CAMBRIDGE INTERNATIONAL EXAMINATIONS

Cambridge International Advanced Level

MARK SCHEME for the May/June 2015 series

9231 FURTHER MATHEMATICS

9231/23 Paper 2, maximum raw mark 100

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge will not enter into discussions about these mark schemes.

Cambridge is publishing the mark schemes for the May/June 2015 series for most Cambridge IGCSE®, Cambridge International A and AS Level components and some Cambridge O Level components.

Mark Scheme Notes

Marks are of the following three types:

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

- When a part of a question has two or more "method" steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly when there are several B marks allocated. The notation DM or DB (or dep*) is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

- The symbol √ implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only. A and B marks are not given for fortuitously "correct" answers or results obtained from incorrect working.

- Note: B2 or A2 means that the candidate can earn 2 or 0.
  B2/1/0 means that the candidate can earn anything from 0 to 2.

  The marks indicated in the scheme may not be subdivided. If there is genuine doubt whether a candidate has earned a mark, allow the candidate the benefit of the doubt. Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored.

- Wrong or missing units in an answer should not lead to the loss of a mark unless the scheme specifically indicates otherwise.

- For a numerical answer, allow the A or B mark if a value is obtained which is correct to 3 s.f., or which would be correct to 3 s.f. if rounded (1 d.p. in the case of an angle). As stated above, an A or B mark is not given if a correct numerical answer arises fortuitously from incorrect working. For Mechanics questions, allow A or B marks for correct answers which arise from taking *g* equal to 9.8 or 9.81 instead of 10.

The following abbreviations may be used in a mark scheme or used on the scripts:

AEF Any Equivalent Form (of answer is equally acceptable)

AG Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)

BOD Benefit of Doubt (allowed when the validity of a solution may not be absolutely clear)

CAO Correct Answer Only (emphasising that no "follow through" from a previous error is allowed)

CWO Correct Working Only – often written by a “fortuitous” answer

ISW Ignore Subsequent Working

MR Misread

PA Premature Approximation (resulting in basically correct work that is insufficiently accurate)

SOS See Other Solution (the candidate makes a better attempt at the same question)

SR Special Ruling (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)

Penalties

MR –1 A penalty of MR –1 is deducted from A or B marks when the data of a question or part question are genuinely misread and the object and difficulty of the question remain unaltered. In this case all A and B marks then become "follow through √" marks. MR is not applied when the candidate misreads his own figures – this is regarded as an error in accuracy. An MR–2 penalty may be applied in particular cases if agreed at the coordination meeting.

PA –1 This is deducted from A or B marks in the case of premature approximation. The PA –1 penalty is usually discussed at the meeting.

Question Number	Mark Scheme Details	Part Mark	Total
1	Find speed u_A [m s^{-1}] of A after initial impulse: $2u_A = 8, \; u_A = 4$ B1 For A & B use conservation of momentum, e.g.: $2v_A + 3v_B = 2u_A \; [= 8]$ M1 Use Newton's law of restitution (consistent signs): $v_B - v_A = eu_A \; [= 4e]$ M1 Combine to find v_A and v_B : $v_A = (2–3e)u_A /5, \; v_B = 2(1+e)u_A /5$ A1 (AEF) $v_A = 4(2 – 3e)/5, \; v_B = 8(1 + e)/5$ A1 Use $v_A < 0$: $2 – 3e < 0, \; e > 2/3$ **AG** B1	 5 1	 **6**
2	*EITHER:* Find/ state comps. of speed after 1st colln.: $u \cos \alpha$ // to A *and* $eu \sin \alpha \perp$ to A M1 A1 Find/ state comps. of speed after 2nd colln.: $eu \sin \alpha$ // to B *and* $eu \cos \alpha \perp$ to A1 Find final angle with barrier B: **AG** $\tan^{-1} (eu \cos \alpha / eu \sin \alpha) = ½\pi – \alpha$ A1 *OR:* Relate angle β_1 after 1st colln. to α: $v_1 \sin \beta_1 = eu \sin \alpha$ $v_1 \cos \beta_1 = u \cos \alpha$ $\tan \beta_1 = e \tan \alpha$ (M1 A1) Relate angle β_2 after 2nd colln. to β_1: $v_2 \sin \beta_2 = ev_1 \cos \beta_1$ $v_2 \cos \beta_2 = v_1 \sin \beta_1$ $\tan \beta_2 = e / \tan \beta_1$ (A1) Find final angle β_2 with barrier B: **AG** $\tan^{-1} (e / e \tan \alpha) = ½\pi – \alpha$ (A1) *EITHER:* Find total loss or gain in KE, e.g.: $½ m\{u^2 – (eu \sin \alpha)^2 – (eu \cos \alpha)^2\}$ M1 A1 Find total loss in KE (A0 if wrong sign): $= ½ m (1 – e^2) u^2$ A1 *OR:* Find or state final speed u_{final} : $u_{\text{final}} = e u$ (B1) Find total loss in KE (A0 if wrong sign): $½ m(u^2 – u_{\text{final}}^2) = ½ m (1 – e^2) u^2$ (M1 A1)	 4 3	 **7**
3	State or imply posn. where max. speed next attained: Centre O M1 EITHER: Find ω from SHM eqn. $x = a \sin\omega t$:, e.g.: $1 = \omega^{-1} \sin^{-1} ½ , \; \omega = \pi/6$ M1 A1 Find period T from $T = 2\pi/\omega$: [ft on ω] $T = 2\pi/(\pi/6) = 12$ [s] B1√ OR: Find ω from SHM eqn. $x = a\cos\omega t$, e.g.: $1 = \omega^{-1} \cos^{-1} 0 – \omega^{-1} \cos^{-1} ½$ $\omega = \pi/2 – \pi/3 = \pi/6$ M1 A1 Find period T from $T = 2\pi/\omega$: [ft on ω] $T = 2\pi/(\pi/6) = 12$ [s] B1√ OR: Find period T from SHM eqn, e.g.: $1 = T/4 – (T/2\pi) \cos^{-1} ½$ M1 A1 $= T (1/4 – 1/6), \; T = 12$ [s] A1 Find distance x from O when speed is half max: $\omega^2 (a^2 – x^2) = ¼ v_{\max}^2$ M1 Use maximum speed: $v_{\max} = a\omega \; [= \pi a/6]$ M1 (note that x is independent of ω) $x^2 = ¾ a^2$ $x = \sqrt{3}a/2$ or $a\sqrt{(3/4)}$ or $0.866a$ A1 EITHER: Find a from $x = a \sin \omega t$: $2.5 = \omega^{-1} \sin^{-1} (\sqrt{2}/a) – \omega^{-1} \sin^{-1} (–½)$ or $1·5 = \omega^{-1} \sin^{-1} (\sqrt{2}/a)$ $\sin^{-1} (\sqrt{2}/a) = \pi/4, \; a = 2$ [m] M1 A1 OR: Find a from $x =$ cos ωt: $2.5 = \omega^{-1} \cos^{-1} (–\sqrt{2}/a) – \omega^{-1} \cos^{-1} ½$ or $1·5 = \omega^{-1} \cos^{-1} (–\sqrt{2}/a) – \omega^{-1} \cos^{-1} 0$ $\cos^{-1} (–\sqrt{2}/a) = 3\pi/4, \; a = 2$ [m] M1 A1	 4 3 2	 **9**

Question Number	Mark Scheme Details			Part Mark	Total
4	Find reaction at C using moments for rod about A:	$R_C \times 2a / \sin 30° = W \times 3a \cos 30°$ $R_C = 3\sqrt{3}W / 8$	M1 A1		
	Find friction F on cube by resolving horizontally:	$F = R_C \sin 30°$ [$= 3\sqrt{3}W / 16$]	B1		
	Find reaction R on cube by resolving vertically:	$R = W + R_C \cos 30°$ [$= 25W / 16$]	B1		
	Use $F \leqslant \mu R$ (allow $F < \mu R$ for M1): **AG** (M0 if $F = \mu R$ with no further explanation)	$3\sqrt{3}/16 \leqslant 25\mu/16$, $\mu \geqslant 3\sqrt{3}/25$	M1 A1		
	Find horizontal force X at A (ignore sign):	$X = F$ [$= 3\sqrt{3}W / 16$]	B1		
	Find vertical force Y at A (ignore sign):	$Y = 2W - R$ *or* $W - R_C \cos 30°$ [$= 7W/16$]	B1		
	Find magnitude of force at A:	$\sqrt{(X^2 + Y^2)} = (\sqrt{19}/8)\, W$ *or* $0.545W$	M1 A1		**10**
5	State or find MI of rod about A:	$I_{\text{rod}} = (1 + ⅓)\, 3M\,(2a)^2$ *or* $⅓\, 3M\,(4a)^2$ [$= 16Ma^2$]	B1		
	Find MI of disc about A:	$I_{\text{disc}} = ½\, 2M\,a^2 + 2M\,(5a)^2$ [$= 51\, Ma^2$]	M1 A1		
	State MI of particle about A:	$I_{\text{particle}} = M\,(ka)^2$	B1		
	Sum to find MI of system about A:	$I = (16 + 51 + k^2)\, Ma^2$ $= (67 + k^2)\, Ma^2$ **AG**	A1	5	
	Use eqn of circular motion to find $\mathrm{d}^2\theta/\mathrm{d}t^2$ where θ is angle of AO with vertical: (A0 if $\cos\theta$ used)	$[-]\, I\, \mathrm{d}^2\theta/\mathrm{d}t^2 = 3Mg \times 2a \sin\theta$ $+ 2Mg \times 5a \sin\theta + Mg \times ka \sin\theta$ $= (16 + k)\, Mga \sin\theta$	M1 A1		
	[Approximate $\sin\theta$ by θ and] find ω^2 in SHM eqn: (A0 if $\cos\theta \approx \theta$ used)	$\omega^2 = (16 + k)g / (67 + k^2)\, a$	M1 A1		
	Equate $2\pi/\omega$ to $4\pi\sqrt{(a/g)}$ to find possible values of k:	$\omega^2 = ¼\, g/a$, $16 + k = ¼(67 + k^2)$ $k^2 - 4k + 3 = 0$, $k = 1, 3$	M1 A1	6	**11**
6	State (at least) null hypothesis (AEF):	H_0: Reliability is independent	B1		
	Find expected values (to 1 d.p.): (lose A1 if rounded to integers)	62.253 57.423 41.323 53.747 49.577 35.677	M1 A1		
	Calculate value of χ^2: (allow 5.03 if 1 d.p. exp. values used)	$\chi^2 = 0.1212 + 0.5416 + 1.6764$ $+ 0.1404 + 0.6274 + 1.9417$ $= 5{\cdot}05$	M1 A1		
	State or use correct tabular χ^2 value (to 3 s.f.):	$\chi_{2,\,0.95}{}^2 = 5.99[1]$	B1		
	Valid method for reaching conclusion:	Accept H_0 if $\chi^2 <$ tabular value	M1		
	Correct conclusion, from correct values: (AEF)	Reliability is indep. of supplier	A1	8	**8**
7	Find gradient b_2 of line x on y from $b_1 b_2 = r^2$:	$3.25\, b_2 = 0.56^2$, $b_2 = 0.0965$	M1 A1		
	Find $\bar{y}$ from line y on x:	$y = 3.25 \times (15.6/10) - 4.27 = 0.8$	M1 A1		
	Find regression line of x on y: (M0 for y on x)	$(x - \bar{x} = b_2\,(y - \bar{y}\,)$ $(x - 1.56) = 0.0965\,(y - 0.8)$	M1 A1		
		$x = 0.0965y + 1.48$	A1		
	State both hypotheses (B0 for r …):	H_0: $\rho = 0$, H_1: $\rho \neq 0$	B1	7	
	State or use correct tabular two–tail r–value:	$r_{10,\,5\%} = 0.632$	*B1		
	State or imply valid method for reaching conclusion:	Accept H_0 if $0.56 <$ r–value (AEF)	M1		
	Correct conclusion (AEF, dep *B1):	There is no non–zero correlation	A1	4	**11**

Question Number	Mark Scheme Details			Part Mark	Total
8	Calculate sample mean:	$\overline{x}$ = 45.29 / 7 = 6.47	M1		
	Estimate population variance:	s_1^2 = (295.36 – 45.29^2/7) / 6			
	(allow biased here: 0.3334 *or* 0.5774^2)	= 0.389 *or* 0.6236^2	M1		
	State hypotheses: (AEF; B0 for $\overline{x}$)	H_0: μ = 6·2, H_1: μ > 6.2	B1		
	Calculate value of t (or $-t$; to 3 s.f.):	t = ($\overline{x}$ – 6.2)/(s_1/√7) = 1.15	M1 A1		
	State or use correct tabular t–value (to 3 s.f.):	$t_{6,\,0.95}$ = 1.94[3]	B1		
	(or can compare $\overline{x}$ with 6.2 + 0.458 = 6.66)				
	Consistent conclusion (AEF, ft on both t–values):	[Accept H_0]			
		Popn. mean not greater than 6.2	B1√		
	Find estimate of population mean:	½ (5.89 + 6.75) = 6.32	M1 A1	7	
	Find estimate of population variance s_2^2:	t √(s_2^2/8) = ½ (6.75 – 5.89) [= 0.43]	M1		
	SR Allow M1 for t √(s_2^2/7), max 4/5				
	Use of correct tabular value:	$t_{7,\,0.975}$ = 2.36[5]	A1		
	(Use of 2.36 may lose final A1)	[s_2 = 0.5143 *or* 4√2/11]			
	(to 3 d.p.)	s_2^2 = 0.2645 *or* 32/121	A1	5	**12**
9	Find a by equating $\int_2^\infty f(x)dx$ to 1:	$\left[-ae^{-(x-2)}\right]_2^\infty$ = a = 1 **AG**	B1	1	
	Find or state distribution function F(x) for $x \geqslant 2$:	F(x) = ∫f(x) dx = $-ae^{-(x-2)}$ + c			
		F(2) = 0 so F(x) = 1 – $e^{-(x-2)}$	M1 A1		
	State F(x) for x < 2:	F(x) = 0 for x < 2	A1		
	Find median value m from F(m) = ½:	1 – $e^{-(m-2)}$ = ½ , e^{m-2} = 2			
		m = 2 + ln 2 *or* 2 – ln ½ *or* 2.69	M1 A1	5	
(i)	Find or state G(y) from $Y = e^X$ for $x \geqslant 2$:	G(y) = P(Y < y) = P(e^X < y)			
		= P(X < ln y) = F(ln y)			
		= 1 – $e^{-(\ln y-2)}$ = 1 – e^2/y	M1 A1		
	Find g(y) by differentiation (allow e^2 = 7·39):	g(y) = e^2/y^2	A1		
	State corresponding range of y:	for $y \geqslant e^2$ [g(y) = 0 for $y < e^2$]	A1	4	
(ii)	Find P(Y > 10):	P(Y > 10) = 1 – G(10)			
	M0 for G(10) or 1 – g(10)	= e^2/10 *or* 0.739	M1 A1	2	**12**
10A	Use conservation of energy when P is level with A:	$\tfrac12 mv_1^2 = \tfrac12 mu^2 - \tfrac12 mga$	B1		
	Use $F = ma$ radially at this point:	$T_1 = mv_1^2/\tfrac12 a$	B1		
	Eliminate v_1^2 to find T_1:	$T_1 = (2m/a)(u^2 - ag)$ **AG**	B1	3	
(i)	State or imply that string loses contact when vertical		M1		
	Use conservation of energy at this point:	$\tfrac12 mv_2^2 = \tfrac12 mu^2 - mga$	B1		
	Use $F = ma$ radially just before this point:	$T_2 = mv_2^2/\tfrac12 a - mg$	B1		
	Use $F = ma$ radially just after this point:	$T_3 = mv_2^2/(3a/2) - mg$	B1		
	EITHER: Find v_2^2 and hence u^2 from $T_2/T_3 = 5$:	$mv_2^2/\tfrac12 a - mg$ =			
		$5\{mv_2^2/(3a/2) - mg\}$, $v_2^2 = 3ag$			
		$u^2 = 3ag + 2ag = 5ag$ **AG**	M1 A1		
	OR: Eliminate v_2^2 in $T_2/T_3 = 5$ and find u^2:	$(m/a)(2u^2 - 5ag)$ =			
		$5(m/a)(2u^2/3 - 7ag/3)$			
		$4u^2/3 = 20ag/3$, $u^2 = 5ag$ **AG**	(M1 A1)	6	

Question Number	Mark Scheme Details			Part Mark	Total
(ii)	Find $\cos\theta$, where $\theta = \angle AOP$, when P is level with A:	$\cos\theta = a/(3a/2) = 2/3$	B1		
	Use conservation of energy at this point:	$\frac{1}{2}mv_4^2 = \frac{1}{2}mu^2 - \frac{1}{2}mga$ *or* $\frac{1}{2}mu^2 - mg\{(3a/2)\cos\theta - \frac{1}{2}a\}$ *or* $\frac{1}{2}mv_2^2 + \frac{1}{2}mga$ *or* $\frac{1}{2}mv_2^2 + mg(3a/2)(1 - \cos\theta)$	B1		
		$[v_4^2 = 4ag]$			
	Use $F = ma$ radially at this point:	$T_4 = mv_4^2/(3a/2) - mg\cos\theta$	B1		
	Eliminate v_4^2 to find T_4:	$T_4 = 4mag \times 2/3a - 2mg/3$ $= 8mg/3 - 2mg/3$ $= 2mg$	M1 A1	5	**14**
10B	State hypotheses (B0 for $\overline{t_A}$ …), e.g.:	H_0: $\mu_A = \mu_B$, H_1: $\mu_A < \mu_B$	B1		
	Estimate both popn. variances (to 3 d.p..):	$s_A^2 = (215.18 - 102^2/50) / 49$ *and* $s_B^2 = (282.3 - 129^2/60) / 59$	M1 A1		
	(allow biased here: 0.1420 *or* 0.3768^2	$s_A^2 = 0.1449$ *or* $71/490$ *or* 0.3807^2			
	and 0.0825 *or* 0.2872^2)	*and* $s_B^2 = 0.0839$ *or* $99/1180$ *or* 0.2897^2	A1		
	EITHER: Estimate combined variance:	$s^2 = s_A^2/50 + s_B^2/60$	M1		
	(to 3 s.f.)	$= \underline{0.004296}$ *or* 0.06555^2	A1		
	Calculate value of z (or $-z$):	$z = (\overline{t_B} - \overline{t_A}) / s$			
	(to 3 s.f.)	$= (2.15 - 2.04) / s = 1.68$	M1 A1		
	OR: Pooled estimate of common variance: (note s_A^2 and s_B^2 not needed explicitly)	$s^2 = (49\,s_A^2 + 59\,s_B^2) / 108$ *or* $(215.18 - 102^2/50 + 282.3 - 129^2/60) / 108$			
	(to 3 s.f.)	$= 0.1116$ *or* 0.334^2	(M1 A1)		
	Calculate value of z (or $-z$):	$z = (\overline{t_B} - \overline{t_A}) / s\sqrt{(1/50 + 1/60)}$			
	(to 3 s.f.)	$= 0.11 / 0.06396 = 1{\cdot}72$	(M1 A1)		
	State or use correct tabular z value: (to 3 s.f.) (or can compare $\overline{t_B} - \overline{t_A} = 0.11$ with 0.105)	$z_{0.95} = 1.645$(allow 1.658)	*B1		
	Correct conclusion (AEF, ft on z, dep *B1):	$z >$ tabular value so club A takes less time	B1√	10	
	Calculate value of z (either sign):	$z = (0.11 - 0.05) / s = 0.06 / s$			
	(to 3 s.f.)	$= 0.9154$[or 0.9381]	M1 A1		
	Find $\Phi(z)$ and set of possible values of α:	$\Phi(z) = 0.820$[or 0.826]			
	(M0 for $\alpha \sim 82$ or 0.82) (to 1 d.p.)	$\alpha \geqslant$ (*or* $>$) 18.0[or 17.4]	M1 A1	4	
	SR Allow B1 (max 3/4) for $\alpha < 18.0$ [or 17.4]				**14**