Planes and lines

Specifications

Vectors and Three–Dimensional Coordinate Geometry

Applications of vectors to two- and three-dimensional geometry, involving points, lines and planes.

Vector equation of a plane in the form $\mathbf{r}.\ \mathbf{n} = d$ or $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$.
Intersection of a line and a plane.
Angle between a line and a plane and between two planes.

Cartesian coordinate geometry of lines and planes.

To include finding the equation of the line of intersection of two non-parallel planes.

Knowledge of formulae other than those in the formulae booklet will not be expected.

In formulae book: The plane through A with normal vector $\mathbf{n} = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}$ has cartesian equation

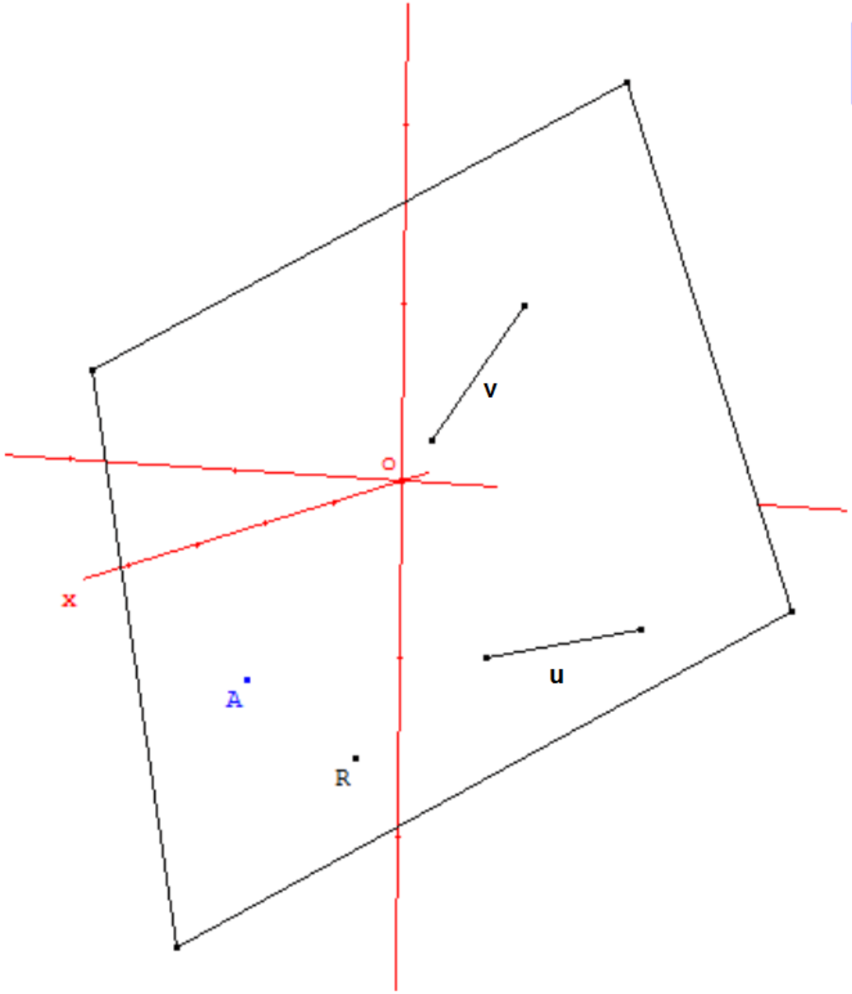
$$n_1x + n_2y + n_3z = d \text{ where } d = \mathbf{a.n}$$

The plane through non-collinear points A, B and C has vector equation

$$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}) + \mu(\mathbf{c} - \mathbf{a}) = (1 - \lambda - \mu)\mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$$

The plane through the point with position vector **a** and parallel to **b** and **c** has equation

$$\mathbf{r} = \mathbf{a} + s\mathbf{b} + t\mathbf{c}$$



Equations of a plane in 3 dimensions

To define a plane, you need

- A point , A for example, or its position vector a
- TWO direction vectors LINEARLY INDEPENDENT, for example **u** and **v.**

An equation of the plane is an expression that is satisfied by any vector position **r** of a point R belonging to that plane.

A point R belongs to the plane

if and only if the vector $\overrightarrow{AR}$ is a linear combination of **u** and **v**

$$\Leftrightarrow \overrightarrow{AR} = \lambda\mathbf{u} + \mu\mathbf{v} \Leftrightarrow \mathbf{r} - \mathbf{a} = \lambda\mathbf{u} + \mu\mathbf{v}$$

$$\Leftrightarrow \mathbf{r} = \mathbf{a} + \lambda\mathbf{u} + \mu\mathbf{v} \qquad \lambda, \mu \in \mathbb{R}$$

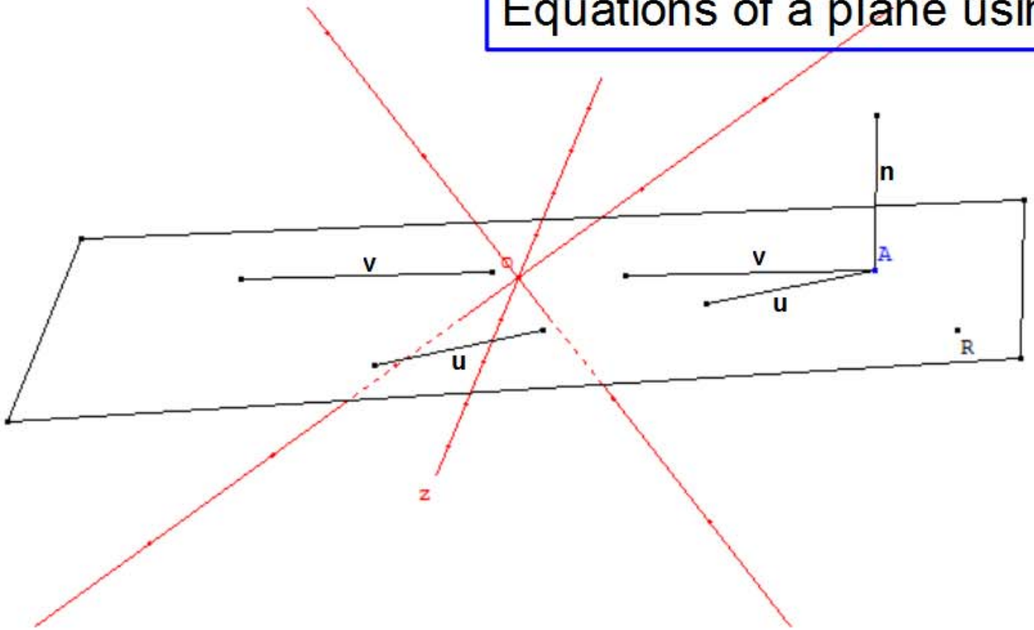
Using the components:

$$(\mathbf{r} =) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \mu \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

Example:

Find, in the form $\mathbf{r}=\mathbf{a}+\lambda\mathbf{u}+\mu\mathbf{v}$, an equation of the plane that passes through the points $A(2,2,-1)$, $B(3,2,-1)$ *and* $C(4,3,5)$

Equations of a plane using NORMAL vectors



A point R belongs to the plane

if and only if the vector $\overrightarrow{AR}$ is a linear combination of **u** and **v**

$\Leftrightarrow$ The vector $\overrightarrow{AR}, \mathbf{u}$ *and* $\mathbf{v}$ are coplanar

This means, using the triple scalar product that :

$$\overrightarrow{AR}.(\mathbf{u} \times \mathbf{v}) = 0$$

The vector $\mathbf{u} \times \mathbf{v}$ is perpendicular to **u** and **v**, perpendicular to the plane.

Such a vector , usually named **n**,is called a NORMAL vector to the plane.

The relationship $\overrightarrow{AR}.(\mathbf{u} \times \mathbf{v}) = 0$ becomes

$\Leftrightarrow \quad \overrightarrow{AR}.\mathbf{n} = 0$

$\Leftrightarrow \quad (\mathbf{r} - \mathbf{a}).\mathbf{n} = 0$

$\Leftrightarrow \quad \mathbf{r}.\mathbf{n} = \mathbf{a}.\mathbf{n}$ *where* **a.n** is to work out.

- Find an equation of the plane going through A(2, 1, 1) with normal vector $\mathbf{n} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ in the form $\mathbf{r}.\mathbf{n} = d$

- Find an equation of the plane going through A(0, 1, 1) with direction vectors $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ in the form $\mathbf{r}.\mathbf{n} = d$

Exercises:

1 Find, in the form $\mathbf{r.n} = p$, an equation of the plane that passes through the point with position vector $\mathbf{a}$ and is perpendicular to the vector $\mathbf{n}$ where

a $\mathbf{a} = \mathbf{i} - \mathbf{j} - \mathbf{k}$ and $\mathbf{n} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$

b $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{n} = 5\mathbf{i} - \mathbf{j} - 3\mathbf{k}$

c $\mathbf{a} = 2\mathbf{i} - 3\mathbf{k}$ and $\mathbf{n} = \mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$

d $\mathbf{a} = 4\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{n} = 4\mathbf{i} + \mathbf{j} - 5\mathbf{k}$

2 Find, in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$ an equation of the plane that passes through the points

a (1, 2, 0), (3, 1, −1) and (4, 3, 2)

b (3, 4, 1), (−1, −2, 0) and (2, 1, 4)

c (2, −1, −1), (3, 1, 2) and (4, 0, 1)

d (−1, 1, 3), (−1, 2, 5) and (0, 4, 4).

3 Find the equation of the plane containing the two lines $\mathbf{r} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + t\begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$ and $\mathbf{r} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} + s\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

1 **a** $\mathbf{r}.(2\mathbf{i} + \mathbf{j} + \mathbf{k}) = 0$
b $\mathbf{r}.(5\mathbf{i} - \mathbf{j} - 3\mathbf{k}) = 0$
c $\mathbf{r}.(\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) = -10$
d $\mathbf{r}.(4\mathbf{i} + \mathbf{j} - 5\mathbf{k}) = 9$

2 **a** $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \lambda(2\mathbf{i} - \mathbf{j} - \mathbf{k}) + \mu(3\mathbf{i} + \mathbf{j} + 2\mathbf{k})$
b $\mathbf{r} = 3\mathbf{i} + 4\mathbf{j} + \mathbf{k} + \lambda(-4\mathbf{i} - 6\mathbf{j} - \mathbf{k}) + \mu(-\mathbf{i} - 3\mathbf{j} + 3\mathbf{k})$
or $\mathbf{r} = 3\mathbf{i} + 4\mathbf{j} + \mathbf{k} + \lambda'(4\mathbf{i} + 6\mathbf{j} + \mathbf{k}) + \mu(-\mathbf{i} - 3\mathbf{j} + 3\mathbf{k})$
c $\mathbf{r} = 2\mathbf{i} - \mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + \mu(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$
d $\mathbf{r} = -\mathbf{i} + \mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{j} + 2\mathbf{k}) + \mu(\mathbf{i} + 3\mathbf{j} + \mathbf{k})$

3 $\mathbf{r}.\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = 7$

Cartesian equation of a plane

Consider the equation of a plane $\mathbf{r}.\mathbf{n} = d$

by writing $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ *and* $\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$, *we have*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} . \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = d \Leftrightarrow n_1 x + n_2 y + n_3 z = d$$

A cartesian equation of a plane can be written as

$$n_1 x + n_2 y + n_3 z = d \quad \textit{where } \mathbf{n} = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k} \text{ is a normal vector}$$

Note: If $d=0$, it means that the plane goes through the origin O(0,0,0).

Examples:

a) Work out a cartesian equation of the plane going thhrough A(1,0,2) with normal vector $\mathbf{n} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

b)i) Find a normal vector of the plane with equation $3x - 2y + 3z = 2$.

ii) Work out the coordinates of the point P, intersection of this plane and the y-axis.

iii) Work out cartesian equations of the line going through P and perpendicular to the line.

Exercises:

Find a Cartesian equation of the plane that passes through the points

a (0, 4, 2), (1, 1, 2) and (−1, 5, 0)

b (1, 1, 0), (2, 3, −3) and (3, 7, −2)

c (3, 0, 0), (2, 0, −1) and (4, 1, 3)

d (1, −1, 6), (3, 1, −2) and (4, 1, 0).

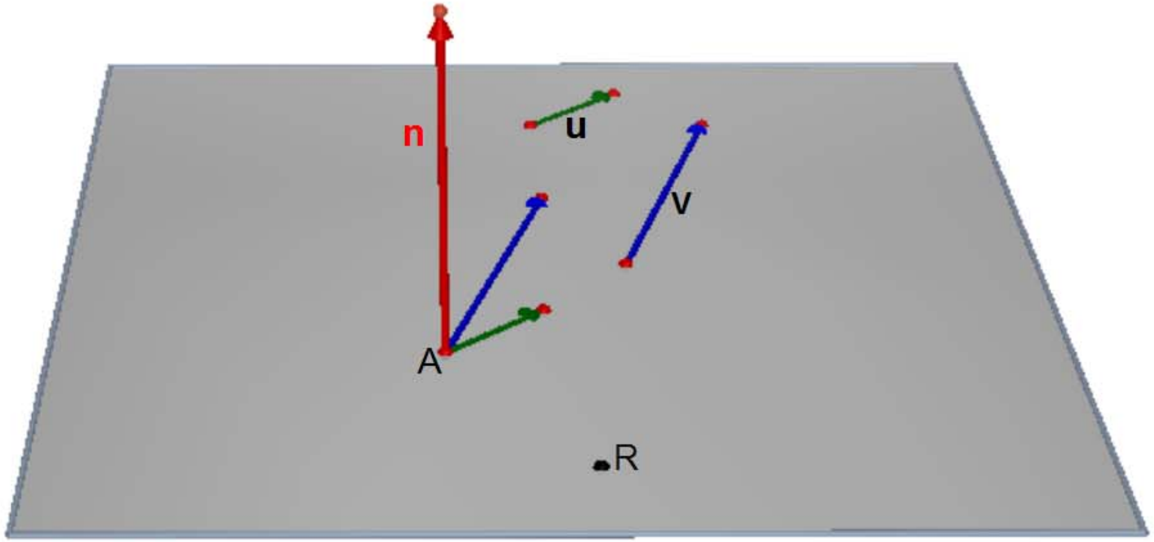
a $3x + y - z = 2$

b $7x - 2y + z = 5$

c $x + 2y - z = 3$

d $2x - 6y - z = 2$

Summary



Consider a plane going through a point A with position vector **a**, and with direction vectors **u** and **v**.

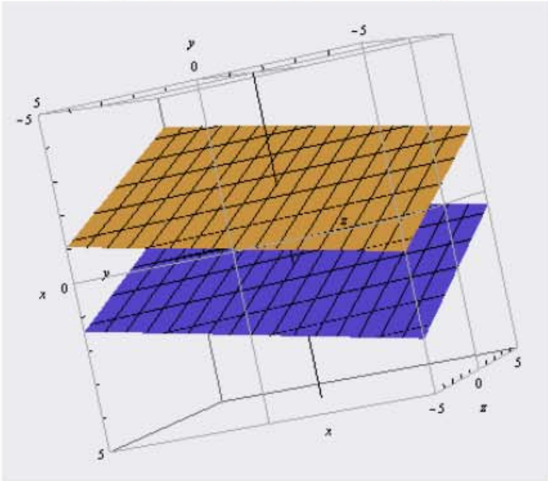
A normal vector to the plane, $\mathbf{n} = n_1\mathbf{i}+n_2\mathbf{j}+n_3\mathbf{k}$, is a multiple of $\mathbf{u} \times \mathbf{v}$.

Equations of this plane can be written:

- $\mathbf{r} = \mathbf{a} + \lambda\mathbf{u} + \mu\mathbf{v} \quad \lambda, \mu \in \mathbb{R}$
- $\mathbf{r}.\mathbf{n} = d \quad \text{where } d = \mathbf{a}.\mathbf{n}$
- $n_1x + n_2y + n_3z = d$

Relative positions of two planes

Two planes can be parallel



Two planes are parallel if their normal vectors are multiples of each other:

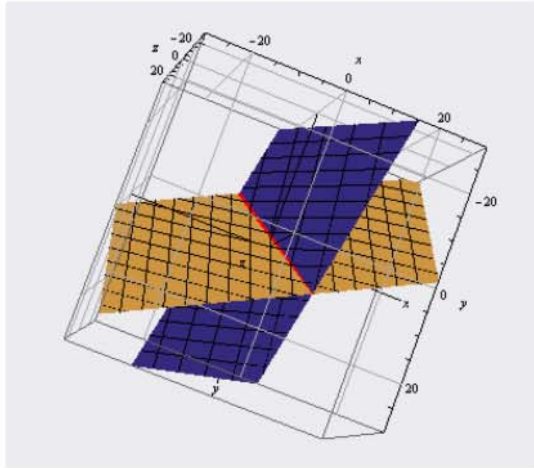
> The plane P_1 with normal vector n_1 and the plane P_2 with normal vector n_2 are parallel if $n_1 = \lambda n_2$

Example:

Show that $P_1 : \mathbf{r}.\begin{pmatrix}1\\-3\\2\end{pmatrix} = 6$ *and* $P_2 : 3x - 9y + 6z = 0$

are parallel.

Two planes can intersect.



The intersection of two planes is a line.
To find an equation of this line, you need

- a point belonging to both planes
- a direction vector

Note:

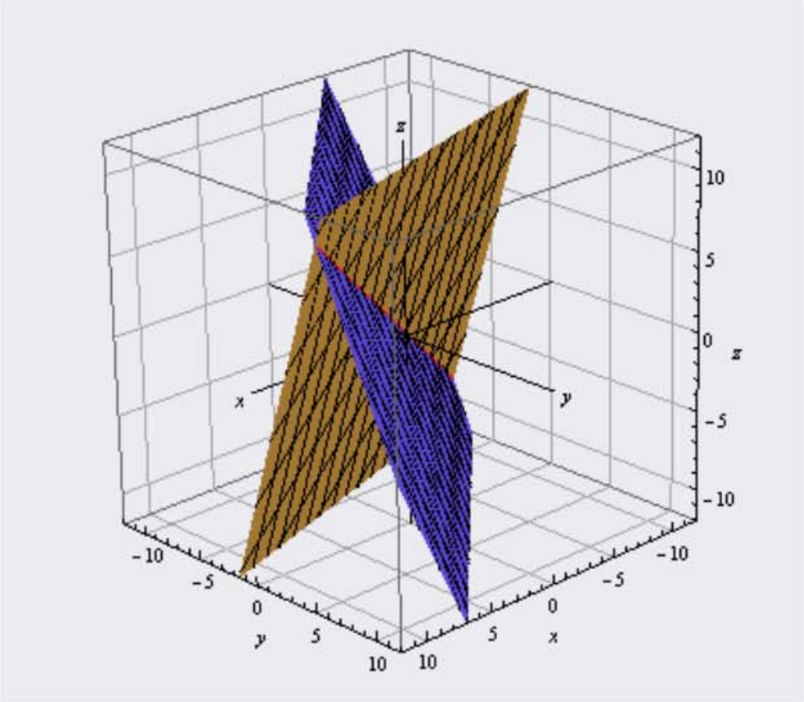
To find a common point, solve the equations of the plane simultaneously

Note :

A direction vector of this line is perpendicular to n_1 and n_2:
a direction vector is $n_1 \times n_2$

Example:

Find the equation of the line of intersection of the planes Π_1 and Π_2 where Π_1 has equation $\mathbf{r}.(2\mathbf{i} - 2\mathbf{j} - \mathbf{k}) = 2$ and Π_2 has equation $\mathbf{r}.(\mathbf{i} - 3\mathbf{j} + \mathbf{k}) = 5$.



Method:

a) Write cartesian equations of the two planes.

b) Consider the equations simultaneously. Choose an arbitrary value for x (*for example*), substitute in the equations and solve to find y and z.

(Choosing $x = 0$ is an easy option but other values can be considered)

This will give you a point P belonging to both plane, i.e the line of intersection.

c) Work out $\mathbf{u}=\mathbf{n}_1 \times \mathbf{n}_2$ *where* $\mathbf{n}_1$ *and* $\mathbf{n}_2$ are normal vectors to the planes.

d) The line of intersection goes through P and has direction vector $\mathbf{u}$: work out an equation of this line (parametric, vector or cartesian).

Practice:

Find the equation of the line of intersection of the planes Π_1 and Π_2 where

a Π_1 has equation $\mathbf{r}.(3\mathbf{i} - 2\mathbf{j} - \mathbf{k}) = 5$ and Π_2 has equation $\mathbf{r}.(4\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = 5$

b Π_1 has equation $\mathbf{r}.(5\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = 16$ and Π_2 has equation $\mathbf{r}.(16\mathbf{i} - 5\mathbf{j} - 4\mathbf{k}) = 53$

c Π_1 has equation $\mathbf{r}.(\mathbf{i} - 3\mathbf{j} + \mathbf{k}) = 10$ and Π_2 has equation $\mathbf{r}.(4\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}) = 1$.

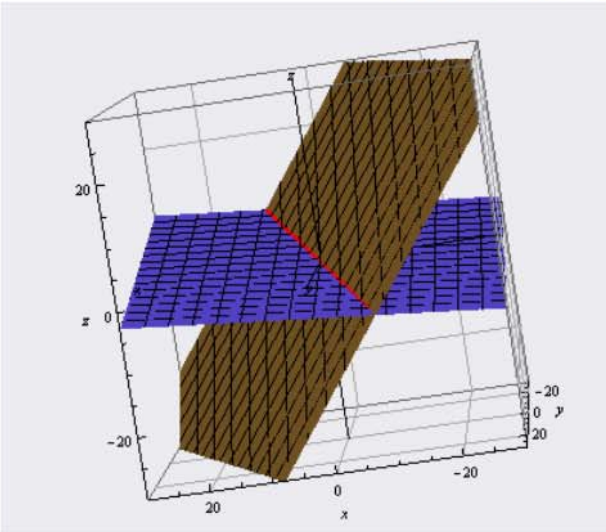
a $\mathbf{r} = (\frac{5}{2}\mathbf{i} + \frac{5}{2}\mathbf{k}) + \lambda(\frac{3}{2}\mathbf{i} + \mathbf{j} + \frac{5}{2}\mathbf{k})$

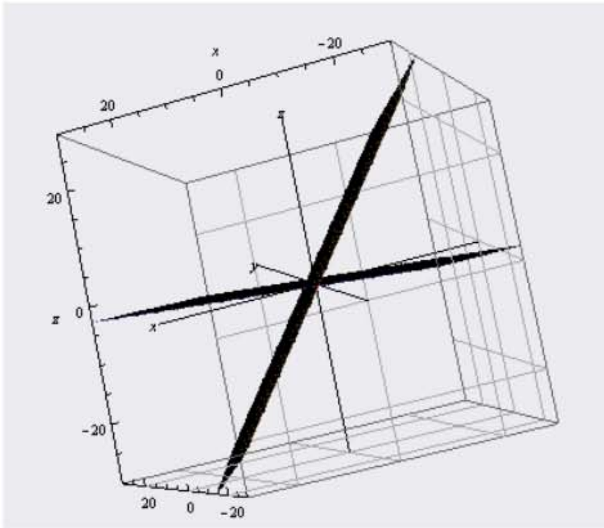
b $\mathbf{r} = (3\mathbf{i} - \mathbf{j}) + \lambda(\frac{2}{3}\mathbf{i} + \frac{4}{3}\mathbf{j} + \mathbf{k})$

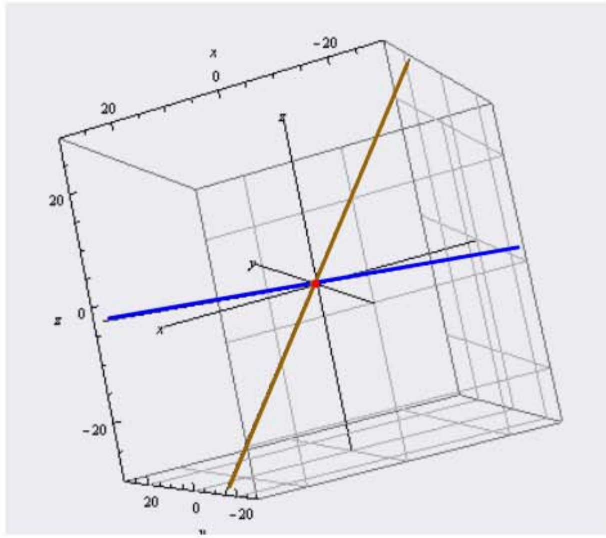
c $\mathbf{r} = (-3\mathbf{i} - \frac{13}{3}\mathbf{j}) + \lambda(\mathbf{i} + \frac{2}{3}\mathbf{j} + \mathbf{k})$

Angles between two intersecting planes

Different views/representations of two planes intersecting

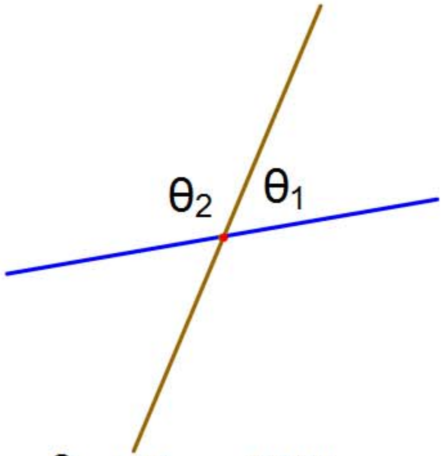






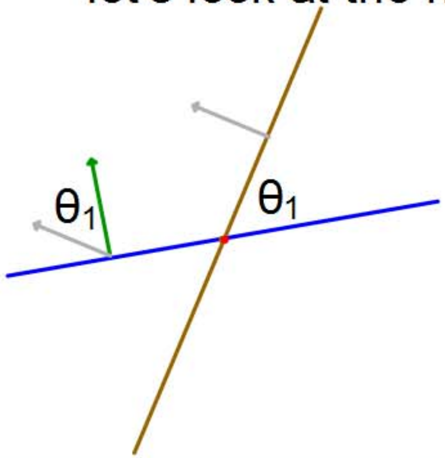
View in the direction of the line of intersection

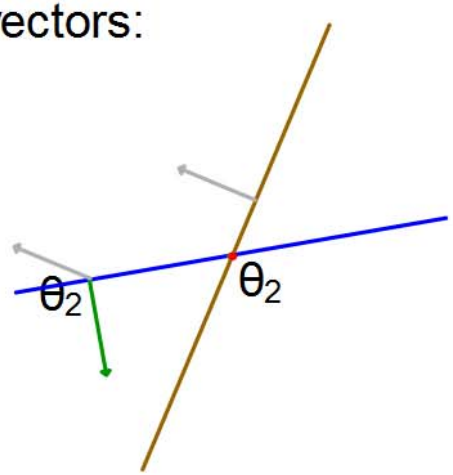
There are two angles to consider:
one acute and one obtuse



$\theta_2 + \theta_1 = 180^o$

To work out these angles,
let's look at the normal vectors:

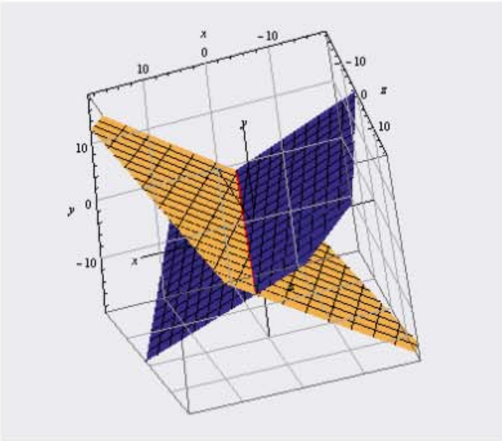


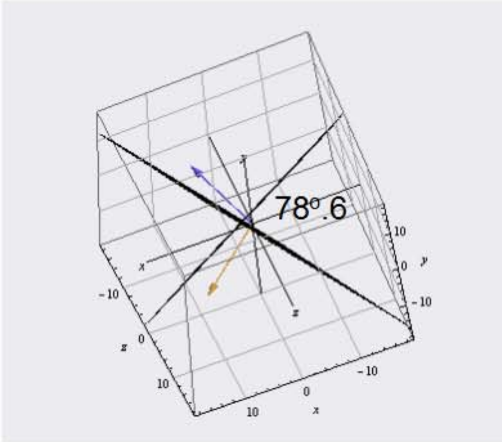


To work out an angle between two planes, work out the angle between the normal vectors.

Examples:

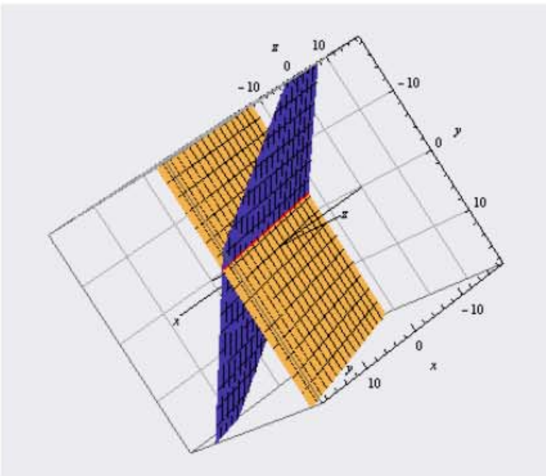
Find the acute angle between the planes with equations $\mathbf{r}.(4\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}) = 13$ and $\mathbf{r}.(7\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}) = 6$ respectively.

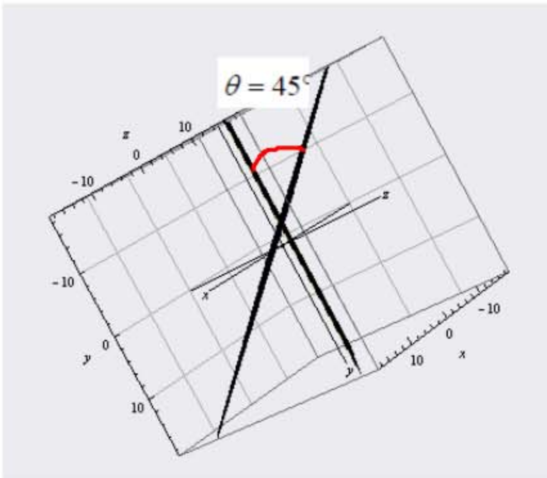




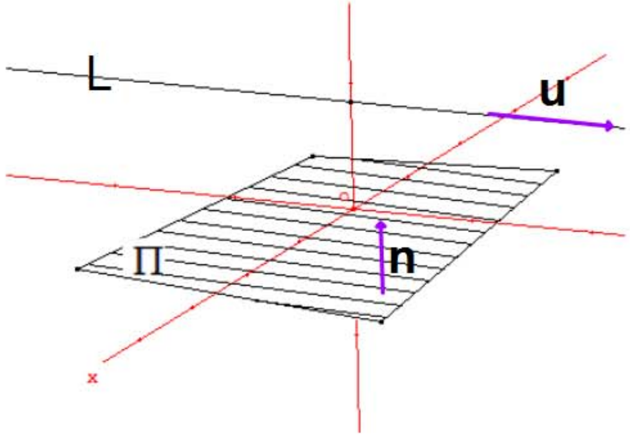
Find the acute angle between the planes

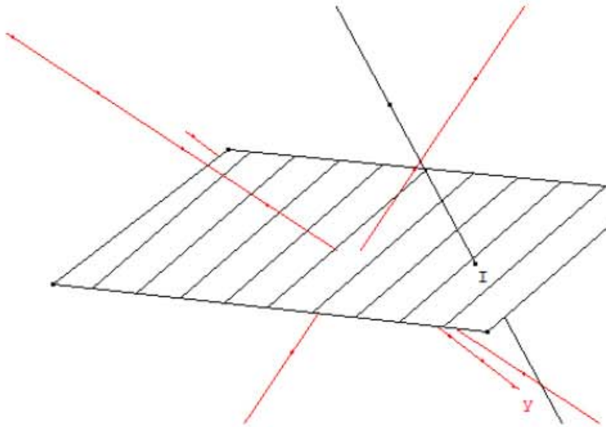
$$3x - 5y - 4z = 8 \qquad \text{and} \qquad 3x - 4z = -7.$$





Lines and planes





The plane Π has vector normal **n**

The line L has direction vector **u**

The line is parallel to the plane

⇔ **n** and **u** are perpendicular

⇔ **n**.**u** = 0

If a line is not parallel to a plane,
They intersect at a point:

- To find the coordinates of the point of intersection between a plane and a line, solve the equations simultaneously.

Exercises:

In each case establish whether the line *l* meets the plane *Π* and, if they meet, find the coordinates of their point of intersection.

a l: $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k})$

Π: $\mathbf{r}.(\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = 1$

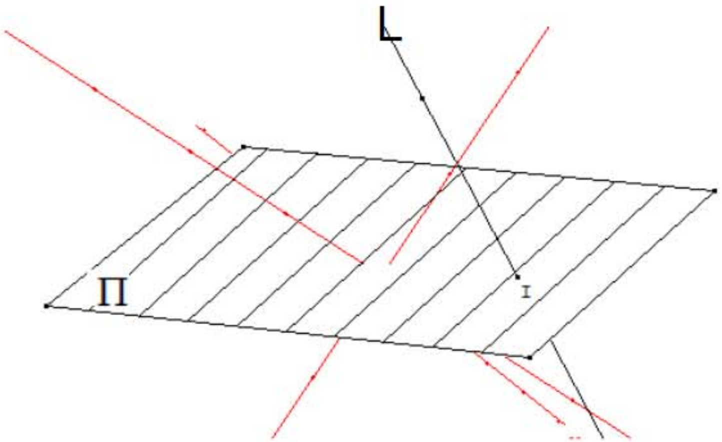
b l: $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(2\mathbf{j} - 2\mathbf{k})$

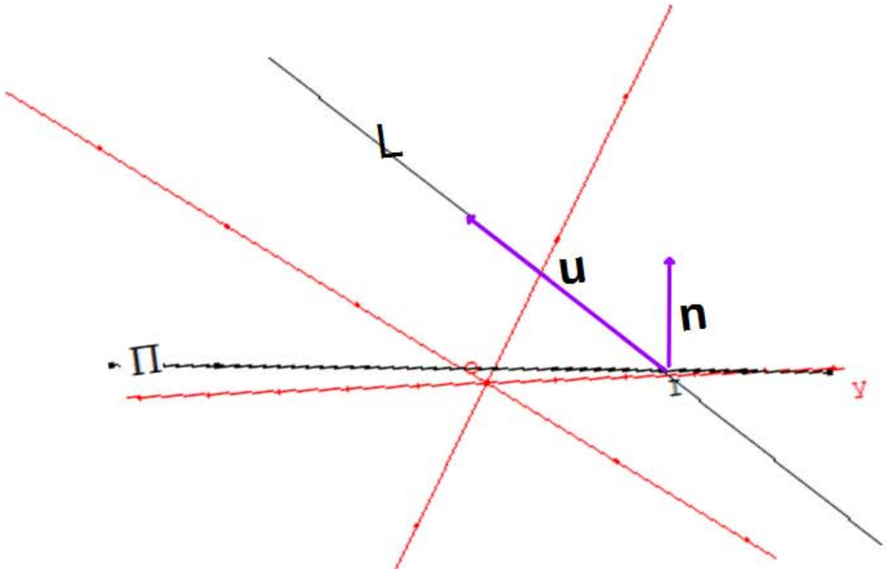
Π: $\mathbf{r}.(3\mathbf{i} - \mathbf{j} - 6\mathbf{k}) = 1$

b There are no values of λ for which the line meets the plane.
The line is parallel to the plane.

c (1, 2, 0)

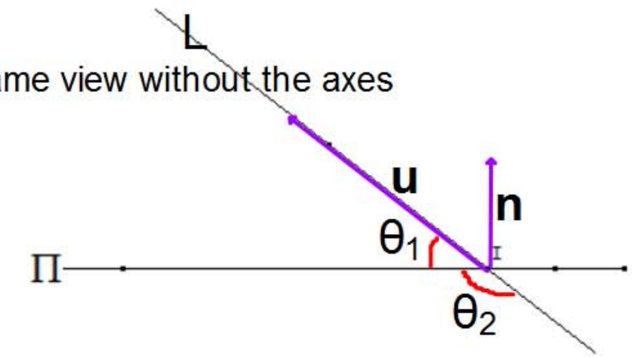
Angles between a line and a plane

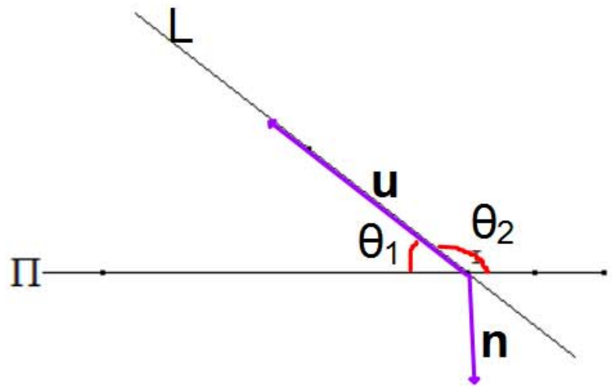




View parallel to the plane.

Same view without the axes





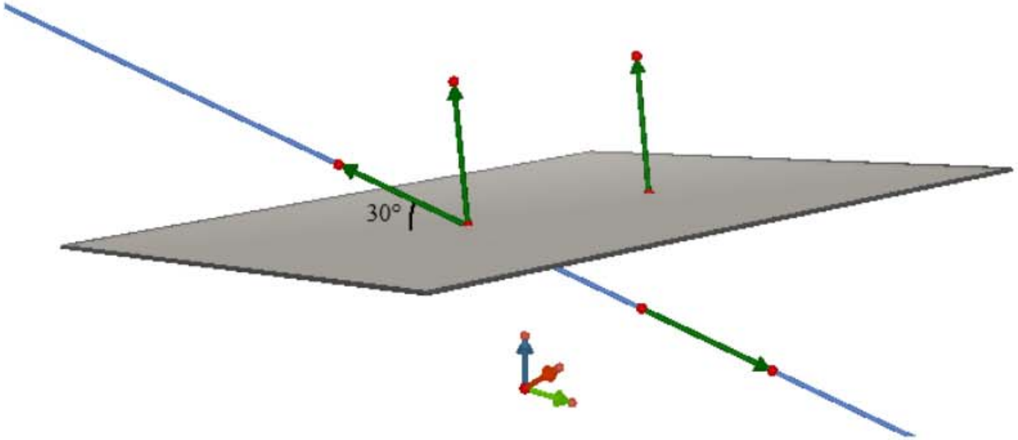
Two angles can be considered between a line and a plane one acute and one obtuse

If we call **α the ACUTE** angle between **u** and **n**, then **90º - α** is a measure of the angle between the line and the plane.

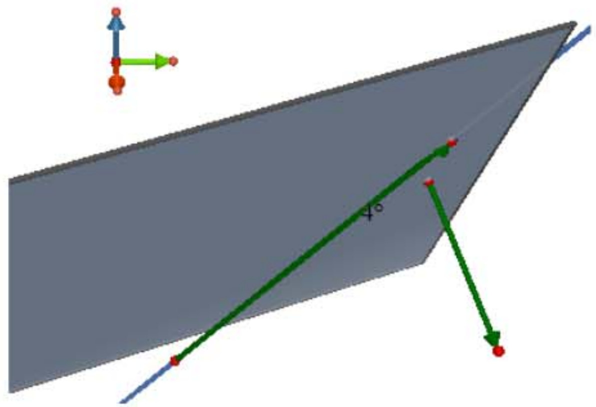
Examples:

- Find the angle between the line

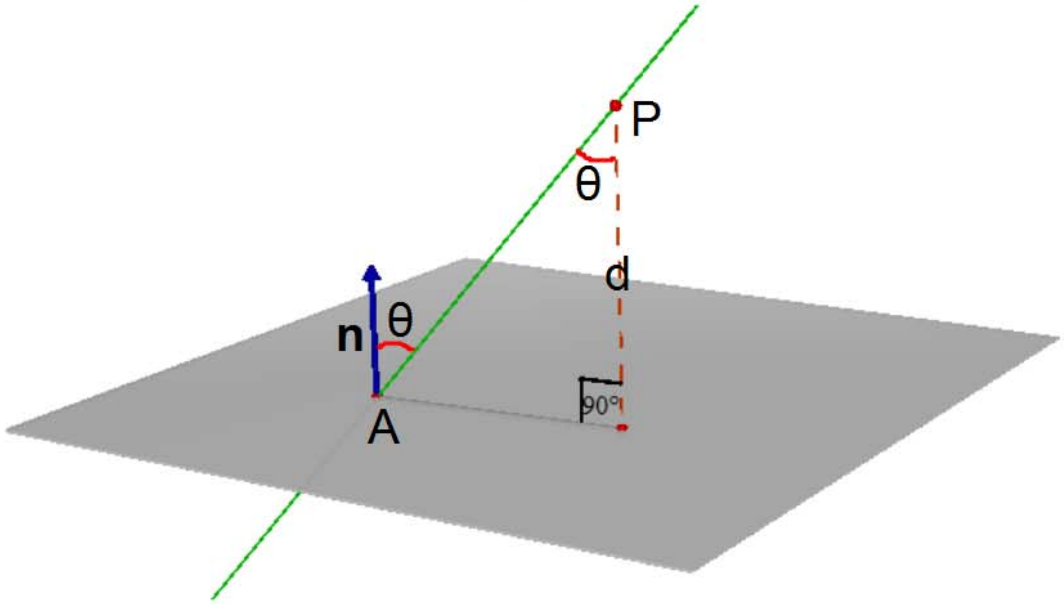
$$\mathbf{r} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \text{ and the plane } x - y + 2z = 7.$$



- Find the acute angle between the line with equation $\mathbf{r} = 2\mathbf{i} + \mathbf{j} - 5\mathbf{k} + \lambda(4\mathbf{i} + 4\mathbf{j} + 7\mathbf{k})$ and the plane with equation $\mathbf{r}.(2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = 13$.



Shortest distance to a plane



A plane Π is defined by a point A and a normal vector $\mathbf{n}$.
Consider a point P which does not belong to the plane.

The shortest distance from P the the plane Π is

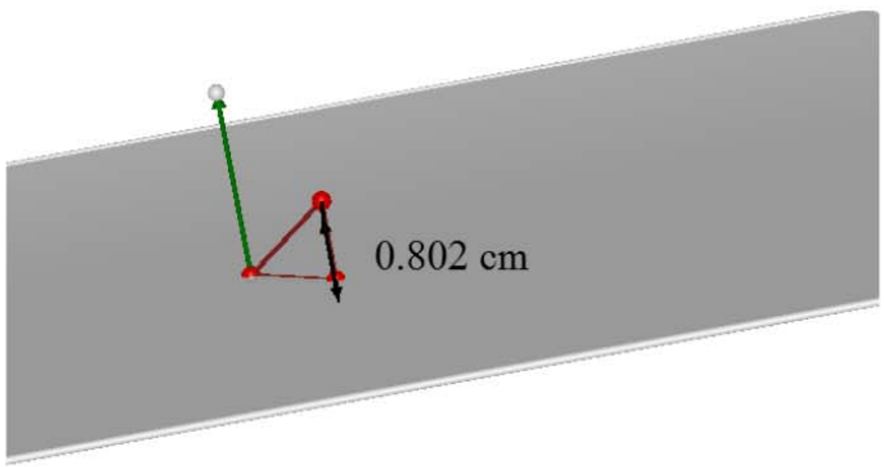
$$d = \mathrm{AP} \times |Cos\theta| \quad \text{where } \theta \text{ is the angle between } \overrightarrow{\mathrm{AP}} \text{ and } \mathbf{n}$$

Using the dot product:

$$\mathrm{Cos}\theta = \frac{\overrightarrow{AP}.\mathbf{n}}{|\overrightarrow{AP}|.|\mathbf{n}|} = \frac{\overrightarrow{AP}.\mathbf{n}}{AP.|\mathbf{n}|} \; so \; d = AP|Cos\theta| = AP\frac{|\overrightarrow{AP}.\mathbf{n}|}{AP.|\mathbf{n}|} =$$

$$d = \frac{|\overrightarrow{AP}.\mathbf{n}|}{|\mathbf{n}|}$$

Work out the distance between the point P(0, 1 2) and the plane Π with equation $2x + 3y - z = 4$.



Miscellaneous questions

1. Find the equation of the line joining (3, 1, 1) to (-2, 3, 5)
   (i) in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$
   (ii) in the form $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = 0$
   (iii) in the direction ratios form.

2. (i) Write the equation of the line $\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}$ in the direction ratios form.

   (ii) Find the direction cosines of this line, and give their geometrical significance.

3. Write the equation of the line $\dfrac{x-1}{1} = \dfrac{y+2}{4} = \dfrac{z-3}{2}$ in the form $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = 0$.

4. Find, in Cartesian form, the equation of the plane containing the point (2, -3, 1) and perpendicular to the vector $\begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix}$.

5. A plane contains the points A (3, 0, 2), B (1, -1, 1) and C (2, 3, -1). Find the equation of the plane
   (i) in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$
   (ii) in the form $\mathbf{r}.\mathbf{n} = d$

6. Show that the plane $\mathbf{r}.(3\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = 1$ contains the line
   $\mathbf{r} = 3\mathbf{i} + 3\mathbf{j} - 2\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} - \mathbf{k})$.

7. A plane has equation $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$.

   Write the equation of this plane in Cartesian form.

Solutions to Exercise

1. (i) Direction vector $= \begin{pmatrix}3\\1\\1\end{pmatrix} - \begin{pmatrix}-2\\3\\5\end{pmatrix} = \begin{pmatrix}5\\-2\\-4\end{pmatrix}$

Alternatively the negative of this vector could be used

Vector equation of line is $\underset{\sim}{r} = \begin{pmatrix}3\\1\\1\end{pmatrix} + \lambda\begin{pmatrix}5\\-2\\-4\end{pmatrix}$

The position vector of the point (-2, 3, 5) could be used instead of (3, 1, 1)

(ii) The vector product form of this line is $\left(\underset{\sim}{r} - \begin{pmatrix}3\\1\\1\end{pmatrix}\right) \times \begin{pmatrix}5\\-2\\-4\end{pmatrix} = 0$

(iii) Direction ratios form of this line is $\dfrac{x-3}{5} = \dfrac{y-1}{-2} = \dfrac{z-1}{-4}$

2. (i) $\dfrac{x-3}{-1} = \dfrac{y-1}{2} = \dfrac{z+3}{-2}$

(ii) $|\underset{\sim}{b}| = \sqrt{(-1)^2 + 2^2 + (-2)^2} = \sqrt{9} = 3$

Direction cosines are $-\frac{1}{3}, \frac{2}{3}$ and $-\frac{2}{3}$.

These are the cosines of the angles between the line and the x-axis, y-axis and z-axis respectively.

3. $\underset{\sim}{a} = \begin{pmatrix}1\\-2\\3\end{pmatrix}$, $\underset{\sim}{b} = \begin{pmatrix}1\\4\\2\end{pmatrix}$

In vector product form, equation of line is $\left(\underset{\sim}{r} - \begin{pmatrix}1\\-2\\3\end{pmatrix}\right) \times \begin{pmatrix}1\\4\\2\end{pmatrix} = 0$.

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7. The normal vector is perpendicular to vectors $\begin{pmatrix}2\\-1\\3\end{pmatrix}$ and $\begin{pmatrix}-1\\3\\2\end{pmatrix}$.

$$\begin{pmatrix}2\\-1\\3\end{pmatrix} \times \begin{pmatrix}-1\\3\\2\end{pmatrix} = \begin{pmatrix}(-1\times 2)-(3\times 3)\\(3\times -1)-(2\times 2)\\(2\times 3)-(-1\times -1)\end{pmatrix} = \begin{pmatrix}-11\\-7\\5\end{pmatrix}$$

The Cartesian equation has form $11x + 7y - 5z = d$

The point (1, 2, 0) lies on the plane (when $\lambda = \mu = 0$):

$11 \times 1 + 7 \times 2 - 5 \times 0 = d$

$11 + 14 + 0 = d$

$d = 25$

The equation of the plane is $11x + 7y - 5z = 25$

4. The normal vector is $\begin{pmatrix}5\\1\\-2\end{pmatrix}$, so the equation has the form $5x + y - 2z = d$.

The plane contains the point (2, -3, 1), so $5 \times 2 - 3 - 2 \times 1 = d$

$10 - 3 - 2 = d$

$d = 5$

The equation of the plane is $5x + y - 2z = 5$.

5. (i) The vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ lie in the plane.

$\overrightarrow{AB} = \begin{pmatrix}1\\-1\\1\end{pmatrix} - \begin{pmatrix}3\\0\\2\end{pmatrix} = \begin{pmatrix}-2\\-1\\-1\end{pmatrix}$ $\quad \overrightarrow{BC} = \begin{pmatrix}2\\3\\-1\end{pmatrix} - \begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}1\\4\\-2\end{pmatrix}$

The point A (3, 0, 2) lies in the plane, so an equation for the plane is

$\underset{\sim}{r} = \begin{pmatrix}3\\0\\2\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\1\end{pmatrix} + \mu\begin{pmatrix}1\\4\\-2\end{pmatrix}$

There are other correct answers – other vectors in the plane could be used for **b** and **c**, and any of the points could be used for **a**.

(ii) A normal vector to the plane is $\underset{\sim}{n} = \begin{pmatrix}2\\1\\1\end{pmatrix} \times \begin{pmatrix}1\\4\\-2\end{pmatrix} = \begin{pmatrix}(1\times -2)-(1\times 4)\\(1\times 1)-(2\times -2)\\(2\times 4)-(1\times 1)\end{pmatrix} = \begin{pmatrix}-6\\5\\7\end{pmatrix}$

The equation of the plane is $\underset{\sim}{r}.\underset{\sim}{n} = \underset{\sim}{a}.\underset{\sim}{n}$, where $\underset{\sim}{a}$ is the position vector of a point on the plane. Taking $\underset{\sim}{a} = \begin{pmatrix}3\\0\\2\end{pmatrix}$:

$$\underset{\sim}{r}.\begin{pmatrix}-6\\5\\7\end{pmatrix} = \begin{pmatrix}3\\0\\2\end{pmatrix}.\begin{pmatrix}-6\\5\\7\end{pmatrix} = -18 + 0 + 14 = -4$$

The equation of the plane is $\underset{\sim}{r}.\begin{pmatrix}-6\\5\\7\end{pmatrix} = -4$.

6. If the plane contains the line, then all points on the line satisfy the equation of the plane.

Substituting the equation of the line into the LHS of the equation of the plane:

$\left[3\underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k} + \lambda(\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k})\right].(3\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k})$

$= (3\underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k}).(3\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}) + \lambda(\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}).(3\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k})$

$= 9 - 6 - 2 + \lambda(3 - 2 - 1)$

$= 1$

So the equation of the plane is satisfied for all values of λ, and therefore the line lies within the plane.

Intersections and angles

1. (i) Find the acute angle between the planes $5x-3y+2z=1$ and $2x-3z=4$.
   (ii) Find the equation of the line of intersection of these two planes.

2. (i) Find the vector equation of the plane which contains the point (2, -6, 1) and is perpendicular to the vector $3\mathbf{i}+\mathbf{j}+3\mathbf{k}$.
   (ii) Write the equation in Cartesian form.
   (iii) Find the coordinates of the point where the plane and the line $\mathbf{r} = \mathbf{j} - 2\mathbf{k} + \lambda(2\mathbf{i} - 5\mathbf{j} + \mathbf{k})$ intersect.
   (iv) Find the acute angle between the line and the plane.

3. (i) Find the line of intersection of the planes $\mathbf{r}.(2\mathbf{i} - \mathbf{j} - 3\mathbf{k}) = 10$ and $\mathbf{r}.(\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) = -9$.
   (ii) Find the acute angle between these two planes.

4. (i) Find the position vector of the point of intersection of the line
   $\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$ and the plane $\mathbf{r}.\begin{pmatrix} 5 \\ -1 \\ 7 \end{pmatrix} = 9$
   (ii) Find the acute angle between the line and the plane.

5. A line passes through the points A (6, -5, 1) and B(3, 1, -8).
   (i) Find a vector equation for the line.
   (ii) Show that the line is perpendicular to the plane $\mathbf{r}.\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = -9$.
   (iii) Find the point of intersection of the line and the plane.
   (iv) Find the shortest distance from point A to the plane.

Solutions to Exercise

1)

The acute angle between the planes is 79.6° (3 s.f.)

The equation of the line of intersection is $\underset{\sim}{r} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 9 \\ 19 \\ 6 \end{pmatrix}$

2)

The equation of the plane is $\underset{\sim}{r} . \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix} = 3$

(ii) $3x + y + 3z = 3$

$\lambda = 2$

The point of intersection is (4, -9, 0)

Angle between line and plane $= 90^\circ - 80.4^\circ = 9.6^\circ$

3)

The equation of the line of intersection is $\underset{\sim}{r} = \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 11 \\ 1 \\ 7 \end{pmatrix}$.

The angle between the planes is 69.1°.

4)

Position vector of point of intersection is $\underset{\sim}{r} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 9 \\ -1 \end{pmatrix}$

Angle between line and plane $= 90^\circ - 85.8^\circ = 4.2^\circ$

5)

An equation for the line is $\underset{\sim}{r} = \begin{pmatrix} 6 \\ -5 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$

The direction vector of the line is parallel to the vector $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$, which is the vector perpendicular to the plane, so the line is perpendicular to the plane.

Coordinates of point of intersection are (4, -1, -5)

$$\text{Distance} = \sqrt{(6-4)^2 + (-5-(-1))^2 + (1-(-5))^2}$$
$$= \sqrt{2^2 + (-4)^2 + 6^2}$$
$$= \sqrt{56}$$