Modified Enlarged 18pt

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Tuesday 13 June 2023 – Afternoon

A Level Mathematics A

H240/02 Pure Mathematics and Statistics

Time allowed: 2 hours
plus your additional time allowance

YOU MUST HAVE:
the Printed Answer Booklet or any suitable paper provided by the centre. The Printed Answer Booklet may be enlarged by the centre.
a scientific or graphical calculator
Insert for Question 8(a) (with this document)

READ INSTRUCTIONS OVERLEAF

DC (CJ/CB) 332624/1

INSTRUCTIONS

Use black ink. You can use an HB pencil, but only for graphs and diagrams.

If you use the Printed Answer Booklet write your answer to each question in the space provided in the PRINTED ANSWER BOOKLET. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.

If you use the Printed Answer Booklet fill in the boxes on the front of the Printed Answer Booklet.

Answer ALL the questions.

Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.

Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question.

The acceleration due to gravity is denoted by $g\,\mathrm{m\,s^{-2}}$. When a numerical value is needed use $g = 9.8$ unless a different value is specified in the question.

Do NOT send this Question Paper for marking. Keep it in the centre or recycle it.

INFORMATION

The total mark for this paper is 100.

The marks for each question are shown in brackets [ ].

ADVICE

Read each question carefully before you start your answer.

FORMULAE
A Level Mathematics A (H240)

Arithmetic series

$$S_n = \tfrac{1}{2}n(a+l) = \tfrac{1}{2}n\{2a+(n-1)d\}$$

Geometric series

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_\infty = \frac{a}{1-r} \quad \text{for } |r| < 1$$

Binomial series

$$(a+b)^n = a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + \ldots + {}^nC_r a^{n-r}b^r + \ldots + b^n \quad (n \in \mathbb{N}),$$

where ${}^nC_r = {}_nC_r = \binom{n}{r} = \dfrac{n!}{r!\,(n-r)!}$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots \quad (|x| < 1, n \in \mathbb{R})$$

Differentiation

f(x)	f′(x)
$\tan kx$	$k\sec^2 kx$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\mathrm{cosec}^2 x$
$\mathrm{cosec}\, x$	$-\mathrm{cosec}\, x \cot x$

Quotient rule $y = \frac{u}{v}, \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Differentiation from first principles

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Integration

$$\int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)| + c$$

$$\int f'(x)\big(f(x)\big)^n\, dx = \frac{1}{n+1}\big(f(x)\big)^{n+1} + c$$

Integration by parts $\int u\frac{dv}{dx}\, dx = uv - \int v\frac{du}{dx}\, dx$

Small angle approximations

$\sin\theta \approx \theta$, $\cos\theta \approx 1 - \frac{1}{2}\theta^2$, $\tan\theta \approx \theta$ where θ is measured in radians

Trigonometric identities

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \qquad \left(A \pm B \neq \left(k + \tfrac{1}{2}\right)\pi\right)$$

Numerical methods

Trapezium rule:

$$\int_a^b y\,\mathrm{d}x \approx \tfrac{1}{2}h\left\{\left(y_0 + y_n\right) + 2\left(y_1 + y_2 + \ldots + y_{n-1}\right)\right\},$$

where $h = \dfrac{b-a}{n}$

The Newton-Raphson iteration for solving $\mathrm{f}(x) = 0$:

$$x_{n+1} = x_n - \frac{\mathrm{f}\left(x_n\right)}{\mathrm{f}'\left(x_n\right)}$$

Probability

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$$

$$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B \mid A) = \mathrm{P}(B)\mathrm{P}(A \mid B)$$

OR $\mathrm{P}(A \mid B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$

Standard deviation

$$\sqrt{\frac{\sum(x-\bar{x})^2}{n}} = \sqrt{\frac{\sum x^2}{n} - \bar{x}^2} \quad \text{OR} \quad \sqrt{\frac{\sum f(x-\bar{x})^2}{\sum f}} = \sqrt{\frac{\sum fx^2}{\sum f} - \bar{x}^2}$$

The binomial distribution

If $X \sim B(n, p)$ then $P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$,

mean of X is np, variance of X is $np(1-p)$

Hypothesis test for the mean of a normal distribution

If $X \sim \mathrm{N}(\mu, \sigma^2)$ then $\overline{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$ and $\frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim \mathrm{N}(0, 1)$

Percentage points of the normal distribution

If Z has a normal distribution with mean 0 and variance 1 then, for each value of p, the table gives the value of z such that $P(Z \leqslant z) = p$.

p	0.75	0.90	0.95	0.975	0.99	0.995
z	0.674	1.282	1.645	1.960	2.326	2.576

p	0.9975	0.999	0.9995
z	2.807	3.090	3.291

Kinematics

Motion in a straight line

$v = u + at$

$s = ut + \frac{1}{2}at^2$

$s = \frac{1}{2}(u + v)t$

$v^2 = u^2 + 2as$

$s = vt - \frac{1}{2}at^2$

Motion in two dimensions

$\mathbf{v} = \mathbf{u} + \mathbf{a}t$

$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$

$\mathbf{s} = \frac{1}{2}(\mathbf{u} + \mathbf{v})t$

$\mathbf{s} = \mathbf{v}t - \frac{1}{2}\mathbf{a}t^2$

SECTION A

Pure Mathematics

1 (a) (i) Express $x^2 - 8x + 11$ in the form $(x - a)^2 + b$ where a and b are constants. [2]

(ii) Hence write down the minimum value of $x^2 - 8x + 11$. [1]

(b) Determine the value of the constant k for which the equation $x^2 - 8x + 11 = k$ has two equal roots. [2]

2 The points O and A have position vectors $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 0 \\ 8 \end{pmatrix}$ respectively. The point P is such that $\overrightarrow{OP} = k\overrightarrow{OA}$, where k is a non-zero constant.

(a) Find, in terms of k, the length of OP. [1]

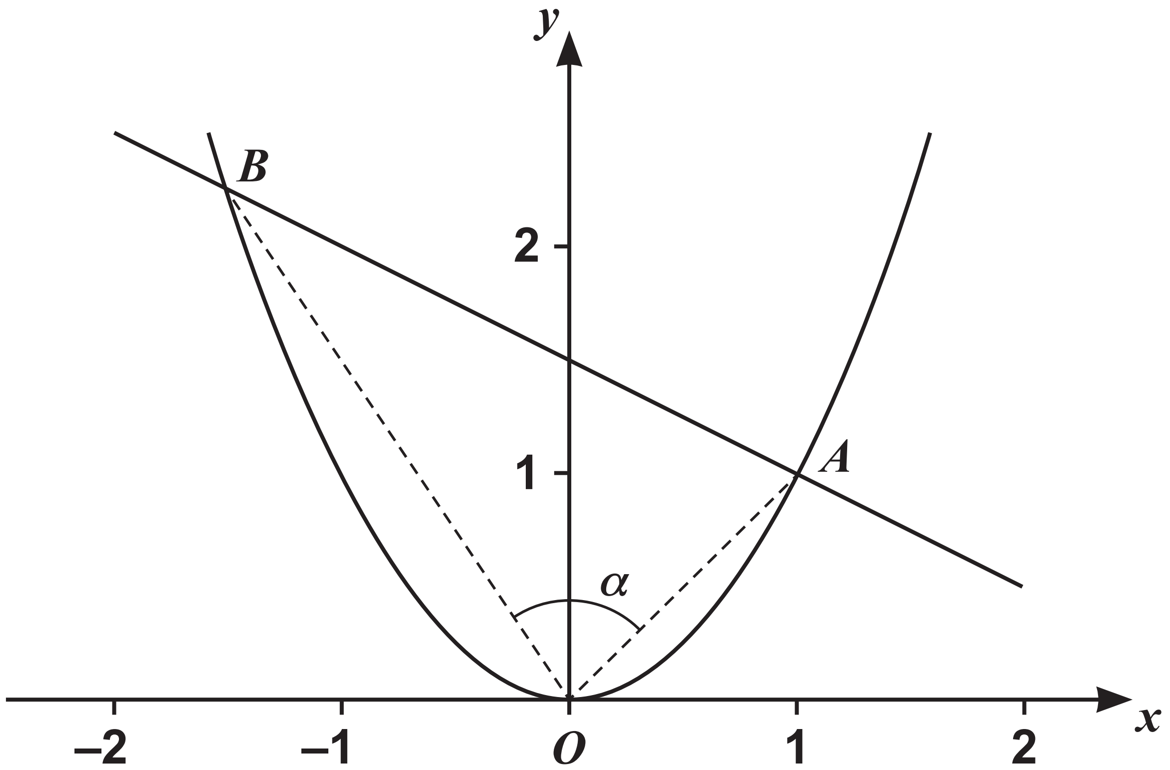
Point B has position vector $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and angle OPB is a right angle.

(b) Determine the value of k. [4]

3 **In this question you must show detailed reasoning.**

Find the exact area of the region enclosed by the curve $y = \frac{1}{x+2}$, the two axes and the line $x = 2.5$. **[3]**

4 **The diagram shows part of the graph of $y = x^2$. The normal to the curve at the point $A(1, 1)$ meets the curve again at B. Angle AOB is denoted by α.**



(a) **Determine the coordinates of B.** **[6]**

(b) **Hence determine the exact value of $\tan\alpha$.** **[3]**

5 **In this question you must show detailed reasoning.**

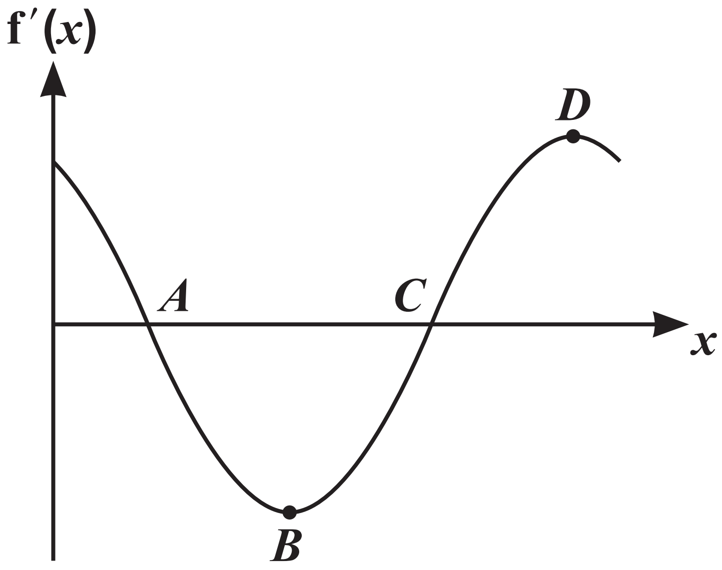
The function f is defined by $f(x) = \cos x + \sqrt{3}\sin x$ with domain $0 \leqslant x \leqslant 2\pi$.

(a) **Solve the following equations.**

(i) $f'(x) = 0$ **[4]**

(ii) $f''(x) = 0$ **[3]**

The diagram shows the graph of the gradient function $y = f'(x)$ for the domain $0 \leqslant x \leqslant 2\pi$.

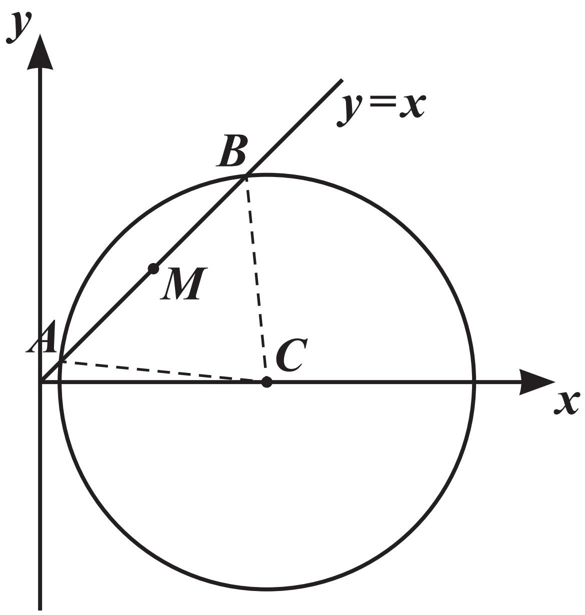


(b) **Use your answers to parts (a)(i) and (a)(ii) to find the coordinates of points A, B, C and D. [2]**

(c) **(i)** **Explain how to use the graph of the gradient function to find the values of x for which $f(x)$ is increasing. [1]**

(ii) **Using set notation, write down the set of values of x for which $f(x)$ is increasing in the domain $0 \leqslant x \leqslant 2\pi$. [2]**

6 A circle has centre C which lies on the x-axis, as shown in the diagram. The line $y = x$ meets the circle at A and B. The midpoint of AB is M.



The equation of the circle is $x^2 - 6x + y^2 + a = 0$, where a is a constant.

(a) In this question you must show detailed reasoning.

Show that the area of triangle ABC is $\frac{3}{2}\sqrt{9 - 2a}$. [7]

(b) (i) Find the value of a when the area of triangle ABC is zero. [1]

(ii) Give a geometrical interpretation of the case in part (b)(i). [1]

(c) Give a geometrical interpretation of the case where $a = 5$. [1]

7 A student wishes to prove that, for all positive integers a and b, $a^2 - 4b \neq 2$.

(a) Prove that $a^2 - 4b = 2 \Rightarrow a$ is even. [2]

(b) Hence or otherwise prove that, for all positive integers a and b, $a^2 - 4b \neq 2$. [3]

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SECTION B

Statistics

8 The stem-and-leaf diagram shows the heights, in centimetres, of 15 plants.

0	2				
1	0				
2	4				
3	0	2	4	9	
4	1	2	4	7	9
5	3	7			
6	2				

Key: | 2 | 5 means 25 cm.

(a) Draw a box-and-whisker plot to illustrate the data. Use the Answer Booklet or the insert provided. [4]

A statistician intends to analyse the data, but wants to ignore any outliers before doing so.

(b) Discuss briefly whether there are any heights in the diagram which the statistician should ignore. [3]

9 A school contains 500 students in years 7 to 11 and 250 students in years 12 and 13. A random sample of 20 students is selected to represent the school at a parents' evening. The number of students in the sample who are from years 12 and 13 is denoted by X.

(a) State a suitable binomial model for X. [1]

Use your model to answer the following.

(b) (i) Write down an expression for $P(X = x)$. [1]

(ii) State, in set notation, the values of x for which your expression is valid. [1]

(c) Find $P(5 \leqslant X \leqslant 9)$. [2]

(d) State one disadvantage of using a random sample in this context. [1]

10 The mass, in kilograms, of a species of fish in the UK has population mean 4.2 and standard deviation 0.25.

An environmentalist believes that the fish in a particular river are smaller, on average, than those in other rivers in the UK.

A random sample of 100 fish of this species, taken from the river, has sample mean 4.16 kg.

Stating a necessary assumption, test at the 5% significance level whether the environmentalist is correct. [8]

11 **The random variable Y has the distribution $\mathrm{N}(\mu, \sigma^2)$.**

(a) Find $\mathrm{P}(Y > \mu - \sigma)$. [1]

(b) Given that $\mathrm{P}(Y > 45) = 0.2$ and $\mathrm{P}(Y < 25) = 0.3$, determine the values of μ and σ. [6]

The random variables U and V have the distributions $\mathrm{N}(10, 4)$ and $\mathrm{N}(12, 9)$ respectively.

(c) It is given that $\mathrm{P}(U < b) = \mathrm{P}(V > c)$, where $b > 10$ and $c < 12$.

Determine b in terms of c. [2]

12 **A student has an ordinary six-sided dice. The student suspects that it is biased against six, so that when it is thrown, it is less likely to show a six than if it were fair.**

In order to test this suspicion, the student plans to carry out a hypothesis test at the 5% significance level.

The student throws the dice 100 times and notes the number of times, X, that it shows a six.

(a) **Determine the largest value of X that would provide evidence at the 5% significance level that the dice is biased against six. [3]**

Later another student carries out a similar test, at the 5% significance level. This student also throws the dice 100 times.

(b) **It is given that the dice is fair.**

Find the probability that the conclusion of the test is that there is significant evidence that the dice is biased against six. [1]

13 The scatter diagram opposite uses information about all the Local Authorities (LAs) in the UK, taken from the 2011 census.

For each LA it shows the percentage (x) of employees who used public transport to travel to work and the percentage (y) who used motorised private transport.

"Public transport" includes train, bus, minibus, coach, underground, metro and light rail. "Motorised private transport" includes car, van, motorcycle, scooter, moped, taxi and passenger in a car or van.

(a) Most of the points in the diagram lie on or near the line with equation $x + y = k$, where k is a constant.

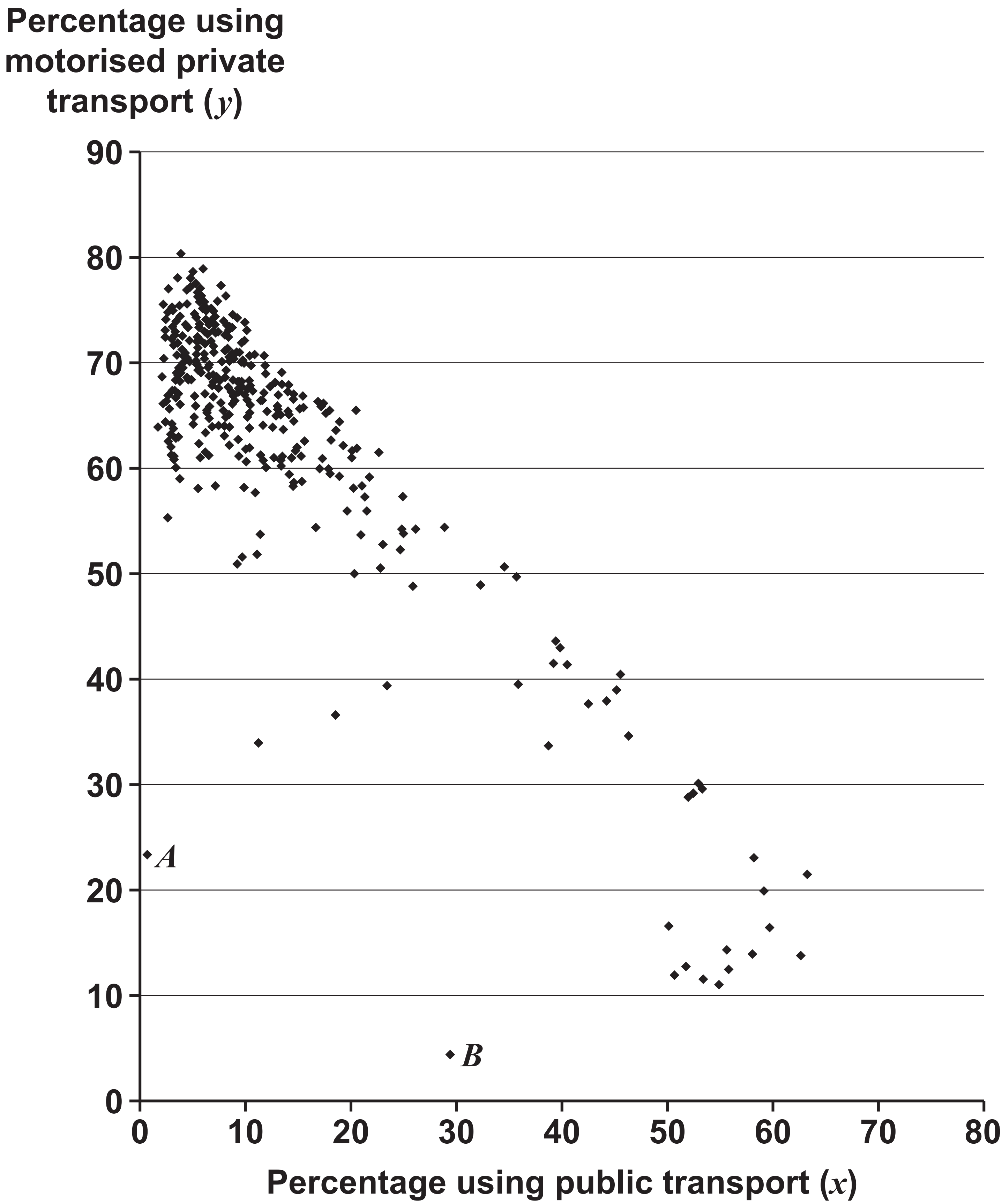
(i) Give a possible value for k. [1]

(ii) Hence give an approximate value for the percentage of employees who either worked from home or walked or cycled to work. [1]

(b) The average amount of fuel used per person per day for travelling to work in any LA is denoted by F.

Consider the two groups of LAs where the percentages using motorised private transport are highest and lowest.

(i) Using only the information in the diagram, suggest, with a reason, which of these two groups will have greater values of F than the other group. [1]


Percentage using motorised private transport (y)
90
80
70
60
50
40
30
20
10
0
A
B
0
10
20
30
40
50
60
70
80
Percentage using public transport (x)

A student says that it is not possible to give a reliable answer to part (b)(i) without some further information.

(ii) Suggest two kinds of further information which would enable a more reliable answer to be given. [2]

(c) Points A and B in the diagram are the most extreme outliers. Use their positions on the diagram to answer the following questions about the two LAs represented by these two points.

(i) The two LAs share a certain characteristic.

Describe, with a justification, this characteristic. [2]

(ii) The environments in these two LAs are very different.

Describe, with a justification, this difference. [2]

(d) A student says that it is difficult to extract detailed information from the scatter diagram.

Explain whether you agree with this criticism. [1]

14 In this question you must show detailed reasoning.

A disease that affects trees shows no visible evidence for the first few years after the tree is infected.

A test has been developed to determine whether a particular tree has the disease. A positive result to the test suggests that the tree has the disease. However, the test is not 100% reliable, and a researcher uses the following model.

If the tree has the disease, the probability of a positive result is 0.95.

If the tree does not have the disease, the probability of a positive result is 0.1.

(a) It is known that in a certain county, A, 35% of the trees have the disease. A tree in county A is chosen at random and is tested.

Given that the result is positive, determine the probability that this tree has the disease. [3]

A forestry company wants to determine what proportion of trees in another county, B, have the disease. They choose a large random sample of trees in county B.

Each tree in the sample is tested and it is found that the result is positive for 43% of these trees.

(b) By carrying out a calculation, determine an estimate of the proportion of trees in county B that have the disease. [4]

END OF QUESTION PAPER

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