Vectors

Scalar product - Vector product - Triple scalar product

Specification

Vectors and Three–Dimensional Coordinate Geometry

Definition and properties of the vector product. Calculation of vector products.	Including the use of vector products in the calculation of the area of a triangle or parallelogram.
Calculation of scalar triple products.	Including the use of the scalar triple product in the calculation of the volume of a parallelepiped and in identifying coplanar vectors. Proof of the distributive law and knowledge of particular formulae is not required.

Linear Independence

Linear independence and dependence of vectors.

In formulae booklet

Vector product: $\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}} = \begin{vmatrix} \mathbf{i} & a_1 & b_1 \\ \mathbf{j} & a_2 & b_2 \\ \mathbf{k} & a_3 & b_3 \end{vmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$

Introduction and definitions

In year 11, we have introduced vectors to describe a translation.

The vector represents a DISPLACEMENT from one point to another.

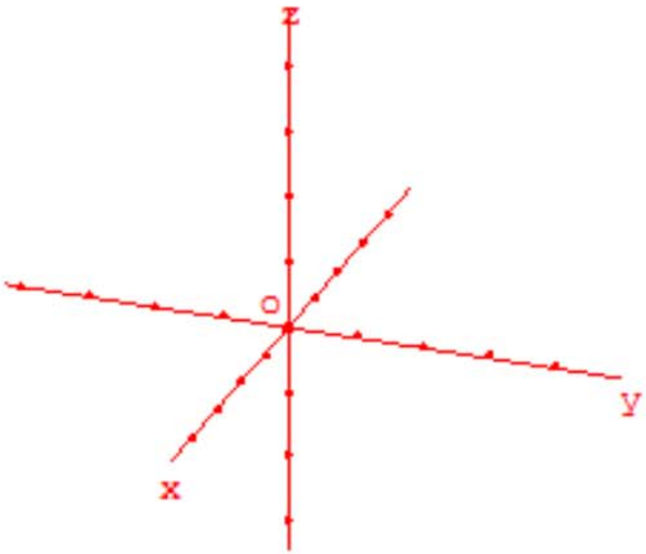
We can extend this notion in 3 dimensions:

$$\mathbf{v} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} \textit{ means } \text{ move -2-units in the x-direction, 3 units in the y-direction}$$

and 1 unit in the z-direction

Notation: In the exam paper, a vector will be written as a bold lower case letter.

other notations for vectors are $\mathbf{v} = \underline{v} = \underset{\sim}{v}$ *or* $\vec{v}$



A vector has a DIRECTION.
This direction is represented by an arrow

$\underline{v}$ and $-\underline{v}$ are opposite vectors,
they have opposite direction

Magnitude/modulus

The modulus/magnitude of a vector
is the length of the displacement it represents
("the size of the vector").

Notation: If the vector $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, its modulus / magnitude

is noted $|\mathbf{v}| = \sqrt{x^2 + y^2 + z^2}$

Equivalent vectors

Two vectors are equivalent (equal)
if they represent the same displacement
Consequence: Unless you are given the starting point (or "the tail") of a vector,
there are an infinite numbers of way to draw/represent this vector.

Unit vectors

A vector is a unit vector if its modulus is 1

Notation : $\hat{\mathbf{a}}$ is the unit vector in the director of the vector **a**.

$$\hat{\mathbf{a}} = \frac{1}{|\mathbf{a}|} \times \mathbf{a}$$

Example: Work out the unit vector in the direction $2\mathbf{i}+3\mathbf{j}+6\mathbf{k}$

Base vectors

In a standard set of axis, we use three directions

the x-direction , the y-direction and the z-direction.

The unit vector in the x-direction is called **i**

The unit vector in the y-direction is called **j**

The unit vector in the z-direction is called **k**

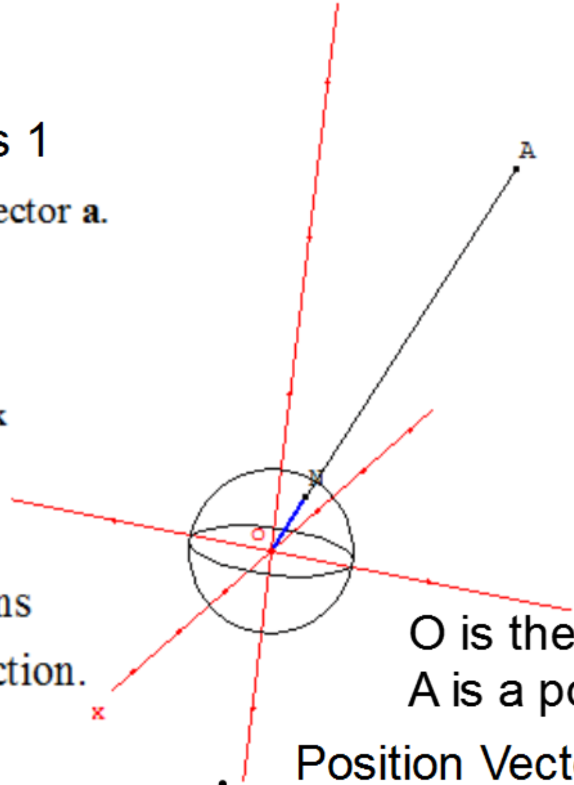
$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} , \quad \mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} , \quad \mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Component of vectors

Any vector $\mathbf{v} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ can be written in terms of the base vectors

$$\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{j}$$

We say that **v** is a LINEAR COMBINATION of **i**, **j** and **k**



O is the Origin : O(0,0)

A is a point with coordinates (x,y,z)

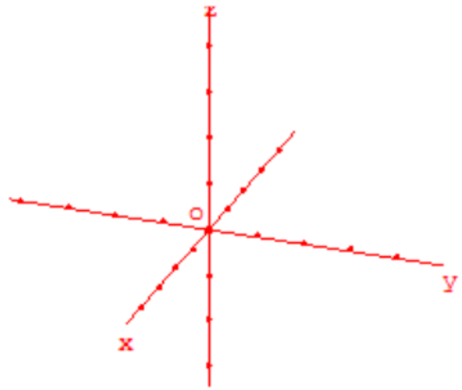
Position Vector

The position vector associated to the point $A(x, y, z)$

is the vector $\mathbf{a} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

It is the translation vector which maps O onto A.

We can also write $\mathbf{a} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.



Operations with vectors

Consider two points $A(x_A, y_A, z_A)$ *and* $B(x_B, y_B, z_B)$

with the position vectors $\mathbf{a} = \begin{pmatrix} x_A \\ y_A \\ z_A \end{pmatrix}$ *and* $\mathbf{b} = \begin{pmatrix} x_B \\ y_B \\ z_B \end{pmatrix}$ *respectively*

- The triangle rule

For any point C, the following identity is true:

$$\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB}$$

- Additions and subtractions

The vector $\overrightarrow{OA} + \overrightarrow{OB} = \mathbf{a} + \mathbf{b} = \begin{pmatrix} x_A + x_B \\ y_A + y_B \\ z_A + z_B \end{pmatrix}$

The vector $\overrightarrow{OA} - \overrightarrow{OB} = \mathbf{a} - \mathbf{b} = \begin{pmatrix} x_A - x_B \\ y_A - y_B \\ z_A - z_B \end{pmatrix}$

- The vector $\overline{\mathrm{AB}}$, (*displacement from A to B*) *is*

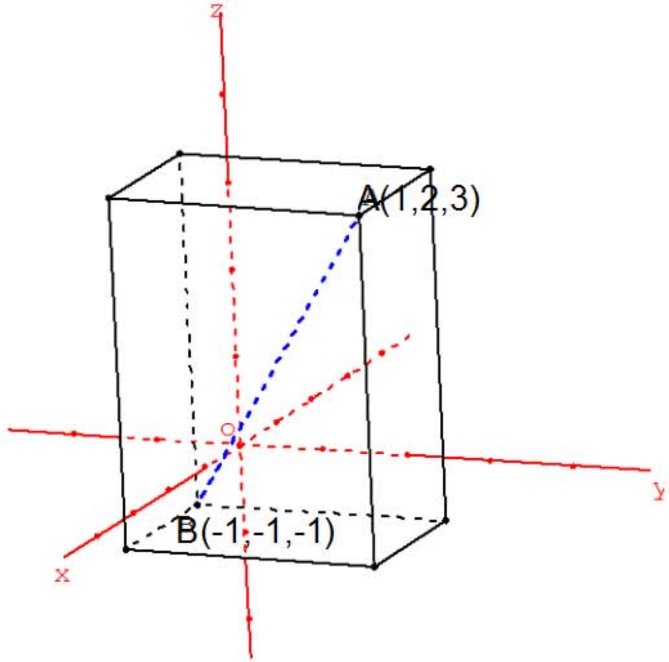
$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} \qquad \overrightarrow{AB} = \begin{pmatrix} x_B - x_A \\ y_B - y_A \\ z_B - z_A \end{pmatrix}$$

- Multiplication by a scalar (number)

λ is a real number.

The vector $\lambda\overrightarrow{OA} = \lambda\mathbf{a} = \begin{pmatrix} \lambda x_A \\ \lambda y_A \\ \lambda y_B \end{pmatrix}$



Consequence:

The distance between the points A and B

is $\mathrm{AB} = |\mathbf{b} - \mathbf{a}| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$

Exercises:

1 Find the modulus of:

a $3\mathbf{i}+5\mathbf{j}+\mathbf{k}$ **b** $4\mathbf{i}-2\mathbf{k}$ **c** $\mathbf{i}+\mathbf{j}-\mathbf{k}$

d $5\mathbf{i}-9\mathbf{j}-8\mathbf{k}$ **e** $\mathbf{i}+5\mathbf{j}-7\mathbf{k}$

2 Given that $\mathbf{a}=\begin{pmatrix}5\\0\\2\end{pmatrix}$, $\mathbf{b}=\begin{pmatrix}2\\1\\-3\end{pmatrix}$ and $\mathbf{c}=\begin{pmatrix}7\\-4\\2\end{pmatrix}$, find in column matrix form:

a $\mathbf{a}+\mathbf{b}$ **b** $\mathbf{b}-\mathbf{c}$ **c** $\mathbf{a}+\mathbf{b}+\mathbf{c}$

d $3\mathbf{a}-\mathbf{c}$ **e** $\mathbf{a}-2\mathbf{b}+\mathbf{c}$ **f** $|\mathbf{a}-2\mathbf{b}+\mathbf{c}|$

3 The position vector of the point A is $2\mathbf{i}-7\mathbf{j}+3\mathbf{k}$ and $\overrightarrow{AB}=5\mathbf{i}+4\mathbf{j}-\mathbf{k}$.
Find the position of the point B.

4 Given that $\mathbf{a}=t\mathbf{i}+2\mathbf{j}+3\mathbf{k}$, and that $|\mathbf{a}|=7$, find the possible values of t.

5 Given that $\mathbf{a}=5t\mathbf{i}+2t\mathbf{j}+t\mathbf{k}$, and that $|\mathbf{a}|=3\sqrt{10}$, find the possible values of t.

6 The points A and B have position vectors $\begin{pmatrix}2\\9\\t\end{pmatrix}$ and $\begin{pmatrix}2t\\5\\3t\end{pmatrix}$ respectively.

a Find $\overrightarrow{AB}$.

b Find, in terms of t, $|\overrightarrow{AB}|$.

c Find the value of t that makes $|\overrightarrow{AB}|$ a minimum.

d Find the minimum value of $|\overrightarrow{AB}|$.

7 The points A and B have position vectors $\begin{pmatrix}2t+1\\t+1\\3\end{pmatrix}$ and $\begin{pmatrix}t+1\\5\\2\end{pmatrix}$ respectively.

a Find $\overrightarrow{AB}$.

b Find, in terms of t, $|\overrightarrow{AB}|$.

c Find the value of t that makes $|\overrightarrow{AB}|$ a minimum.

d Find the minimum value of $|\overrightarrow{AB}|$.

1 **a** $\sqrt{35}$ **b** $\sqrt{20}=2\sqrt{5}$ **c** $\sqrt{3}$
d $\sqrt{170}$ **e** $\sqrt{75}=5\sqrt{3}$

2 **a** $\begin{pmatrix}7\\1\\-1\end{pmatrix}$ **b** $\begin{pmatrix}-5\\5\\-5\end{pmatrix}$ **c** $\begin{pmatrix}14\\-3\\1\end{pmatrix}$
d $\begin{pmatrix}8\\4\\4\end{pmatrix}$ **e** $\begin{pmatrix}8\\-6\\10\end{pmatrix}$ **f** $10\sqrt{2}$

3 $7\mathbf{i}-3\mathbf{j}+2\mathbf{k}$

4 6 or −6

5 $\sqrt{3}$ or $-\sqrt{3}$

6 **a** $\begin{pmatrix}2t-2\\-4\\2t\end{pmatrix}$ **b** $\sqrt{8t^2-8t+20}$
c $t=\frac{1}{2}$ **d** $3\sqrt{2}$

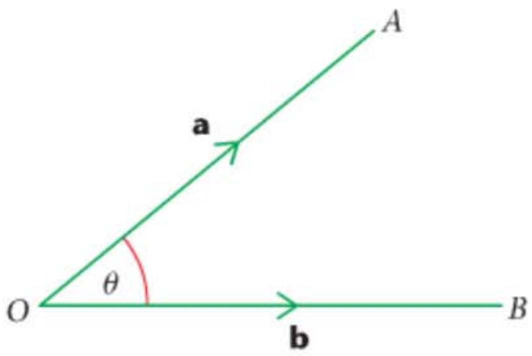
7 **a** $\begin{pmatrix}-t\\4-t\\-1\end{pmatrix}$ **b** $\sqrt{2t^2-8t+17}$
c $t=2$ **d** 3

Scalar product of two vectors

- The scalar product of two vectors **a** and **b** is written as **a**.**b** (say 'a dot b'), and defined by

  $\mathbf{a}.\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta$

  where θ is the angle between **a** and **b**.



You can see from this diagram that if **a** and **b** are the position vectors of A and B, then the angle between **a** and **b** is $\angle AOB$.

- If **a** and **b** are the position vectors of the points A and B, then

  $$\cos AOB = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}$$

- The non-zero vectors **a** and **b** are perpendicular if and only if $\mathbf{a}.\mathbf{b} = 0$.

Also, because $\cos 0° = 1$,

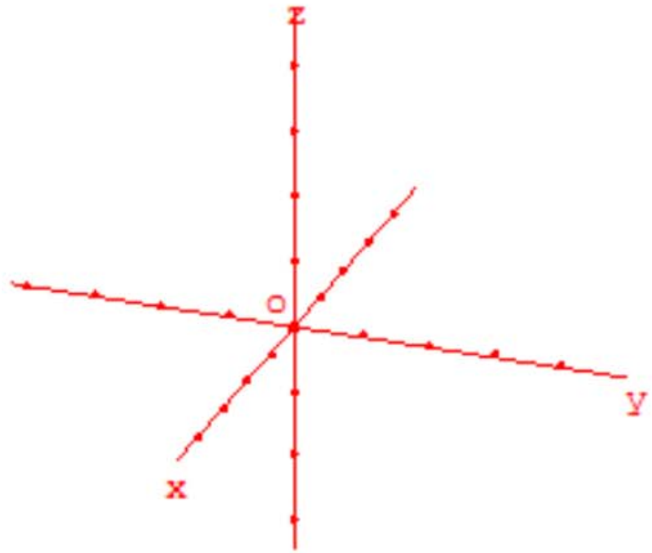
- If **a** and **b** are parallel, $\mathbf{a}.\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|$. ← $|\mathbf{a}|\,|\mathbf{b}| \cos 0°$
  - In particular, $\mathbf{a}.\mathbf{a} = |\mathbf{a}|^2$. ← $|\mathbf{a}|\,|\mathbf{a}| \cos 0°$

Properties of the scalar product

$\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{c}$ are three vectors, λ is a scalar

- $(\lambda\mathbf{a}).\mathbf{b} = \mathbf{a}.(\lambda\mathbf{b}) = \lambda(\mathbf{a}.\mathbf{b})$
- $\mathbf{a}.\mathbf{b} = \mathbf{b}.\mathbf{a}$
- Distributivity

$$\mathbf{a}.(\mathbf{b}+\mathbf{c}) = \mathbf{a}.\mathbf{b} + \mathbf{a}.\mathbf{c}$$



Scalar product using the components of the vectors

The vector **a** and **b** are given as

$$\mathbf{a} = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \quad and \quad \mathbf{b} = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$$

Let's work out $\mathbf{a}.\mathbf{b}$

Use the results for parallel and perpendicular unit vectors:

$\mathbf{i}.\mathbf{i} = \mathbf{j}.\mathbf{j} = \mathbf{k}.\mathbf{k} = 1$

$\mathbf{i}.\mathbf{j} = \mathbf{i}.\mathbf{k} = \mathbf{j}.\mathbf{i} = \mathbf{j}.\mathbf{k} = \mathbf{k}.\mathbf{i} = \mathbf{k}.\mathbf{j} = 0$

Scalar product

- If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$,

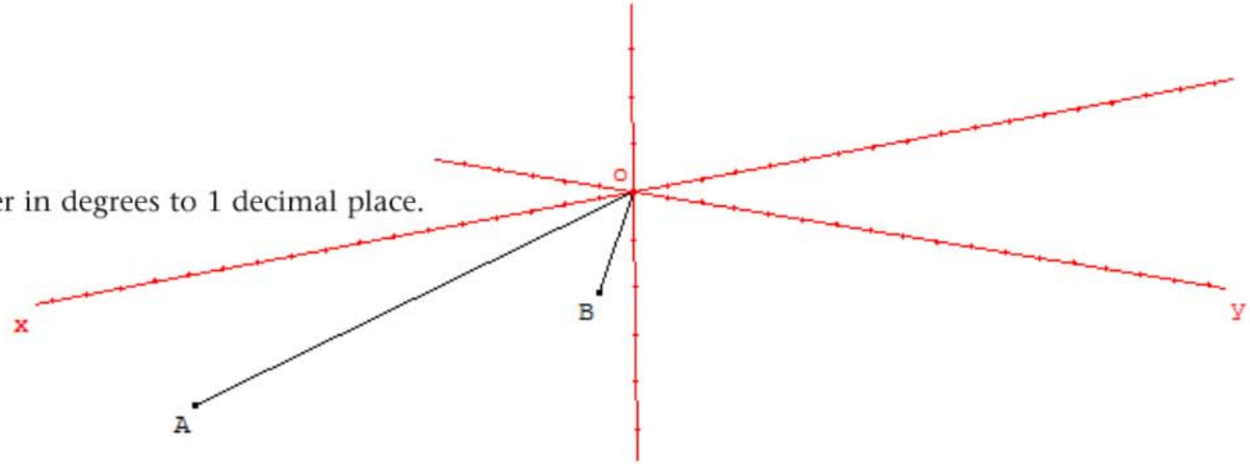
$$\mathbf{a}.\mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} . \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3$$

Typical exercises:

Given that $\mathbf{a} = 8\mathbf{i} - 5\mathbf{j} - 4\mathbf{k}$ and $\mathbf{b} = 5\mathbf{i} + 4\mathbf{j} - \mathbf{k}$:

a Find $\mathbf{a}.\mathbf{b}$

b Find the angle between **a** and **b**, giving your answer in degrees to 1 decimal place.



Given that the vectors $\mathbf{a} = 2\mathbf{i} - 6\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 5\mathbf{i} + 2\mathbf{j} + \lambda\mathbf{k}$ are perpendicular, find the value of λ.

Exercises:

1 The vectors **a** and **b** each have magnitude 3 units, and the angle between **a** and **b** is 60°. Find **a**.**b**.

2 In each part, find **a**.**b**:

a $\mathbf{a} = 5\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$

b $\mathbf{a} = 10\mathbf{i} - 7\mathbf{j} + 4\mathbf{k}$, $\mathbf{b} = 3\mathbf{i} - 5\mathbf{j} - 12\mathbf{k}$

c $\mathbf{a} = \mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{b} = -\mathbf{i} - \mathbf{j} + 4\mathbf{k}$

d $\mathbf{a} = 2\mathbf{i} - \mathbf{k}$, $\mathbf{b} = 6\mathbf{i} - 5\mathbf{j} - 8\mathbf{k}$

e $\mathbf{a} = 3\mathbf{j} + 9\mathbf{k}$, $\mathbf{b} = \mathbf{i} + 12\mathbf{j} - 4\mathbf{k}$

3 In each part, find the angle between **a** and **b**, giving your answer in degrees to 1 decimal place:

a $\mathbf{a} = 3\mathbf{i} + 7\mathbf{j}$, $\mathbf{b} = 5\mathbf{i} + \mathbf{j}$

b $\mathbf{a} = 2\mathbf{i} - 5\mathbf{j}$, $\mathbf{b} = 6\mathbf{i} + 3\mathbf{j}$

c $\mathbf{a} = \mathbf{i} - 7\mathbf{j} + 8\mathbf{k}$, $\mathbf{b} = 12\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

d $\mathbf{a} = -\mathbf{i} - \mathbf{j} + 5\mathbf{k}$, $\mathbf{b} = 11\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$

e $\mathbf{a} = 6\mathbf{i} - 7\mathbf{j} + 12\mathbf{k}$, $\mathbf{b} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$

f $\mathbf{a} = 4\mathbf{i} + 5\mathbf{k}$, $\mathbf{b} = 6\mathbf{i} - 2\mathbf{j}$

g $\mathbf{a} = -5\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - 2\mathbf{j} + 11\mathbf{k}$

h $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$

4 Find the value, or values, of λ for which the given vectors are perpendicular:

a $3\mathbf{i} + 5\mathbf{j}$ and $\lambda\mathbf{i} + 6\mathbf{j}$

b $2\mathbf{i} + 6\mathbf{j} - \mathbf{k}$ and $\lambda\mathbf{i} - 4\mathbf{j} - 14\mathbf{k}$

c $3\mathbf{i} + \lambda\mathbf{j} - 8\mathbf{k}$ and $7\mathbf{i} - 5\mathbf{j} + \mathbf{k}$

d $9\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$ and $\lambda\mathbf{i} + \lambda\mathbf{j} + 3\mathbf{k}$

e $\lambda\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\lambda\mathbf{i} + \lambda\mathbf{j} + 5\mathbf{k}$

5 Find, to the nearest tenth of a degree, the angle that the vector $9\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$ makes with:

a the positive x-axis **b** the positive y-axis

6 Find, to the nearest tenth of a degree, the angle that the vector $\mathbf{i} + 11\mathbf{j} - 4\mathbf{k}$ makes with:

a the positive y-axis **b** the positive z-axis

7 The angle between the vectors $\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $2\mathbf{i} + \mathbf{j} + \mathbf{k}$ is θ. Calculate the exact value of $\cos\theta$.

8 The angle between the vectors $\mathbf{i} + 3\mathbf{j}$ and $\mathbf{j} + \lambda\mathbf{k}$ is 60°.

Show that $\lambda = \pm\sqrt{\frac{13}{5}}$.

9 Simplify as far as possible:

a $\mathbf{a}.(\mathbf{b} + \mathbf{c}) + \mathbf{b}.(\mathbf{a} - \mathbf{c})$, given that **b** is perpendicular to **c**.

b $(\mathbf{a} + \mathbf{b}).(\mathbf{a} + \mathbf{b})$, given that $|\mathbf{a}| = 2$ and $|\mathbf{b}| = 3$.

c $(\mathbf{a} + \mathbf{b}).(2\mathbf{a} - \mathbf{b})$, given that **a** is perpendicular to **b**.

10 Find a vector which is perpendicular to both **a** and **b**, where:

a $\mathbf{a} = \mathbf{i} + \mathbf{j} - 3\mathbf{k}$, $\mathbf{b} = 5\mathbf{i} - 2\mathbf{j} - \mathbf{k}$

b $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$, $\mathbf{b} = \mathbf{i} - 6\mathbf{j} + 3\mathbf{k}$

c $\mathbf{a} = 4\mathbf{i} - 4\mathbf{j} - \mathbf{k}$, $\mathbf{b} = -2\mathbf{i} - 9\mathbf{j} + 6\mathbf{k}$

11 The points A and B have position vectors $2\mathbf{i} + 5\mathbf{j} + \mathbf{k}$ and $6\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ respectively, and O is the origin.
Calculate each of the angles in $\triangle OAB$, giving your answers in degrees to 1 decimal place.

12 The points A, B and C have position vectors $\mathbf{i} + 3\mathbf{j} + \mathbf{k}$, $2\mathbf{i} + 7\mathbf{j} - 3\mathbf{k}$ and $4\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$ respectively.

a Find, as surds, the lengths of AB and BC.

b Calculate, in degrees to 1 decimal place, the size of $\angle ABC$.

Answers

1 $\frac{9}{2}$

2 **a** 2 **b** 17 **c** −6 **d** 20 **e** 0

3 **a** 55.5° **b** 94.8° **c** 87.4° **d** 79.0° **e** 100.9° **f** 53.7° **g** 132.2° **h** 70.5°

4 **a** −10 **b** 5 **c** $2\frac{3}{5}$ **d** $-2\frac{1}{2}$ **e** −5 or 2

5 **a** 32.9° **b** 117.8°

6 **a** 20.5° **b** 109.9°

7 $\frac{2\sqrt{2}}{3}$

9 **a** $2\mathbf{a}.\mathbf{b} + \mathbf{a}.\mathbf{c}$ **b** $13 + 2\mathbf{a}.\mathbf{b}$ **c** $2|\mathbf{a}|^2 - |\mathbf{b}|^2$

10 **a** $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ **b** $3\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ **c** $3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$

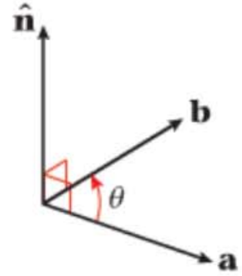
11 64.7°, 64.7°, 50.6°

12 **a** $\sqrt{33}$, $\sqrt{173}$ **b** 29.1°

The vector product or cross product

The scalar product combines two vectors and returns a scalar/number.

The vector product combines two vectors and returns a third vector

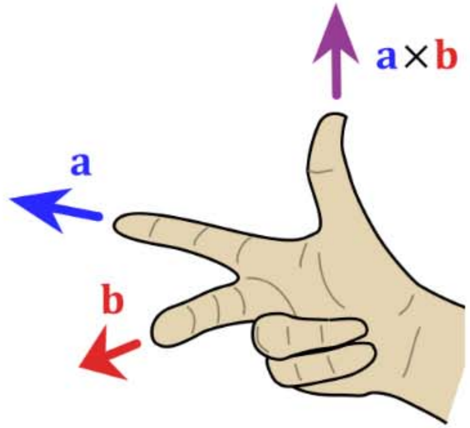


■ **The vector (or cross) product of the vectors a and b is defined as**

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}},$$

This is a key fact which you should learn.

where again θ is the angle between **a** and **b**, and where $\hat{\mathbf{n}}$ is a unit vector perpendicular to both **a** and **b**. The direction of $\hat{\mathbf{n}}$ is that in which a right-handed screw would move when turned from **a** to **b**.

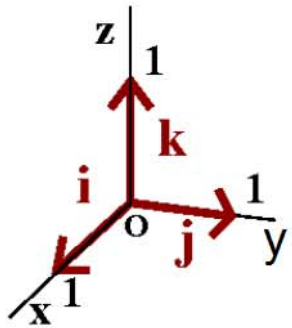


Using the base vectors

Complete

- $\mathbf{i} \times \mathbf{i} =$ ____
- $\mathbf{j} \times \mathbf{j} =$ ____
- $\mathbf{k} \times \mathbf{k} =$ ____

- $\mathbf{i} \times \mathbf{j} =$ ____ and $\mathbf{j} \times \mathbf{i} =$ ____
- $\mathbf{j} \times \mathbf{k} =$ ____ and $\mathbf{k} \times \mathbf{j} =$ ____
- $\mathbf{k} \times \mathbf{i} =$ ____ and $\mathbf{i} \times \mathbf{k} =$ ____



Properties of the vector product

a , **b** *and* **c** are three vectors and λ is a scalar

- $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$
- $(\lambda\mathbf{a})\times\mathbf{b} = \mathbf{a}\times(\lambda\mathbf{b}) = \lambda(\mathbf{a}\times\mathbf{b})$
- Distributivity

  $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{c}$

$\mathbf{a}\times\mathbf{b} = 0 \Leftrightarrow \mathbf{a} = 0$ or $\mathbf{b} = 0$ or **a** and **b** are parallel

Application:

Find the following vector products.

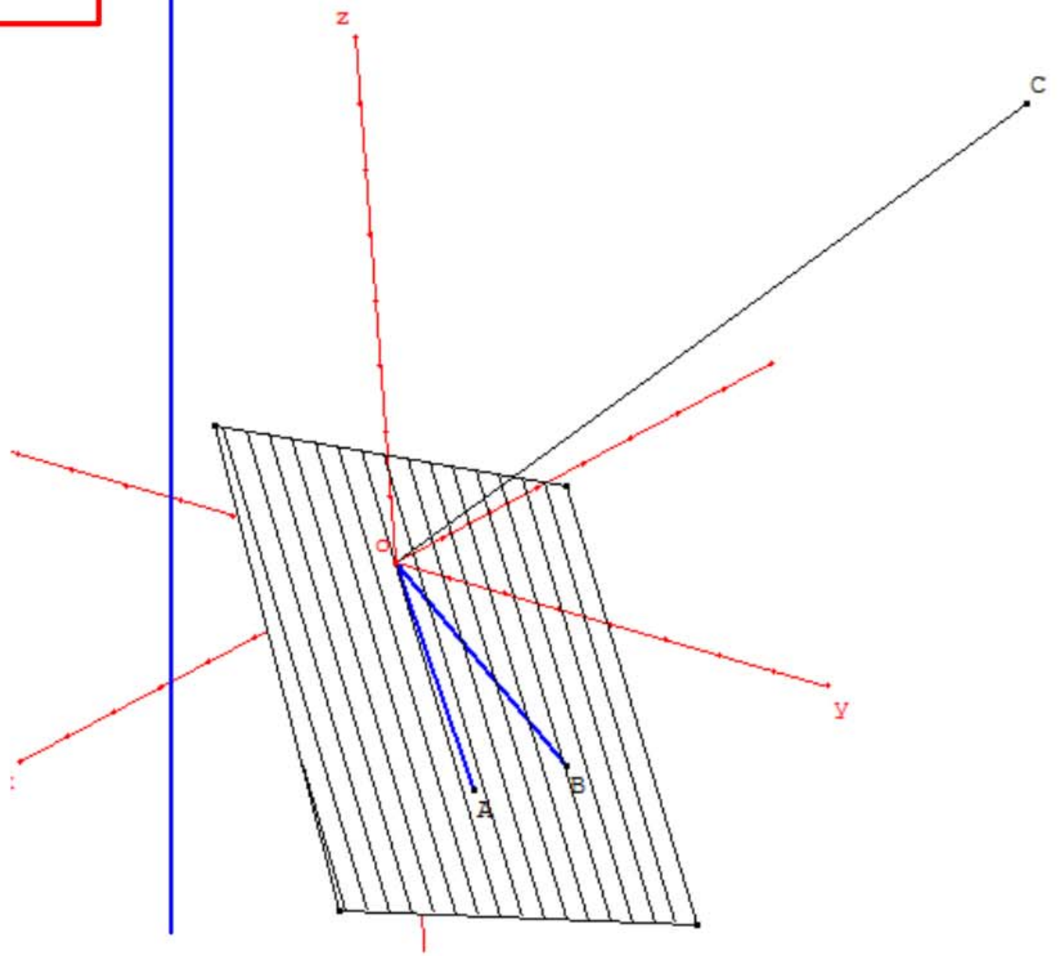
1. $\mathbf{i}\times(\mathbf{i}+\mathbf{j}+\mathbf{k})$.

2. $(\mathbf{i}+\mathbf{j}+\mathbf{k})\times\mathbf{i}$.

3. $(3\mathbf{i}+\mathbf{j})\times 2\mathbf{k}$.

4. $(\mathbf{i}+\mathbf{j})\times(\mathbf{i}+\mathbf{k})$.

Use these properties to work out $\mathbf{a}\times\mathbf{b}$ with $\mathbf{a} = 2\mathbf{i}+3\mathbf{j}-2\mathbf{k}$ and $\mathbf{b} = -\mathbf{i}+2\mathbf{j}-3\mathbf{k}$



Generalisation and alternative method :

The distributivity of the vector product over addition can be used to find the vector product of any two vectors, $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$.

Conclusion

$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$.

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

This formula is not easy to remember, but there is an alternative, using determinants.

Determinants method:

2×2 *matrices*

The determinant of the matrix $M=\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$

is the number noted $\det(M)=\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}=a_1b_2-a_2b_1$

(This number is the area scale factor of the matrix transformation M)

Examples:

$M=\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ *then* $\det(M)=$

$A=\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ *then* $\det(A)=$

Working out the components of the vector product

two vectors, $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ and $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$

Be careful with this "-" sign

Write the determinant: $\mathbf{a}\times\mathbf{b}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i}-\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j}+\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}$

In practice: Method 1

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \qquad \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \qquad \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

OR

Method 2

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 & a_1 & a_2 & a_3 & a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 & b_1 & b_2 & b_3 & b_1 & b_2 & b_3 \end{vmatrix}$$

Numerical examples:

- Work out the components of $\mathbf{a} \times \mathbf{b}$ in each case

$$a)\ \mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \ \textit{and} \ \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \qquad b)\ \mathbf{a} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} \ \textit{and} \ \mathbf{b} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

- Find a unit vector perpendicular to both $(4\mathbf{i} + 3\mathbf{j} + 2\,\mathbf{k})$ and $(8\mathbf{i} + 3\mathbf{j} + 3\mathbf{k})$.

Note:
When writing the determinant, it does not matter if you write the components of the vectors in lines or in columns

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & a_1 & b_1 \\ \mathbf{j} & a_2 & b_2 \\ \mathbf{k} & a_3 & b_3 \end{vmatrix}$$

More practice

Work out the components of these vectors
Give your answers as a column matrix

(a) $\begin{pmatrix}1\\2\\3\end{pmatrix} \times \begin{pmatrix}-2\\3\\-2\end{pmatrix}$ (b) $\begin{pmatrix}-2\\3\\-1\end{pmatrix} \times \begin{pmatrix}1\\-5\\0\end{pmatrix}$ (c) $\begin{pmatrix}4\\-5\\7\end{pmatrix} \times \begin{pmatrix}2\\-3\\6\end{pmatrix}$

(d) $\begin{pmatrix}5\\-1\\2\end{pmatrix} \times \begin{pmatrix}4\\0\\0\end{pmatrix}$ (e) $\begin{pmatrix}1\\0\\0\end{pmatrix} \times \begin{pmatrix}0\\1\\0\end{pmatrix}$ (f) $\begin{pmatrix}1\\0\\2\end{pmatrix} \times \begin{pmatrix}0\\1\\2\end{pmatrix}$

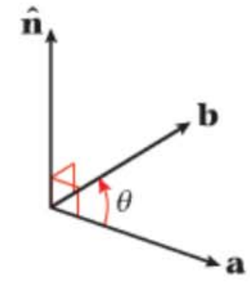
Answers

(a) $\begin{pmatrix}-13\\-4\\7\end{pmatrix}$ (b) $\begin{pmatrix}-5\\-1\\7\end{pmatrix}$ (c) $\begin{pmatrix}-9\\-10\\-2\end{pmatrix}$

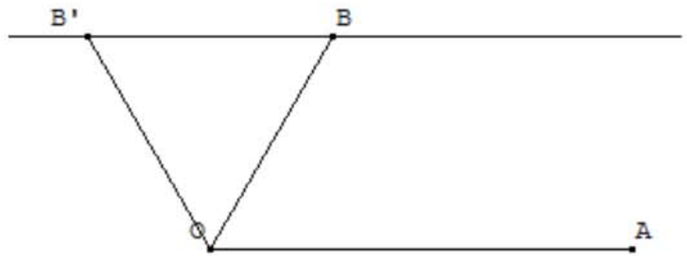
(d) $\begin{pmatrix}0\\8\\4\end{pmatrix}$ (e) $\begin{pmatrix}0\\0\\1\end{pmatrix}$ (f) $\begin{pmatrix}-2\\-2\\1\end{pmatrix}$

Vector product and angles

$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}}$ so $|\mathbf{a} \times \mathbf{b}| =$



Note that without anymore information, it is not possible to determine a measure of θ



Typical question:

Find the sine of the acute angle between the vectors $\mathbf{a} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = -3\mathbf{j} + 4\mathbf{k}$.

Exercises:

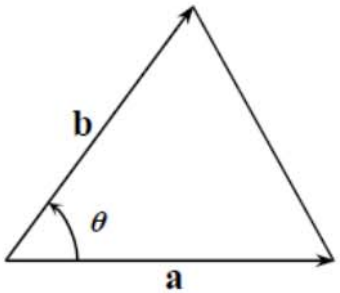
Find the sine of the angle between **a** and **b** in each of the following. You may leave your answers as surds, in their simplest form.

a $\mathbf{a} = 3\mathbf{i} - 4\mathbf{j}$, $\mathbf{b} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

b $\mathbf{a} = \mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 5\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$

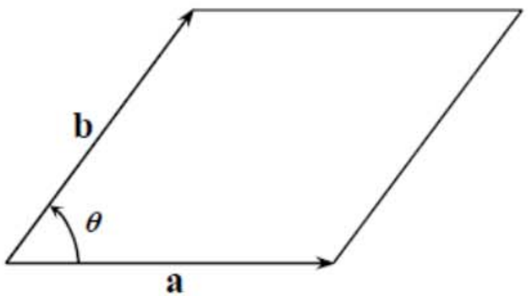
c $\mathbf{a} = 5\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 4\mathbf{i} + 4\mathbf{j} + \mathbf{k}$

Application of vector products to areas



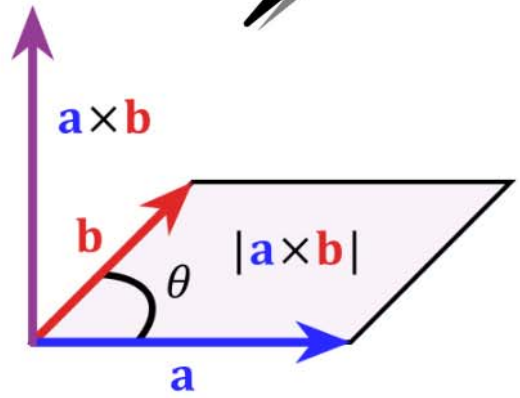
Area $= \frac{1}{2}ab\sin\theta = \frac{1}{2}|\mathbf{a}\times\mathbf{b}|$

The area of a triangle is $\frac{1}{2}|\mathbf{a}\times\mathbf{b}|$



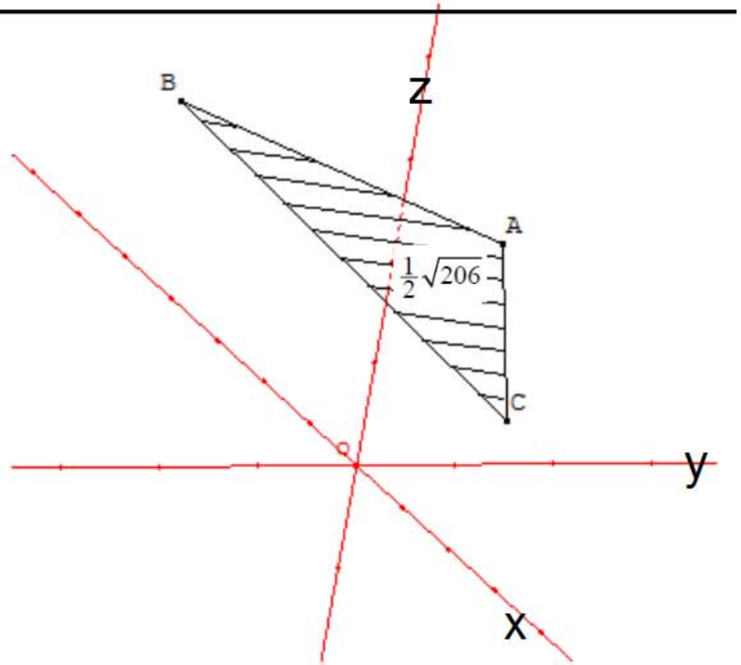
Area $= ab\sin\theta = |\mathbf{a}\times\mathbf{b}|$

The area of a parallelogram is $|\mathbf{a}\times\mathbf{b}|$



Let's try it...

Find the area of triangle ABC, where A is (2, 0, 3), B is (1, –3, 4) and C is (–1, 2, 0).



Exercises:

Find the area of triangle OAB, where O is the origin, A is the point with position vector **a** and B is the point with position vector **b** in the following cases.

1 $\mathbf{a} = \mathbf{i} + \mathbf{j} - 4\mathbf{k}$ $\quad\quad$ $\mathbf{b} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$

2 $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$ $\quad\quad$ $\mathbf{b} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

3 $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$ $\quad\quad$ $\mathbf{b} = \begin{pmatrix} 2 \\ 6 \\ -9 \end{pmatrix}$

4 Find the area of the triangle with vertices $A(0, 0, 0)$, $B(1, -2, 1)$ and $C(2, -1, -1)$.

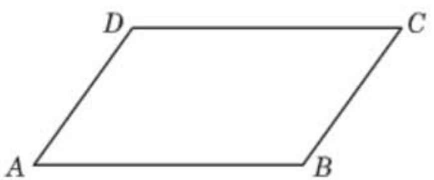
5 Find the area of triangle ABC, where the position vectors of A, B and C are **a**, **b** and **c** respectively, in the following cases:

i $\mathbf{a} = \mathbf{i} - \mathbf{j} - \mathbf{k}$ $\quad$ $\mathbf{b} = 4\mathbf{i} + \mathbf{j} + \mathbf{k}$ $\quad$ $\mathbf{c} = 4\mathbf{i} - 3\mathbf{j} + \mathbf{k}$

ii $\mathbf{a} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ $\mathbf{c} = \begin{pmatrix} 2 \\ 0 \\ -10 \end{pmatrix}$

6 Find the area of the triangle with vertices $A(1, 0, 2)$, $B(2, -2, 0)$ and $C(3, -1, 1)$.

7 Find the area of the triangle with vertices $A(-1, 1, 1)$, $B(1, 0, 2)$ and $C(0, 3, 4)$.



8 Find the area of the parallelogram $ABCD$, shown in the figure, where the position vectors of A, B and D are $\mathbf{i} + \mathbf{j} + \mathbf{k}$, $-3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$ and $2\mathbf{i} - \mathbf{j}$ respectively.

1 4.5
2 $\frac{5\sqrt{2}}{2}$
3 16.5
4 $\frac{3}{2}\sqrt{3}$
5 **i** $2\sqrt{13}$ **ii** 8.5
6 $\frac{3}{2}\sqrt{2}$
7 $\frac{5}{2}\sqrt{3}$
8 $5\sqrt{2}$

The triple scalar product

The scalar product and the vector product involve only two vectors.

There are "operators" which involve 3 vectors, one is the triple vector product: **a×(b×c)**.

Another one is the triple scalar product: **a**.(**b×c**)

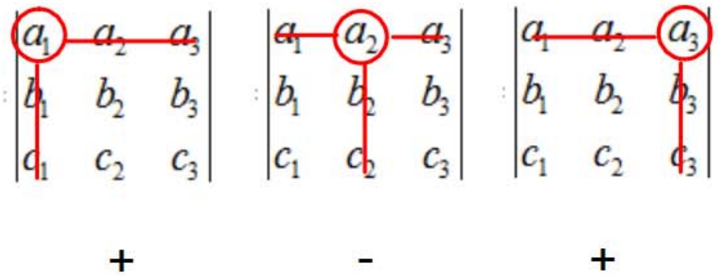
Note: the triple scalar product returns a scalar (number).

The triple scalar product formula

The three vectors $\mathbf{a}=\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \mathbf{b}=\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ *and* $\mathbf{c}=\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$

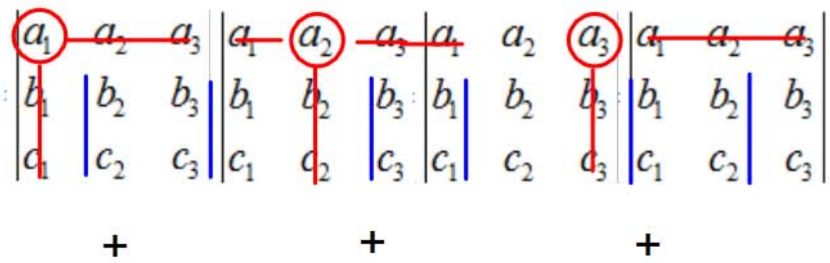
$$\mathbf{a}.(\mathbf{b}\times\mathbf{c})=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}-a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}+a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

In practice: Method 1



OR

Method 2



Numerical examples:

Given that $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 4\mathbf{k}$, $\mathbf{b} = \mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{c} = 2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$

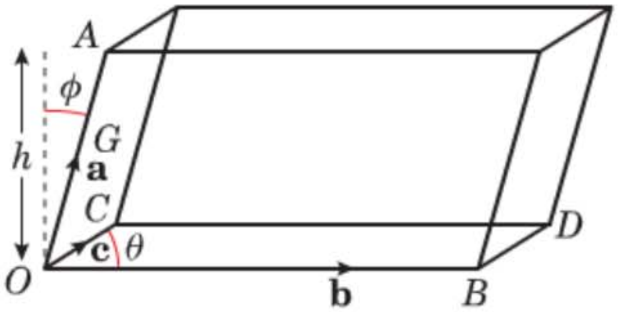
Find $\mathbf{a}.(\mathbf{b} \times \mathbf{c})$

Given that $\mathbf{a} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{b} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{c} = 3\mathbf{i} + 4\mathbf{k}$

find $\mathbf{b}.(\mathbf{c} \times \mathbf{a})$

Interpretations and uses of the scalar triple product

Volume of Parallelepipeds (parallelogram prism)



Proof

Volume = area base × height

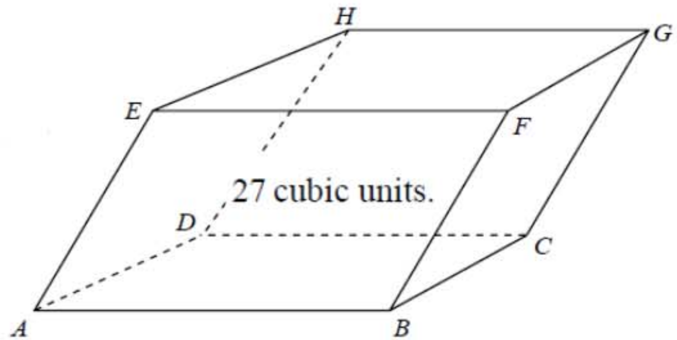
■ **The volume of the parallelepiped is given by $|\mathbf{a}.(\mathbf{b} \times \mathbf{c})|$**

(**a**, **b** and **c** are adjacent edges)

Note that : $\mathbf{a}.\mathbf{b} \times \mathbf{c} = \mathbf{b}.\mathbf{c} \times \mathbf{a} = \mathbf{c}.\mathbf{a} \times \mathbf{b}$

Application:

Find the volume of the parallelepiped $ABCDEFGH$, where A is (1, –1, 4), B is (2, 0, 7), D is (5, 0, –4) and E is (6, 1, 8)



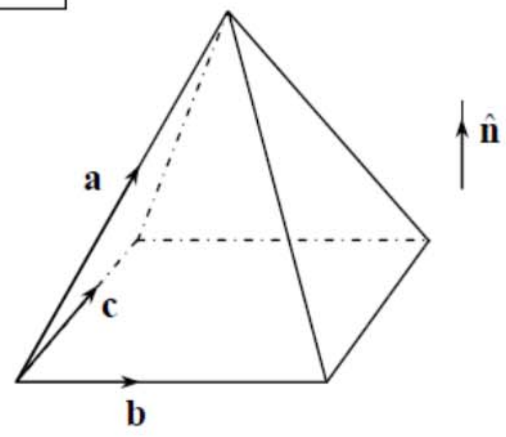
Other volumes

$\mathbf{a.b} \times \mathbf{c}$ means $\mathbf{a.(b} \times \mathbf{c)}$

- **Pyramid**

Volume = $\frac{1}{3}$ × area of base × perpendicular height
(the base is a rectangle or parallelogram)

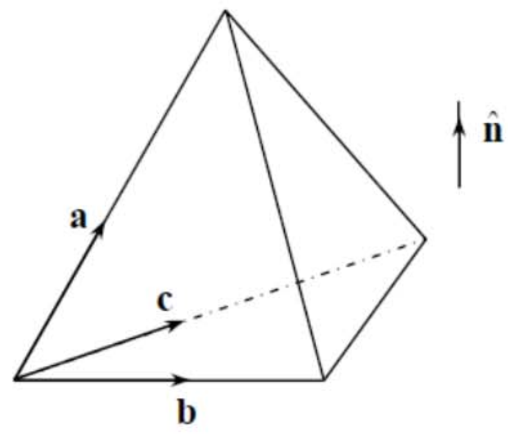
  - volume $= \frac{1}{3}|\mathbf{a.b} \times \mathbf{c}|$



- **Tetrahedron**

Volume = $\frac{1}{3}$ × area of base × perpendicular height
(the base is a triangle)

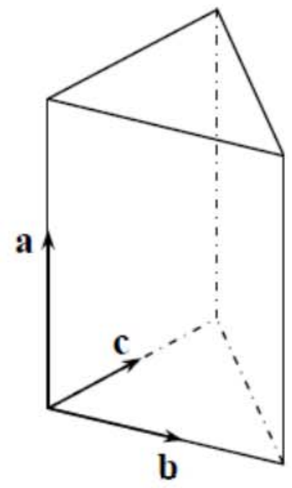
  - volume $= \frac{1}{6}|\mathbf{a.b} \times \mathbf{c}|$



- **Triangular prism**

Volume = area of base × height

  - area of base $= \frac{1}{2}|\mathbf{a.b} \times \mathbf{c}|$



Exercises:

5 A tetrahedron has vertices at $A(1, 2, 3)$, $B(4, 3, 4)$, $C(1, 3, 1)$ and $D(3, 1, 4)$.
Find the volume of the tetrahedron.

6 A tetrahedron has vertices at $A(2, 2, 1)$, $B(3, -1, 2)$, $C(1, 1, 3)$ and $D(3, 1, 4)$.

a Find the area of base BCD.

b Find a unit vector normal to the face BCD.

c Find the volume of the tetrahedron.

7 A tetrahedron has vertices at $A(0, 0, 0)$, $B(2, 0, 0)$, $C(1, \sqrt{3}, 0)$ and $D\left(1, \frac{\sqrt{3}}{3}, \frac{2\sqrt{6}}{3}\right)$.

a Show that the tetrahedron is regular.

b Find the volume of the tetrahedron.

8 A tetrahedron $OABC$ has its vertices at the points
$O(0, 0, 0)$, $A(1, 2, -1)$, $B(-1, 1, 2)$ and $C(2, -1, 1)$.

a Write down expressions for $\overrightarrow{AB}$ and $\overrightarrow{AC}$ in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ and find $\overrightarrow{AB} \times \overrightarrow{AC}$.

b Deduce the area of triangle ABC.

c Find the volume of the tetrahedron.

9 The points A, B, C and D have position vectors

$\mathbf{a} = (2\mathbf{i} + \mathbf{j})$ $\mathbf{b} = (3\mathbf{i} - \mathbf{j} + \mathbf{k})$ $\mathbf{c} = (-2\mathbf{j} - \mathbf{k})$ $\mathbf{d} = (2\mathbf{i} - \mathbf{j} + 3\mathbf{k})$

respectively.

a Find $\overrightarrow{AB} \times \overrightarrow{BC}$ and $\overrightarrow{BD} \times \overrightarrow{DC}$.

b Hence find

i the area of triangle ABC

ii the volume of the tetrahedron $ABCD$.

5 $\frac{2}{3}$

6 **a** 3 **b** $\pm\frac{1}{3}(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$ **c** $2\frac{1}{3}$

7 **b** $\frac{2}{3}\sqrt{2}$

8 **a** $\overrightarrow{AB} = -2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$
$\overrightarrow{AC} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$
$\overrightarrow{AB} \times \overrightarrow{AC} = 7\mathbf{i} + 7\mathbf{j} + 7\mathbf{k}$

b $\frac{7\sqrt{3}}{2}$

c $\frac{7}{3}$

9 **a** $5\mathbf{i} - \mathbf{j} - 7\mathbf{k}$
$2\mathbf{i} - 8\mathbf{j} + \mathbf{k}$

b **i** $\frac{5}{2}\sqrt{3}$

ii $\frac{19}{6}$

Coplanarity of vectors - Linear independence

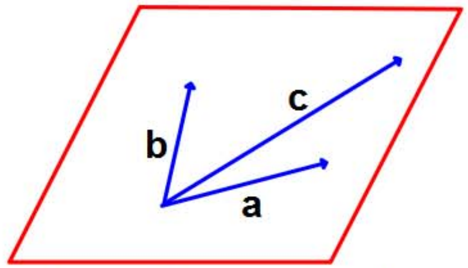
Three vectors are coplanar if one is the combination of the two other ones.
In this case, the vectors are said the be LINEARLY DEPENDENT.

$$\mathbf{a}, \mathbf{b} \text{ and } \mathbf{c} \text{ are coplanar } \Leftrightarrow |\mathbf{a}.\mathbf{b}\times\mathbf{c}| = 0$$

meaning

Three vectors $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ and $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ are linearly dependent if, and only if,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$



$\mathbf{c}$ is a combination of $\mathbf{a}$ and $\mathbf{b}$:

$$\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b}$$

Determine whether or not the following sets of vectors are coplanar.

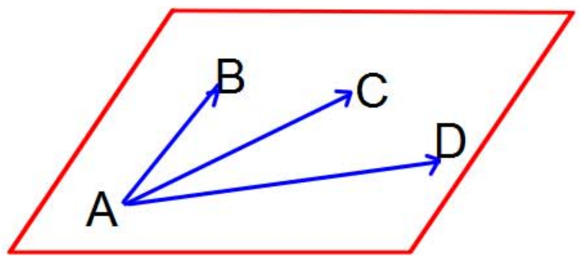
(a) $2\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}, \quad 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}, \quad 7\mathbf{j} - 17\mathbf{k}$

(b) $3\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}, \quad 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}, \quad 5\mathbf{j} + 4\mathbf{k}$

Four points are coplanar if they belongs to the same plane.
(Note: three points are always coplanar)

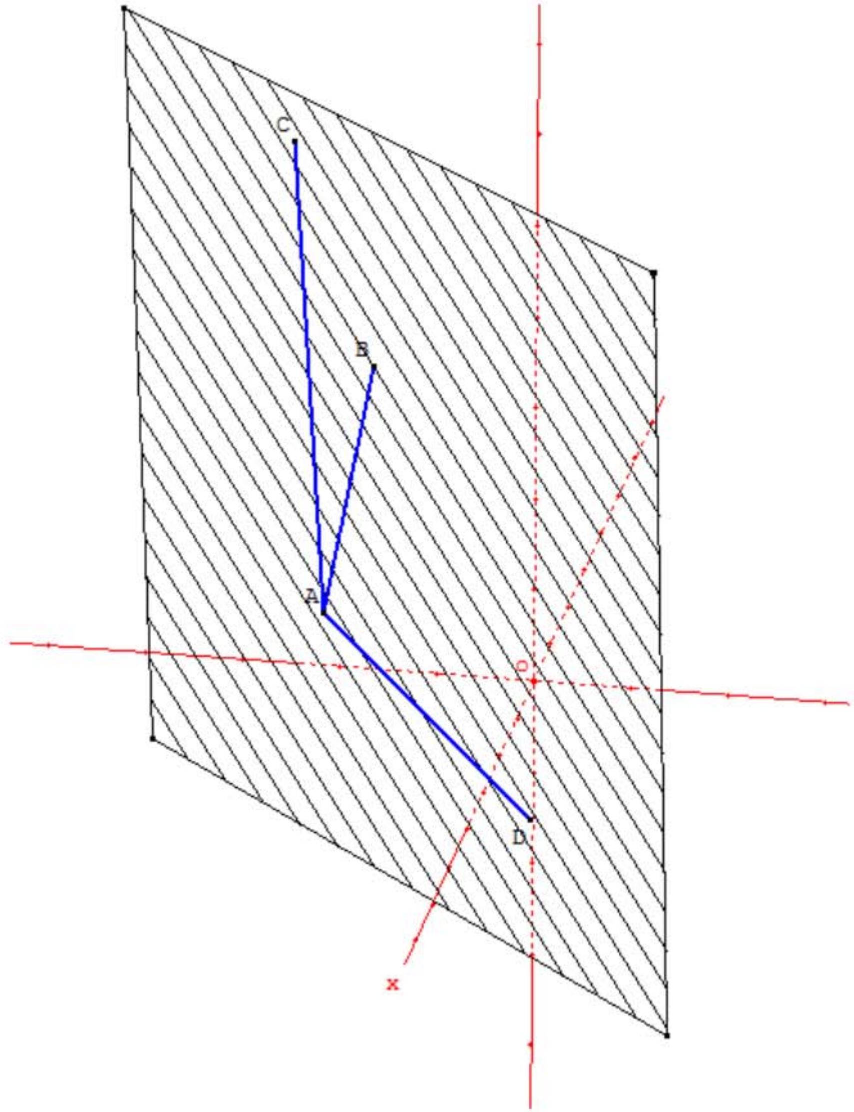
The four distincts points A, B, C *and* D *are coplanar*
if and only if

$$\overrightarrow{AB}.\left(\overrightarrow{AC}\times\overrightarrow{AD}\right) = 0$$



Exercises:

Determine whether the four points (1, –2, 1), (4, –1, 5), (3, –2, 7) and (6, 1, 1) lie in a plane or not.



Summary of key points

1 The vector (or cross) product of the vectors **a** and **b** is defined as

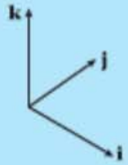
$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}},$$

where θ is the angle between **a** and **b**, and where $\hat{\mathbf{n}}$ is a unit vector perpendicular to both **a** and **b**. The direction of $\hat{\mathbf{n}}$ is that in which a right handed screw would move when turned from **a** to **b**.

2 $\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$
The vector product is not commutative. The order matters.

3 If **i**, **j** and **k** are unit vectors in the x, y and z directions respectively, then

$\mathbf{i} \times \mathbf{i} = 0$
$\mathbf{j} \times \mathbf{j} = 0$
$\mathbf{k} \times \mathbf{k} = 0$
also $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ and $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$
$\mathbf{j} \times \mathbf{k} = \mathbf{i}$ and $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$
$\mathbf{k} \times \mathbf{i} = \mathbf{j}$ and $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$



4 If $\mathbf{a} \times \mathbf{b} = 0$, then either
$\mathbf{a} = 0$ or $\mathbf{b} = 0$ or **a** and **b** are parallel.

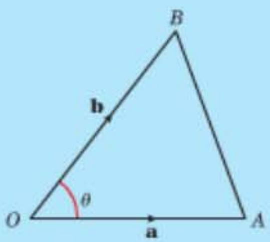
5 In Cartesian form when $\mathbf{a} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})$ and $\mathbf{b} = (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

i.e. $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

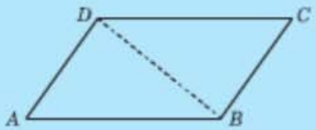
6 After evaluating a cross product take the dot product of each of the given vectors with your answer vector. Both answers should be zero as $\mathbf{a} \times \mathbf{b}$ is perpendicular to each of **a** and **b**. **This is a useful check.**

7 Area of triangle $OAB = \frac{1}{2}|\mathbf{a} \times \mathbf{b}|$



8 Area of triangle $ABC = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| \quad = \frac{1}{2}|(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})|$
$= \frac{1}{2}|(\mathbf{a} \times \mathbf{b}) + (\mathbf{b} \times \mathbf{c}) + (\mathbf{c} \times \mathbf{a})|$

9 Area of parallelogram $ABCD = |\overrightarrow{AB} \times \overrightarrow{AD}|$
$= |(\mathbf{b} - \mathbf{a}) \times (\mathbf{d} - \mathbf{a})|$
$= |(\mathbf{a} \times \mathbf{b}) + (\mathbf{b} \times \mathbf{d}) + (\mathbf{d} \times \mathbf{a})|$



10 In Cartesian form when $\mathbf{a} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})$, $\mathbf{b} = (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$ and $\mathbf{c} = (c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k})$.

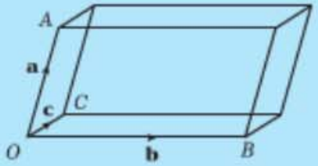
$$\mathbf{a}.(\mathbf{b} \times \mathbf{c}) = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$$

This can also be written as

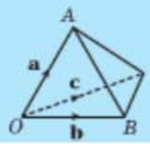
$$\mathbf{a}.(\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

11 Note that $\mathbf{a}.(\mathbf{b} \times \mathbf{c}) = \mathbf{b}.(\mathbf{c} \times \mathbf{a}) = \mathbf{c}.(\mathbf{a} \times \mathbf{b})$
Also $\mathbf{a}.(\mathbf{a} \times \mathbf{x}) = 0$ for any vector **x**.

12 The volume of the parallelepiped is given by $|\mathbf{a}.(\mathbf{b} \times \mathbf{c})|$.



13 The volume of the tetrahedron is given by $|\frac{1}{6}\mathbf{a}.(\mathbf{b} \times \mathbf{c})|$.



The scalar product

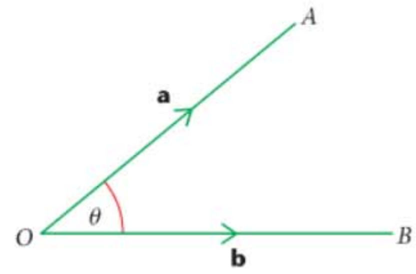
The vector **a** and **b** are given as

$$\mathbf{a} = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$$

$$\mathbf{a.b} = |\mathbf{a}||\mathbf{b}|\,Cos\,\theta$$

and

$$\mathbf{a.b} = a_1a_2 + b_1b_2 + c_1c_2$$



$$Consequence: \; Cos\theta = \frac{\mathbf{a.b}}{|\mathbf{a}||\mathbf{b}|}$$