

# FURTHER MATHEMATICS

Paper 9231/11

Paper 11

## General Comments

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There was no evidence to suggest that candidates had any difficulty completing the paper in the time allowed. A very high proportion of scripts had substantial attempts at all eleven questions. Once again there were few misreads and few rubric infringements.

Candidates displayed a sound knowledge of most topics on the syllabus. Candidates tackled most questions very confidently. It was good to see that the questions on complex numbers and polar coordinates were answered well, whereas in the past this has not always been the case. Candidates lost marks, from time to time, because their work lacked sufficient rigour, especially when an answer was given on the question paper.

## Comments on Specific Questions

### Question 1

This question was well done and so candidates got off to a good start. Nearly all candidates realised that they needed to sum the series using the method of differences. A small number found  $\sum_{r=1}^n u_r$  and then most of these candidates used  $\sum_{r=13}^n u_r = \sum_{r=1}^n u_r - \sum_{r=1}^{12} u_r$  to obtain the correct result. When small slips occurred, the mark for the sum to infinity could often be earned on a follow through basis.

$$\text{Answers: } \frac{1}{5} - \frac{1}{\sqrt{2n+1}}, \frac{1}{5}$$

### Question 2

This question proved to be very accessible to the vast majority of candidates. Only a very small number made errors with the initial differentiation. A small number then tried, in vain, to use  $\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$ . The remainder knew the appropriate form for parametric equations and generally obtained the correct answer, except for a small proportion who made errors with trigonometric formulae or errors involving the square root.

$$\text{Answer: } \sqrt{2} \left( e^{\frac{1}{2\pi}} - 1 \right) (= 5.39)$$

### Question 3

The vast majority of candidates earned the first two marks and were then able to conjecture an expression for  $S_n$  successfully. The candidates who thought that  $S_n = r \times r!$  were not able to progress, but those who had the correct conjecture were able to produce a correct proof by mathematical induction in a very satisfactory manner.

$$\text{Answers: } 1, 1, 1, 1; S_n = (n + 1)! - 1$$

#### Question 4

Nearly all candidates were able to find the equations of the asymptotes of  $C$  correctly. In the second part of the question the majority considered the sign of the discriminant of a relevant quadratic equation in  $x$  with coefficients in terms of  $y$ . This led them to conclude that there were no points on  $C$  for which  $(y - 1)(y - 9) < 0$ . The final mark could only be awarded to those who produced a thumbnail sketch, or equivalent, to show how the inequality was solved, since the answer was a given result. A substantial minority attempted the second part of the question using differentiation, in order to find a local maximum and a local minimum. Those who did not produce an appropriate test for each, and did not consider the discontinuity at  $x = 1$ , lost marks.

Answers:  $x = 1$ ,  $y = 2x + 3$

#### Question 5

In the first part of the question those who considered the determinant equal to zero, in order to find the value of  $a$ , invariably gained all three marks. Those who tried to work with equations were far more likely to score one or two marks as their working was either insufficient, or muddled, or both. In part (i), either using row reduction of an augmented matrix, or by working with equations, many obtained  $x - y + 2z = 4$  and  $5z = 9$  but did not go further and produce a general solution. In part (ii), only a handful of candidates recognised the correct geometrical interpretation of the situation.

Answers:  $a = -1$ , (i)  $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0 \\ 1.8 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$  (oe), (ii) planes form a prism (oe)

#### Question 6

The first part of the question was done extremely well by many candidates. The majority used the binomial expansion of  $(c + is)^5$  and equated the real part to  $\cos 5\theta$ , whereupon the use of  $\cos^2 \theta = 1 - \sin^2 \theta$  led to the correct result. A small minority used  $(z + z^{-1})^n = 2 \cos n\theta$ , where  $z = \cos \theta + i \sin \theta$ , but were seldom able to complete the proof successfully. In the second part of the question, candidates who showed insufficient reasoning in reaching the displayed result lost one or two marks. It was necessary to show the solutions of  $\cos 5\theta = 0$  and to explain how to match one of these solutions to the value  $\frac{3 - \sqrt{5}}{8}$  which had been obtained by solving the quadratic equation in  $\sin^2 \theta$ .

#### Question 7

Nearly all candidates were able to use integration by parts to obtain the given reduction formula correctly, with only a small minority showing insufficient working or making a sign error. Most then went on to use this reduction formula. A sizeable number of candidates made errors in finding  $I_0$ , or  $I_1$ , with the result that their value for  $I_4$  was incorrect. Those who obtained the correct value could easily see that  $0 < I_4 < 1$  led to the displayed result, usually earning two of the three marks. The third mark was awarded to those who provided a correct sketch of the function in the interval  $(0, 1)$  or who explained, clearly, how the function behaved in this interval.

Answer:  $I_4 = 24e - 65$

#### Question 8

Nearly all candidates gained the first three marks for sketching the circle and the cardioid correctly on the same diagram. A surprising number of candidates wrote the coordinates of the points of intersection in the wrong order, thereby losing one of the two marks. Others gave values for the angle outside the range specified in the question, while still others gave a mixture of Cartesian and polar coordinates, writing 0 when they should have written  $a$ . Candidates usually identified correctly the region whose area was to be determined. Inevitably some lost accuracy marks because of incorrect limits for the integration, sign errors with the trigonometry, or failure to add half the area of the circle to the result of their integration.

Answers:  $\left(a, \frac{\pi}{2}\right)$ ,  $\left(a, \frac{3\pi}{2}\right)$

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The majority of candidates adopted the easier approach in the first part of the question. This involved finding  $\frac{dv}{dx}$  and  $\frac{d^2v}{dx^2}$  from  $v = xy$ , rather than finding  $\frac{dy}{dx}$  and  $\frac{d^2y}{dx^2}$  from  $y = vx^{-1}$ , before substituting in the differential equation to demonstrate the required result. Large numbers of candidates found correctly the complementary function and particular integral. Most of them found the correct general solution for  $y$  in terms of  $x$ , but a small number stopped at  $v$  in terms of  $x$ .

$$\text{Answer: } y = \frac{1}{x} (Ae^{-3x} + Be^x + 2e^{2x})$$

### Question 10

Much good work was demonstrated in all three parts of this question. In part (i), a surprising number of candidates did not produce the most direct method of solution, preferring to find a two parameter expression for the vector  $\overrightarrow{PQ}$ . They then found a pair of simultaneous equations by equating to zero the scalar product formed between  $\overrightarrow{PQ}$  and each direction vector of the lines  $l_1$  and  $l_2$ . These equations had awkward coefficients and even more awkward solutions. Nevertheless, most candidates' algebra and arithmetic was sufficiently good to emerge with the correct answer. The preceding method sometimes appeared in part (ii) as a means of getting the direction of the line  $PQ$ , when it had not been found in part (i). In part (iii) some candidates used the formula for the distance of a point from a line, while other candidates used the triple scalar product method.

$$\text{Answers: (i) } \frac{4}{\sqrt{3}} \text{ (= 2.31), (ii) } 4x + 5y + z = -15 \text{ (oe), (iii) } \frac{38}{\sqrt{42}} \text{ (oe) (= 5.86)}$$

### Question 11 Either

Those choosing this alternative did the preliminary parts extremely well, apart from the occasional sign error. In general the required method was known for each of the four parts. In the final part of the question, most candidates realised that the substitution  $x = y - 1$  was needed and a considerable number obtained the correct quartic equation  $y^4 - 4y^2 + 4 = 0$ . Most who got to this stage were able to solve for  $y$  correctly. A substantial minority then did not find  $x$ . A large number of candidates did not indicate that a quartic equation has four roots, in this case two pairs of equal roots, thus losing the final mark. Rather than pointing out the repeated roots, a small number of candidates earned the final mark by finding the product of two linear factors from the two roots which they had obtained, then dividing the quartic expression by this product and factorising the quotient.

$$\text{Answers: (i) } -4, \text{ (ii) } 12, \text{ (iii) } 4, \text{ (iv) } 12, x = \pm\sqrt{2} - 1 \text{ (twice)}$$

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Those who chose this somewhat less popular alternative generally produced good work. Some lost several of the early marks, particularly on the first small proof, usually by producing an incomplete argument. The second small proof was generally done rather better, but again some arguments skipped a stage or two. A number of candidates did not appreciate that, with zeros below the leading diagonal, the eigenvalues of the matrix  $\mathbf{A}$  could immediately be written down. They usually obtained the correct values from the characteristic equation or by some other factorisation method. Eigenvectors were generally found accurately, thus giving the matrix  $\mathbf{P}$ . Most candidates deduced the eigenvalues for  $\mathbf{A} + 3\mathbf{I}$  correctly. Weaker candidates thought that these were the eigenvalues for  $\mathbf{B}$ , but most found the reciprocals of them and correctly wrote the matrix  $\mathbf{D}$ .

$$\text{Answers: } \mathbf{P} = \begin{pmatrix} 1 & 2 & -6 \\ 0 & 1 & 25 \\ 0 & 0 & 20 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{6} \end{pmatrix}$$

# FURTHER MATHEMATICS

Paper 9231/12

Paper 12

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Paper 9231/13

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# FURTHER MATHEMATICS

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Paper 9231/21

Paper 21

## Key Messages

To score full marks in the paper candidates must be well versed in both Mechanics and Statistics, though any preference between these two areas can be exercised in the choice of the final optional question.

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## Comments on Specific Questions

### **Question 1**

Almost all candidates formulated the two equations for the speeds of the spheres after the collision by means of conservation of momentum and Newton's restitution equation. If these are solved to give the speed of *A*, then requiring that *A*'s direction of motion is reversed in the collision yields the required minimum value of the coefficient of restitution *e*. Its maximum value is 1, but the Examiners did not necessarily expect this to be

stated. Care must be taken over the signs of the various terms in the equations as these depend on the directions in which the spheres are taken to move.

Answer:  $e > \frac{5}{7}$ .

### Question 2

Most candidates were aware that the components of  $P$ 's speed along and perpendicular to the barrier after the collision are  $4 \cos \alpha$  and  $0.4 \times 4 \sin \alpha$  respectively, but some were unable to progress further. The key to the next step is that the sum of the squares of these two components is effectively given to be  $2^2$ , leading to a value for either  $\sin^2 \alpha$  or  $\cos^2 \alpha$  and hence  $\alpha$ . It is of course invalid to assume that  $P$  leaves the barrier at the same angle  $\alpha$ .

Answer:  $70.9^\circ$  or  $1.24$  rad.

### Question 3

Applying Newton's second law of motion to the particle  $P$  in a radial direction at both  $A$  and  $B$  yields  $R_A$  and  $R_B$  respectively, and equating  $R_B$  to  $10R_A$  gives an equation involving  $P$ 's unknown speed  $v_B$  at  $B$ . A second equation for  $v_B$  results from conservation of energy from  $A$  to  $B$ , allowing its elimination to show that  $u^2 = ag$  as required. Taking the normal reaction to be zero when  $P$  loses contact with the sphere yields one equation for the speed at that point, with a second again coming from conservation of energy. In this case elimination of the speed produces the required value of  $\cos \theta$ . The question was generally answered well, though care is needed in both parts when finding both the magnitude and sign of the change of potential energy between the relevant points.

Answer:  $\frac{11}{15}$ .

### Question 4

It is essential that candidates realise that the compressive force  $S$  exerted by the spring on the rod is in a vertical direction and not, for example, normal to the rod. Most candidates readily deduced that the friction  $F_A$  at  $A$  is related to the normal reaction  $R_A$  there by  $F_A = \frac{1}{3}R_A$ , and took moments for the rod about at least one point. There are various possible choices for this point, of which  $C$  was marginally the most popular, perhaps because the moment equation does not then involve  $S$ . Some choices may involve a little less effort than others. Thus choosing  $B$ , for example, does not introduce the normal reaction at  $B$ . Even better, choosing the point of intersection of the horizontal line through  $B$  and the vertical line of the spring gives  $R_A$  almost immediately.  $S$  may then be found, most easily by resolving forces vertically. The underlying principle in the second part was widely understood to be Hooke's Law, which here takes the form  $S = \lambda x/L$  with  $\lambda$  given to be  $2mg$ . The principal difficulty in applying it to find the natural length,  $L$ , of the spring appeared to be formulating  $x$ . Since the spring is in compression,  $x$  is here  $L - CD$ , and not  $CD - L$ . Since  $CD = 9/20$ , the value of  $L$  may then be found. In addition to ensuring that the meaning of symbols used for the unknown forces is clear, candidates should state which point any moment equation is being taken about since this is not always immediately obvious from their equation, particularly when an error is present.

Answer: (i)  $\frac{1}{2}mg$ ; (ii)  $3/5$ .

### Question 5

Candidates rarely had difficulty showing that  $AO = 3.6a$ , which requires that the tensions in the two strings be found by Hooke's Law and equated since  $P$  is in equilibrium at  $O$ . The next part again requires the two tensions to be found, but here when  $P$  is at a general point. This enables the net force on  $P$  to be equated to the product of its mass and acceleration to produce an SHM equation in which  $\omega^2 = \frac{5g}{2a}$ . In order to produce

the standard form of the SHM equation, the distance of  $P$  from the centre  $O$  of its motion should be used. Not all the candidates who measured from another point, most commonly  $C$ , removed the resulting constant term from their equation of motion by an appropriate change of variable, which should not only be done but

preferably also explained. The period  $T$  is readily found using the standard SHM formula  $T = \frac{2\pi}{\omega}$ , and similarly the greatest speed of  $P$  from  $0.2a\omega$ .

Answer: (ii)  $2\pi\sqrt{\left(\frac{2a}{5g}\right)}$ ; (iii)  $\sqrt{a}$ .

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The two given estimates for the population variances may be used to produce an unbiased estimate

$s^2 = \frac{s_x^2}{50} + \frac{s_y^2}{60}$  of the population variance of the combined sample, and a z-value of 1.869 then calculated

from  $\frac{1.8}{s}$ . Candidates must then use the complete normal distribution table to find the corresponding value

0.9692 of  $\Phi(z)$ ; the smaller table of critical values gives insufficient accuracy. Finally candidates should realise that this is a two-sided test when finding the set of possible values of  $\alpha$ . A large number of candidates effectively assumed that  $X$  and  $Y$  have the same population variance and used a pooled estimate of this common variance to produce a slightly different result for  $\alpha$ , but it was rare to see this assumption stated explicitly.

Answer:  $\alpha \leq 6.2$ .

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The expected and median values of  $T$  were almost always produced correctly, the latter by determining the value  $m$  for which the distribution function  $F(m)$  equals  $\frac{1}{2}$ . Most candidates also realised that the final part involves  $F(20)$ , though the probability was often based on  $F(20)$  instead of the correct  $1 - F(20)$ .

Answer: (i) 100; (ii) 69.3, 0.819.

### Question 8

The expected values were usually found correctly from  $\frac{100\lambda^r e^{-r}}{r!}$  with  $\lambda = \frac{225}{100}$ , except that some

candidates evaluated this expression with  $r = 8$  to find the final value instead of subtracting all the previous expected values from the required total of 100. As it happens both numbers are small so this error is unimportant here. The goodness of fit test was also often carried out well, with the last four cells being combined to ensure that none of expected frequencies is less than 5. Comparison of the calculated value 10.8 of  $\chi^2$  with the critical value 11.14 leads to acceptance of the null hypothesis, namely that the Poisson distribution does fit the data.

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Answer: (i)  $y = 0.388x + 3.57$ ; (ii) 0.582; (iii) 7.45; (iv)  $N > 16$ .

### Question 10

When integrating  $f(x)$  to find the distribution functions  $F$  of  $X$  for  $1 \leq x \leq 3$  and hence  $G$  of  $Y$ , candidates should remember to include an appropriate constant of integration to ensure that  $F(1) = 0$  and  $F(3) = 1$ . For completeness,  $G(y)$  should also be stated for values of  $y$  outside the central interval of  $1 \leq y \leq 27$ , and it should similarly be made clear on the graph of  $Y$ 's probability density function  $g$  that it is  $\frac{1}{6y^3}$  within this

central interval and zero outside it. In the final part, the mean of  $Y$  is found in the usual way by integrating  $y g(y)$  over  $1 \leq y \leq 27$ , giving a value of 10. Candidates can save time here by not finding the median value 8, since the required probability can be found from  $G(10) - \frac{1}{2}$  rather than  $G(10) - G(8)$ .

Answer: 0 ( $y < 1$ ),  $\frac{1}{2} (y^{\frac{1}{3}} - 1)$  ( $1 \leq y \leq 27$ ), 1 ( $y > 27$ ); 0.0772.

### Question 11 (Mechanics)

This optional question was attempted by only a very small proportion of the candidates, but most of those who did so made good attempts at the first part. One possible method of solution involves using the parallel and perpendicular axes theorems as appropriate to find the moments of inertia of each of the four rings about  $AB$ , then summing the results. Alternatively the parallel axes theorem can be used as appropriate to find the moments of inertia of each of the rings about an axis through the centre of mass of the composite object and perpendicular to its plane. After summing the four results, application of the perpendicular axis theorem gives the required moment of inertia about  $AB$ . In the second part, candidates should realise that the addition of the particle at  $D$  increases the moment of inertia of the body to  $I = 14(2 + \sqrt{3})mr^2$ . The required angular speed  $\omega$  then follows from applying conservation of energy to give  $\frac{1}{2}I\omega^2 = 9mg \times R \sin 60^\circ$ .

Answer: (ii)  $\sqrt{\frac{9r}{14r}}$ .

### Question 11 (Statistics)

The required 95% confidence interval was found correctly by most candidates, using the unbiased estimate 2.549 of the population variance for  $A$  and a tabular  $t$ -value of 2.365. The most appropriate assumptions in the second part are that the distribution of the lengths is also normal for lake  $B$ , and that its population variance is the same as that for  $A$ 's distribution. As in all such tests, the hypotheses should be stated in terms of the population means and not the sample means. The unbiased pooled estimate 2.436 of the common population variance may be used to calculate a  $t$ -value of 0.712. Since it is a one-tail test, comparison with the tabulated value of 1.782 leads to acceptance of the null hypothesis, namely that the mean length of fish in  $A$  is not greater than that of fish in  $B$ . The final confidence interval is found in the usual way with the same estimate of the common variance but with a tabular  $t$ -value of 2.179.

Answer: [11.9, 14.6]; [-1.24, 2.44].

# FURTHER MATHEMATICS

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Paper 9231/22

Paper 22

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Answer: [11.9, 14.6]; [-1.24, 2.44].

# FURTHER MATHEMATICS

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Paper 9231/23

Paper 23

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Answer:  $70.9^\circ$  or  $1.24$  rad.

### Question 3

Applying Newton's second law of motion to the particle  $P$  in a radial direction at both  $A$  and  $B$  yields  $R_A$  and  $R_B$  respectively, and equating  $R_B$  to  $10R_A$  gives an equation involving  $P$ 's unknown speed  $v_B$  at  $B$ . A second equation for  $v_B$  results from conservation of energy from  $A$  to  $B$ , allowing its elimination to show that  $u^2 = ag$  as required. Taking the normal reaction to be zero when  $P$  loses contact with the sphere yields one equation for the speed at that point, with a second again coming from conservation of energy. In this case elimination of the speed produces the required value of  $\cos \theta$ . The question was generally answered well, though care is needed in both parts when finding both the magnitude and sign of the change of potential energy between the relevant points.

Answer:  $\frac{11}{15}$ .

### Question 4

It is essential that candidates realise that the compressive force  $S$  exerted by the spring on the rod is in a vertical direction and not, for example, normal to the rod. Most candidates readily deduced that the friction  $F_A$  at  $A$  is related to the normal reaction  $R_A$  there by  $F_A = \frac{1}{3}R_A$ , and took moments for the rod about at least one point. There are various possible choices for this point, of which  $C$  was marginally the most popular, perhaps because the moment equation does not then involve  $S$ . Some choices may involve a little less effort than others. Thus choosing  $B$ , for example, does not introduce the normal reaction at  $B$ . Even better, choosing the point of intersection of the horizontal line through  $B$  and the vertical line of the spring gives  $R_A$  almost immediately.  $S$  may then be found, most easily by resolving forces vertically. The underlying principle in the second part was widely understood to be Hooke's Law, which here takes the form  $S = \lambda x/L$  with  $\lambda$  given to be  $2mg$ . The principal difficulty in applying it to find the natural length,  $L$ , of the spring appeared to be formulating  $x$ . Since the spring is in compression,  $x$  is here  $L - CD$ , and not  $CD - L$ . Since  $CD = 9/20$ , the value of  $L$  may then be found. In addition to ensuring that the meaning of symbols used for the unknown forces is clear, candidates should state which point any moment equation is being taken about since this is not always immediately obvious from their equation, particularly when an error is present.

Answer: (i)  $\frac{1}{2}mg$ ; (ii)  $3/5$ .

### Question 5

Candidates rarely had difficulty showing that  $AO = 3.6a$ , which requires that the tensions in the two strings be found by Hooke's Law and equated since  $P$  is in equilibrium at  $O$ . The next part again requires the two tensions to be found, but here when  $P$  is at a general point. This enables the net force on  $P$  to be equated to the product of its mass and acceleration to produce an SHM equation in which  $\omega^2 = \frac{5g}{2a}$ . In order to produce

the standard form of the SHM equation, the distance of  $P$  from the centre  $O$  of its motion should be used. Not all the candidates who measured from another point, most commonly  $C$ , removed the resulting constant term from their equation of motion by an appropriate change of variable, which should not only be done but

preferably also explained. The period  $T$  is readily found using the standard SHM formula  $T = \frac{2\pi}{\omega}$ , and similarly the greatest speed of  $P$  from  $0.2a\omega$ .

Answer: (ii)  $2\pi\sqrt{\left(\frac{2a}{5g}\right)}$ ; (iii)  $\sqrt{a}$ .

### Question 6

The two given estimates for the population variances may be used to produce an unbiased estimate

$s^2 = \frac{s_x^2}{50} + \frac{s_y^2}{60}$  of the population variance of the combined sample, and a z-value of 1.869 then calculated

from  $\frac{1.8}{s}$ . Candidates must then use the complete normal distribution table to find the corresponding value

0.9692 of  $\Phi(z)$ ; the smaller table of critical values gives insufficient accuracy. Finally candidates should realise that this is a two-sided test when finding the set of possible values of  $\alpha$ . A large number of candidates effectively assumed that  $X$  and  $Y$  have the same population variance and used a pooled estimate of this common variance to produce a slightly different result for  $\alpha$ , but it was rare to see this assumption stated explicitly.

Answer:  $\alpha \leq 6.2$ .

### Question 7

The expected and median values of  $T$  were almost always produced correctly, the latter by determining the value  $m$  for which the distribution function  $F(m)$  equals  $\frac{1}{2}$ . Most candidates also realised that the final part involves  $F(20)$ , though the probability was often based on  $F(20)$  instead of the correct  $1 - F(20)$ .

Answer: (i) 100; (ii) 69.3, 0.819.

### Question 8

The expected values were usually found correctly from  $\frac{100\lambda^r e^{-r}}{r!}$  with  $\lambda = \frac{225}{100}$ , except that some

candidates evaluated this expression with  $r = 8$  to find the final value instead of subtracting all the previous expected values from the required total of 100. As it happens both numbers are small so this error is unimportant here. The goodness of fit test was also often carried out well, with the last four cells being combined to ensure that none of expected frequencies is less than 5. Comparison of the calculated value 10.8 of  $\chi^2$  with the critical value 11.14 leads to acceptance of the null hypothesis, namely that the Poisson distribution does fit the data.

### Question 9

The gradient of the required regression line is first found using the standard formula, and then used with the means of the given sample values of  $x$  and  $y$  to find the constant term. These calculations were generally performed well, and the product moment correlation coefficient  $r$  was similarly found without difficulty using in part some of the intermediate results from the calculation of the gradient. Although estimating the value of  $y$  when  $x = 10$  by means of the regression line found earlier rarely presented problems, there was considerable variety in the comments made on its reliability. Acceptable comments include it being unreliable because  $r$  is much smaller than 1, and reliable because 10 is within the range of sample values of  $x$  or is close to the mean value of 7.3. The final part was seemingly found to be more challenging. It requires comparison of the calculated product moment correlation coefficient with the critical values for a one-tail test with significance level 1%, looking for the values of  $N$  for which the coefficient is greater than the critical value.

Answer: (i)  $y = 0.388x + 3.57$ ; (ii) 0.582; (iii) 7.45; (iv)  $N > 16$ .

### Question 10

When integrating  $f(x)$  to find the distribution functions  $F$  of  $X$  for  $1 \leq x \leq 3$  and hence  $G$  of  $Y$ , candidates should remember to include an appropriate constant of integration to ensure that  $F(1) = 0$  and  $F(3) = 1$ . For completeness,  $G(y)$  should also be stated for values of  $y$  outside the central interval of  $1 \leq y \leq 27$ , and it should similarly be made clear on the graph of  $Y$ 's probability density function  $g$  that it is  $\frac{1}{6y^3}$  within this

central interval and zero outside it. In the final part, the mean of  $Y$  is found in the usual way by integrating  $y g(y)$  over  $1 \leq y \leq 27$ , giving a value of 10. Candidates can save time here by not finding the median value 8, since the required probability can be found from  $G(10) - \frac{1}{2}$  rather than  $G(10) - G(8)$ .

Answer: 0 ( $y < 1$ ),  $\frac{1}{2} (y^{\frac{1}{3}} - 1)$  ( $1 \leq y \leq 27$ ), 1 ( $y > 27$ ); 0.0772.

### Question 11 (Mechanics)

This optional question was attempted by only a very small proportion of the candidates, but most of those who did so made good attempts at the first part. One possible method of solution involves using the parallel and perpendicular axes theorems as appropriate to find the moments of inertia of each of the four rings about  $AB$ , then summing the results. Alternatively the parallel axes theorem can be used as appropriate to find the moments of inertia of each of the rings about an axis through the centre of mass of the composite object and perpendicular to its plane. After summing the four results, application of the perpendicular axis theorem gives the required moment of inertia about  $AB$ . In the second part, candidates should realise that the addition of the particle at  $D$  increases the moment of inertia of the body to  $I = 14(2 + \sqrt{3})mr^2$ . The required angular speed  $\omega$  then follows from applying conservation of energy to give  $\frac{1}{2}I\omega^2 = 9mg \times R \sin 60^\circ$ .

Answer: (ii)  $\sqrt{\frac{9r}{14r}}$ .

### Question 11 (Statistics)

The required 95% confidence interval was found correctly by most candidates, using the unbiased estimate 2.549 of the population variance for  $A$  and a tabular  $t$ -value of 2.365. The most appropriate assumptions in the second part are that the distribution of the lengths is also normal for lake  $B$ , and that its population variance is the same as that for  $A$ 's distribution. As in all such tests, the hypotheses should be stated in terms of the population means and not the sample means. The unbiased pooled estimate 2.436 of the common population variance may be used to calculate a  $t$ -value of 0.712. Since it is a one-tail test, comparison with the tabulated value of 1.782 leads to acceptance of the null hypothesis, namely that the mean length of fish in  $A$  is not greater than that of fish in  $B$ . The final confidence interval is found in the usual way with the same estimate of the common variance but with a tabular  $t$ -value of 2.179.

Answer: [11.9, 14.6]; [-1.24, 2.44].