

(b) Sketch the curve C. *(2 marks)*

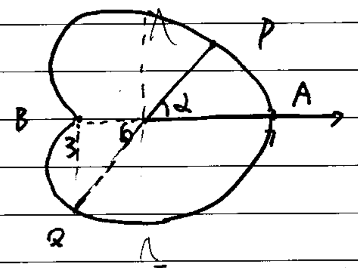
(c) Find the area of the region bounded by the curve C. *(6 marks)*

(d) The point P is the point on the curve C for which $\theta = \alpha$, where $0 < \alpha \leqslant \frac{\pi}{2}$. The point Q lies on the curve such that POQ is a straight line, where the point O is the pole. Find, in terms of α, the area of triangle OQB. *(4 marks)*

Student Response

8	a) when $\theta = 0$.
	$r = 5 + 2\times\cos 0 = 5 + 2\times 1 = 7$ $\therefore$ A lies on the Curve C
	when $\theta = \pi$.
	$r = 5 + 2\times\cos\pi = 5 + 2\times(-1) = 3$ $\therefore$ B lies on the Curve C
	b)

2



B1 B0 | 1

c) $A = \frac{1}{2}\int_{-\pi}^{\pi}(5+2\cos\theta)^2\,d\theta$

$= \frac{1}{2}\int_{-\pi}^{\pi} 25 + 20\cos\theta + 4\cos^2\theta \; d\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta \qquad \cos 2\theta = 2\cos^2\theta - 1$

$\frac{\cos 2\theta + 1}{2} = \cos^2\theta$

$\therefore 4\cos^2\theta = 2\cos 2\theta + 2$

$\therefore = \frac{1}{2}\int_{-\pi}^{\pi} 25 + 20\cos\theta + 2\cos 2\theta + 2 \; d\theta$

$= \frac{1}{2}\left[27\theta + 20\sin\theta + \sin 2\theta\right]_{-\pi}^{\pi}$

$= \frac{1}{2}\left[(27\pi + 20\times 0 + 0) - (-27\pi)\right]$

$= \frac{1}{2}(54\pi) = 27\pi$.

6

d) $OB = 3$ $\quad OQ = 5 + 2\cos(-\pi+\alpha)$ ($\angle POA = \angle BOQ$) M1

B1 m1

area of $OBQ = \frac{1}{2} OB \cdot OQ \times \sin\angle BOQ$

$= \frac{3}{2} \times [5 + 2\cos(\alpha - \pi)] \times \sin\alpha$.

A

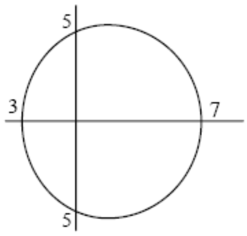
3

(12)

Commentary

In the exemplar the candidate produced full correct solutions to parts (a) and (c). Although the candidate's sketch in part (b) should not have had a 'dent' on the left hand side, this 'error' was condoned, but full marks were not scored because there was no indication of vertical scaling. A '5' at the top of the vertical dotted line would have been sufficient. Only a minority of candidates scored all the four marks in part (d). The candidate in the exemplar produced a very good attempt and found the correct expression for OQ by finding r when □ = −□ + □.

The correct formula for the area of the triangle was then used but the final step, to reach an expression in □ only (not in □ and □), was not carried out. The identity $\cos(A-B) = \cos A\cos B + \sin A\sin B$, or equivalent, should have been used to write cos(□ − □) as −cos□.

8(a)	$\theta = 0,\ r = 5 + 2\cos 0 = 7$ {*A* lies on *C*}	B1		
	$\theta = \pi,\ r = 5 + 2\cos\pi = 3$ {*B* lies on *C*}	B1	2	
(b)		B1		Closed single loop curve, with (indication of) symmetry
		B1	2	Critical values, 3,5,7 indicated
(c)	Area = $\frac{1}{2}\int(5+2\cos\theta)^2\mathrm{d}\theta$	M1		Use of $\frac{1}{2}\int r^2\mathrm{d}\theta$
	$= \frac{1}{2}\int_{-\pi}^{\pi}\left(25+20\cos\theta+4\cos^2\theta\right)\mathrm{d}\theta$	B1 B1		OE for correct expansion of $(5+2\cos\theta)^2$ For correct limits
	$=\frac{1}{2}\int_{-\pi}^{\pi}\left(25+20\cos\theta+2(\cos 2\theta+1)\right)\mathrm{d}\theta$	M1		Attempt to write $\cos^2\theta$ in terms of $\cos 2\theta$
	$=\frac{1}{2}\left[27\theta+20\sin\theta+\sin 2\theta\right]_{-\pi}^{\pi}$	A1F		Correct integration ft wrong non-zero coefficients in $a + b\cos\theta + c\cos 2\theta$
	$= 27\pi$	A1	6	CSO
(d)	Triangle *OBQ* with $OB = 3$ and angle $BOQ = \alpha$	B1		PI
	$OQ = 5 + 2\cos(-\pi + \alpha)$	M1		OE
	Area of triangle $OQB = \frac{1}{2}OB\times OQ\sin\alpha$	m1		Dep. on correct method to find *OQ*
	$=\frac{3}{2}(5-2\cos\alpha)\sin\alpha$	A1	4	CSO
	Total		**14**	