FURTHER MATHEMATICS

Paper 9231/01

Paper 1

General comments

Some scripts of excellent quality and many of good quality were received in response to this examination. There were very few poor scripts. Work was well presented by the vast majority of candidates. Solutions were set out in a clear and logical order. The standard of numerical accuracy was good. There were fewer lapses with algebraic manipulation this year than was the case last year.

There was no evidence to suggest that candidates had any difficulty completing the paper in the time allowed. A very high proportion of scripts had substantial attempts at all twelve questions. Again there were few misreads and few rubric infringements.

The Examiners felt that many candidates had a sound knowledge of most topics on the syllabus. As was the case last year there was some good work on induction. This year there was a choice between complex numbers and linear spaces in **Question 12**, with the vast majority of candidates opting for linear spaces. Those who chose complex numbers, however, were rather more successful at the alternative question that they selected.

Comments on specific questions

Question 1

Most candidates adopted the method suggested in the question and obtained the correct quartic equation $y^4 - y^3 - 3y^2 - 3y - 1 = 0$.

Only the weaker candidates did not recall the formula $\sum \alpha^2 = \left(\sum \alpha\right)^2 - 2\sum \alpha\beta$ and many were able to complete the question.

A small number used the formula $S_{N+4} = S_N + S_{N+3}$ and an iterative procedure, having found, for example, S_{-1} and S_2.

Answer: 7.

Question 2

The introductory result was verified by nearly all candidates and most saw its relevance to the method of differences. A large number of candidates were also able to give the sum to infinity of the series.

Answers: **(i)** $\frac{1}{15} - \frac{1}{(N+3)(2N+5)}$; **(ii)** $\frac{1}{15}$.

Question 3

This question was done very well by the vast majority of candidates, who knew the appropriate formula for the y-coordinate of the centroid, and could accurately evaluate the two integrals involved in order to produce the given result.

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Question 4

Nearly all candidates were able to gain the first two marks on this question for the length of the arc and most could gain the third mark for the expression $2\pi\int_0^1\left(\frac{x^3}{3}(1+x^4)\mathrm{d}x\right)$, although many of the weaker candidates did not see how to split the integral into two parts. Among these weaker candidates there were numerous unsuccessful attempts to integrate by parts.

Question 5

Most candidates made a good attempt to plot the spiral, with only a few losing a mark for a curve that was manifestly not tangential to the initial line at the pole. A number of candidates did not realise that $\theta = \alpha$ was a half line, while other candidates did not distinguish between construction lines and the half line, both of which cost a mark. The calculation which followed was mostly done well. A common error was to get the factor 0.5 on the wrong side of the equation, which the Examiners hoped would have been avoided at this level. This error was astutely avoided by those candidates who equated integrals from 0 to α and α to $\frac{\pi}{2}$. Occasionally the factor 0.5 vanished from the formula for the area. Square roots, rather than cube roots, also appeared in some answers.

Answer: $\frac{\pi}{\sqrt[3]{16}}$.

Question 6

There were many good attempts at this question, even among the weaker candidates. Common errors, such as an initial, unwanted, $\frac{\mathrm{d}y}{\mathrm{d}x} =$ or $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} =$, or failure to differentiate the constant term, were mostly avoided. Some who differentiated implicitly, perfectly correctly, made errors in one or both parts with the arithmetic. Probably slightly more than half of the attempts involved expanding brackets before trying to differentiate. The use of the quotient rule, having obtained an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$, in the second part of the question, was quite infrequent.

Answers: $-\frac{1}{3}$, $-\frac{4}{9}$.

Question 7

Almost all candidates produced a correct derivation of the reduction formula, with only a very few omitting to insert limits. Most candidates obtained some marks in the proof by induction. Weaker candidates usually picked up a mark for the inductive hypothesis and a proof that H_1 was true. Some, unfortunately, produced a proof that H_0 was true. The wording of the question indicates that this is not acceptable. Better candidates were able to prove that $H_k \Rightarrow H_{k+1}$, but many who did so could not complete the proof by writing a satisfactory conclusion. As a minimal case, the Examiners expected to see '$I_n < n!$ for all positive integers *n*'.

Question 8

There were many complete answers to this question. Almost all candidates knew the correct forms for the complementary function and particular integral and were able to make good attempts at finding both. A small number of candidates introduced an erroneous variable, usually in the complementary function. A considerable number of candidates were also able to find the limit of the solution as $x \to \infty$, thus gaining full marks for the question.

Answer: $y = \mathrm{e}^{-\frac{x}{2}}(A\cos 4x + B\sin 4x) + x^2 + 1$.

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Question 9

Nearly all candidates made good attempts at determining a set of eigenvectors, with only a small number making algebraic errors. Many were able to write down the orthogonal matrix **P**, using the eigenvectors. This mark could be gained on a follow-through from incorrect eigenvectors. Many could also find the diagonal matrix **D**. While many gained a substantial number of the first 8 marks, by contrast very few could make much progress in obtaining the final 3 marks. The Examiners hoped to see the initial statement:

$$k^n\mathbf{A}^n = \mathbf{P}\begin{pmatrix} k^n & 0 & 0 \\ 0 & k^n 5^n & 0 \\ 0 & 0 & k^n 7^n \end{pmatrix}\mathbf{P}^{-1}.$$

Answers: $\begin{pmatrix} 17 \\ -6 \\ -7 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}; \begin{pmatrix} 17 & 1 & 1 \\ -6 & -2 & 0 \\ -7 & 1 & 1 \end{pmatrix}; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5^n & 0 \\ 0 & 0 & 7^n \end{pmatrix}; -\frac{1}{7} < k < \frac{1}{7}.$

Question 10

There were many sound attempts at this question. However, a substantial minority of candidates thought that the oblique asymptote was $y = x$ and this was a costly error, since it was not possible to score marks for the sketch graphs after this. The question made it clear that turning points had to be found. Nevertheless, some candidates totally ignored the instruction, while others only found the x-coordinates. There were some sign errors with asymptotes and turning points in the case $\lambda < 0$.

Answers: $x = -\lambda$, $y = x - \lambda$; turning points $(0, 0)$ and $(-2\lambda, -4\lambda)$.

Question 11

This question caused considerable difficulty and only a minority of the best candidates scored in excess of half marks. In the first part, a large number of candidates tried to find P and Q explicitly, but could not solve the equations arising from orthogonality conditions. Those who found the direction of the common normal from a vector product frequently made sign errors. Further sign errors occurred with the scalar product of the unit normal vector with the vector $\mathbf{i} - \mathbf{j}$. Another common error was the omission of the modulus sign in the final answer to part **(i)**. Some used part **(i)** to answer part **(ii)**, while others realised that part **(ii)** could be answered independently and thus 2 marks were scored by a substantial number. Some lost a mark by only giving one value of t, or by answering in degrees. There was a lot of misunderstanding with part **(iii)**, where few could find the correct normal to the plane BPQ. The best candidates realised that they needed the vector product of vectors in the directions of BQ and PQ. It was then easy to use the scalar product method with the unit normal vector. A small number of candidates eliminated parameters from the parametric equation of plane BPQ in order to find its cartesian equation and then used the 'distance of a point from a line' formula.

Answers: **(i)** $\dfrac{|2\sin t - 1|}{\sqrt{4\sin^2 t + 17}}$; **(ii)** $\dfrac{\pi}{6}, \dfrac{5\pi}{6}$; **(iii)** 0.219.

Question 12 EITHER

This question was much less popular than the alternative question. Nevertheless there were some very good answers to it, but some candidates struggled with the algebra in the initial stages. The key was to realise that $\sum_0^{n-1}(1+\mathrm{i}\tan\theta)^k$ can be expressed in two ways: firstly as $\sum_0^{n-1}(\cos k\theta + \mathrm{i}\sin k\theta)\sec^k\theta$, using De Moivre's Theorem, and secondly as $\dfrac{\{1-(1+\mathrm{i}\tan\theta)^{\mathrm{n}}\}}{-\mathrm{i}\tan\theta}$, using the sum of a geometric progression. By equating the real parts of these two expressions, the result follows immediately. A reasonable number of those answering this question saw that the substitution $\theta = \dfrac{\pi}{3}$ was required to obtain the second result. Rather fewer, however, saw that $\theta = \cos^{-1} x$ was needed, along with suitable manipulation, to obtain the final result.

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Question 12 OR

This was much the more popular of the two alternatives, but those candidates trying it found only limited success. For part **(i)**, most of those attempting the question used row or column operations to reduce the matrix to echelon form. Some stopped short of this stage and lost one or both marks. Those who got a correct echelon form mostly found a correct basis for R_1. Similarly in part **(ii),** row operations were used on the matrix, or its transpose, to obtain an echelon form. From this the basis of K_2 was deduced, usually from a set of equations. Those who operated on the transpose frequently wrote results of the operations on the vector $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ beside the matrix. This resulted in $\begin{pmatrix} a \\ b \\ a+b+2c \\ c-d \end{pmatrix}$.

The basis could then be written down from the 3rd and 4th elements, which corresponded to the zero rows in the matrix. Few candidates obtained the fifth mark by showing that each basis vector was a linear combination of the basis vectors of R_1. In part **(iii),** the most straightforward justification for W not being a vector space was to say that it did not contain the zero vector. Only a small number said this. It was not uncommon for statements to appear which could not be substantiated. Only a handful of candidates got anywhere with part **(iv).** One way of tackling the problem was to find $\mathbf{M}_2\mathbf{M}_1 = \begin{pmatrix} 0 & -7 & -14 & -14 \\ 0 & -18 & -36 & -36 \\ 0 & -10 & -20 & -20 \\ 0 & -45 & -90 & -90 \end{pmatrix}$, hence nullity = $4 - r(\mathbf{M}_2\mathbf{M}_1) = 4 - 1 = 3$. Alternatively, for any vector $\mathbf{x}$, $\mathbf{M}_2\mathbf{M}_1\mathbf{x} = \mathbf{M}_2(\alpha\mathbf{b}_1 + \beta\mathbf{b}_2 + \gamma\mathbf{b}_3)$, where $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ are any 3 linearly independent basis vectors of R_1, 2 of which must be basis vectors of K_2. Hence if $\mathbf{b}_1$ and $\mathbf{b}_2$ are basis vectors of K_2, then the dimension of the null space of $T_3 = 4 - 1 = 3$.

Answers: **(i)** e.g. $\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 6 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}\right\}$ or $\left\{\begin{pmatrix} 3 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 0 \\ 7 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}\right\}$; **(ii)** e.g. $\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix}\right\}$ or $\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}\right\}$; **(iv)** 3.

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FURTHER MATHEMATICS

Paper 9231/02

Paper 2

General comments

Since almost all candidates attempted the required number of questions, there seems as in previous years to have been no undue time pressure. Good attempts were seen at most questions, but among the compulsory ones **Questions 1** and **10** were found to be particularly challenging. In the single question which offers an alternative, namely **Question 11**, the Statistics alternative proved more popular, as has usually been the case in the past. As has also usually happened in previous examinations, the paper worked well in discriminating between candidates, producing a range of performance varying from low to very high marks.

Comments on specific questions

Question 1

The key to answering this question is to appreciate that the acceleration of the point P is composed of a tangential component $l \cos \theta \sin \theta$ and a radial component $l \sin^2 \theta$. Combining these gives $l \sin \theta$, which is the magnitude of P's velocity, as required. Very few candidates realised this, and instead wasted time on irrelevant and often invalid derivations. One common error, for example, was to assume that the plane is vertical and OP a pendulum, while another very different type of error was to produce $\cos \theta$ as the result of differentiating $\sin \theta$ with respect to t. As a consequence most candidates gained no credit in this question.

Question 2

Most candidates made a reasonable attempt at finding the frequency ω from $\frac{2\pi}{T}$ and then multiplying this by the amplitude a, here 0.0105 m, to give the required maximum speed. Occasionally a was mistakenly taken as 0.021 m, or there was confusion over the required units. Most candidates found the required speed by using the standard formula $\omega\sqrt{(a^2 - x^2)}$, though an approach using $a\omega \cos \omega t$ is a valid alternative. The most common error here was to take x as 0.005 m rather than 0.0055 m.

Answer: 2.93 ms^{-1}.

Question 3

The lamina can clearly be regarded as a disc of radius $2a$ and mass $\frac{4m}{3}$ from which a concentric disc of radius a and mass $\frac{m}{3}$ has been removed, suggesting that the moments of inertia of these two discs about an axis perpendicular to the lamina at O can be found from the relevant formula given in the *List of Formulae*, and then the difference taken to yield $\frac{5ma^2}{2}$. The perpendicular axes theorem must of course be applied to find the moment of inertia of the lamina about an axis parallel to T, followed by the parallel axes theorem to verify the given result. While most candidates purported to arrive at this given result, many used invalid methods to do so. One such example was to use the same mass m for the two discs, and to add rather than subtract their moments of inertia, which fortuitously gives $\frac{5ma^2}{2}$.

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The second part is solved by considering rotational energy, with the limiting value of ω corresponding to the centre of mass rising a vertical distance $4a$. It is not necessary to consider a general point after rotation through some angle θ, though this will of course yield the desired result provided θ is finally taken to be π. A common error was to take the maximum distance as $2a$ rather than $4a$, or correspondingly θ to be $\frac{1}{2}\pi$. Some candidates tried to relate the angular acceleration to the couple acting on the lamina, but made no useful progress.

Answer: $\omega < \dfrac{3.90}{\sqrt{a}}$.

Question 4

Although most of this question can be solved by systematically taking moments and resolving forces, the majority of candidates were unable to find all of the frictional and normal contact forces at A, B and C correctly. In many cases no diagram was given on the script, and while it is be hoped that these candidates had at least marked all the forces on the diagram given in their question paper, it was not easy to follow their working when arbitrary undefined symbols such as F_1 were used for the various forces. Showing that the three frictional forces have equal magnitudes needs rather more than simply saying this must be so since the system is in equilibrium; perhaps horizontal resolution of the forces on the system, and moments about O for the sphere. The most common fault was to omit one or more forces from an equation, often due to having forces at B acting on the rod but not the sphere, or *vice versa*. Among the minority of candidates who did produce the ratio of the frictional and normal contact forces at each of the points A, B and C, many took the smallest of these ratios as the minimum value of μ, when this must in fact be the largest of the three ratios if equilibrium is to be maintained.

Answer: 1.

Question 5

The first required result, that $v_B < u$, may be shown by either using $v_B = eu$ with the coefficient of restitution e satisfying $e < 1$, or by noting that momentum is conserved but kinetic energy is not. The second required inequality follows from the first by using conservation of momentum. Most candidates made a reasonable attempt at this, though assuming the second inequality in order to prove the first, and then using the first to prove the second is clearly inadequate. The final part of the question may be tackled in a number of ways, but all require that B's speed parallel to the wall is unchanged while its speed normal to the wall is reduced by a factor equal to the required coefficient of restitution e, and that the kinetic energy is reduced by a factor $\frac{1}{3}$ (and not $\frac{2}{3}$ as was often wrongly used). Other common errors were to reduce B's speed rather than just its normal component by the factor e, or to reduce only the component of kinetic energy normal to the wall by a factor $\frac{1}{3}$ on the basis that the speed is unchanged parallel to the wall.

Answer: $\frac{1}{3}$.

Question 6

In general candidates seemed to find this the easiest question on the paper, though many used incorrect tabular values and obtained correspondingly inaccurate answers. The sample mean 495 is immediately obvious by symmetry, and the confidence interval semi-width 14 is reduced by a factor $\dfrac{1.729}{2.093}$, taken from the table of critical values of the t-distribution with 19 degrees of freedom.

Answer: [483, 507].

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Question 7

This question was also generally well done, with most candidates choosing to find the regression line of y on x since x is the independent variable. The best volume of weedkiller to apply is then found by putting $y = 0$ in this equation, and the resulting estimate is probably reliable since the product moment correlation coefficient is close to one in magnitude, and the point is only just outside the range of data.

Answers: **(i)** $y = 59.9 – 102x$; **(ii)** 0.59.

Question 8

Noting that the samples are paired, the first step is to form the differences of the tabulated IQ scores and then estimate the mean and variance for this set of differences, rather than estimating a separate variance for each of the two sets of data. The confidence interval then follows from the usual formula, with the required tabular t-value here 1.86 corresponding to 8 degrees of freedom. The criterion for whether the confidence interval gives no evidence of a difference between the two population means is whether it includes the value 0, which in this case it does and so the claim is supported. This was appreciated by only a minority of candidates.

Answer: [– 5.97, 0.86].

Question 9

Having stated the null and alternative hypotheses, the test requires calculation of the expected frequencies 76, 20, 6 and 98, and hence the value 8.52 of χ^2. Comparison with the tabulated critical value 7.815 leads to the conclusion that the population in Sydney does not conform to the national population. As for the smallest sample size, this is the one below which the smallest expected frequency is reduced to 5, thus requiring that cells be combined and a different critical value employed, and is therefore found from $\frac{5 \times 200}{6}$.

Answer: 167.

Question 10

Some candidates correctly and explicitly related P($T > t$) to the probability of zero hits occurring in t minutes, and hence to the first term of the Poisson distribution with mean $0.8t$. Others, however, did not explain the connection and effectively did little more than write down the result which the question required them to show. There was also considerable confusion over how to show that T has a negative exponential distribution, which hinges on how such a distribution is defined. Here the cumulative distribution function is found from 1 – P($T > t$) and then differentiated to show that the probability density function is in the standard form for a negative exponential distribution. The second part of the question defeated most candidates, some of whom wrongly equated the required probability to that of the time interval between two successive hits exceeding one-fiftieth of one hour. Instead a suitable Normal approximation should be employed, and its use justified as required by the question, for example by commenting on the size of a relevant parameter. Two possibilities for approximating the required probability are either $1 - \Phi\left(\frac{60-\mu}{\sigma}\right)$ with values $\mu = \frac{50}{0.8}$ and $\sigma^2 = \frac{50}{0.8^2}$ or $\Phi\left(\frac{49.5-\mu}{\sigma}\right)$ with $\mu = \sigma^2 = 60 \times 0.8$.

Answer: 0.611 or 0.614.

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Question 11 EITHER

This alternative was attempted by a minority of the candidates, and few produced complete answers to it. The first part is straightforward, involving the application of Newton's law of motion to the particle B, on which the net force acting is $mg - T$. In order to show that B moves in simple harmonic motion, it is necessary to recast the equation into the standard SHM form, which requires a suitable change of variable such as $y = x - \frac{1}{4}a$. It is immediately obvious from this that the centre of motion is when $y = 0$ and hence $x = \frac{1}{4}a$.

The particle A will start to slip when the frictional force reaches its limiting value, and then $T = \frac{1}{3}mg$ and hence $x = \frac{a}{12}$ and $y = -\frac{a}{6}$. Some candidates mistakenly equated the frictional force to the net force acting on B rather than A. The corresponding value of t is most easily found by using the standard SHM form $y = y_0 \cos \omega t$, where here $y_0 = -\frac{1}{4}a$, since the use of the alternative sine form is more complicated as allowance must then be made for the fact that y is initially y_0 rather than 0 when $t = 0$. Substitution for y, y_0 and ω yields the required time t.

Answers: $\frac{1}{4}a$; $0.421\sqrt{\frac{a}{g}}$.

Question 11 OR

Although the hypotheses were almost always stated correctly, the required assumptions of the two populations having normal distributions and a common variance were seen less frequently. Since this common variance is unknown it must be estimated, yielding $s = 0.0215$. A frequent error was to instead regard the two population variances as being known, equal to the estimates given in the question, and to apply an inappropriate formula using them. Most candidates proceeded immediately to the familiar test, comparing a calculated t-value of 2.85 with the critical value 2.262, and concluding that the acidity levels do differ. Only a very small minority understood what was meant by the rejection region for the test, namely those values of $|\bar{x}_E - \bar{x}_W|$ for which the null hypothesis is rejected. Having found the lower bound 0.201 by evaluating $2.262s\sqrt{\frac{1}{6}+\frac{1}{5}}$, comparison with the difference 0.253 of the sample means is of course an equally valid method for reaching the conclusion of the test. The final part of the question requires a similar calculation as for the rejection region, but with a critical t-value of 1.833, yielding 0.163, and then the required largest value of a is found from 0.253 – 0.163. Many candidates wrongly used the critical value 2.262 again.

Answers: $|\bar{x}_E - \bar{x}_W| > 0.201$; 0.09.

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