Vector product and scalar product – Exam questions

Question 1: Jan07 Q3

The points P, Q and R have position vectors $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ respectively relative to an origin O, where

$$\mathbf{p} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \ \mathbf{q} = \begin{bmatrix} -3 \\ 4 \\ 20 \end{bmatrix} \text{ and } \mathbf{r} = \begin{bmatrix} 9 \\ 2 \\ 4 \end{bmatrix}$$

(a) (i) Determine $\mathbf{p} \times \mathbf{q}$. *(2 marks)*

(ii) Find the area of triangle OPQ. *(3 marks)*

(b) Use the scalar triple product to show that $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ are linearly dependent, and interpret this result geometrically. *(3 marks)*

Question 2: Jun11 Q3

Given the vectors $\mathbf{p} = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$, $\mathbf{q} = \begin{bmatrix} 7 \\ -2 \\ 4 \end{bmatrix}$ and $\mathbf{r} = \begin{bmatrix} 2 \\ 3 \\ t \end{bmatrix}$, where t is a scalar parameter, determine the value of t in each of the following cases:

(a) $\mathbf{p} \times \mathbf{q}$ is parallel to $\mathbf{r}$; *(3 marks)*

(b) $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ are linearly dependent. *(3 marks)*

Question 3: Jun08

The vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are given by

$$\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}, \quad \mathbf{b} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} \quad \text{and} \quad \mathbf{c} = -2\mathbf{i} + t\mathbf{j} + 6\mathbf{k}$$

where t is a scalar constant.

(a) Determine, in terms of t where appropriate:

(i) $\mathbf{a} \times \mathbf{b}$; *(2 marks)*

(ii) $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$; *(2 marks)*

(iii) $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$. *(2 marks)*

(b) Find the value of t for which $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are linearly dependent. *(2 marks)*

(c) Find the value of t for which $\mathbf{c}$ is parallel to $\mathbf{a} \times \mathbf{b}$. *(2 marks)*

Question 4: Jan09 Q3

The points X, Y and Z have position vectors

$$\mathbf{x} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 5 \\ 7 \\ 4 \end{bmatrix} \quad \text{and} \quad \mathbf{z} = \begin{bmatrix} -8 \\ 1 \\ a \end{bmatrix}$$

respectively, relative to the origin O.

(a) Find:

(i) $\mathbf{x} \times \mathbf{y}$; *(2 marks)*

(ii) $(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{z}$. *(2 marks)*

(b) Using these results, or otherwise, find:

(i) the area of triangle OXY; *(2 marks)*

(ii) the value of a for which $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are linearly dependent. *(2 marks)*

Question 5: Jun10 Q1

The position vectors of the points P, Q and R are, respectively,

$$\mathbf{p} = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{r} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$$

(a) Show that $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ are linearly dependent. *(2 marks)*

(b) Determine the area of triangle PQR. *(4 marks)*

Question 6: Jan06 Q4

The vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are given by

$$\mathbf{a} = \mathbf{i} - \mathbf{j} - \mathbf{k}, \quad \mathbf{b} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k} \quad \text{and} \quad \mathbf{c} = 4\mathbf{i} - \mathbf{j} + 5\mathbf{k}$$

(a) (i) Evaluate $\begin{vmatrix} 1 & -1 & -1 \\ 2 & 3 & -1 \\ 4 & -1 & 5 \end{vmatrix}$. *(2 marks)*

(ii) Hence determine whether $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are linearly dependent or independent. *(1 mark)*

(b) (i) Evaluate $\mathbf{b} \cdot \mathbf{c}$. *(2 marks)*

(ii) Show that $\mathbf{b} \times \mathbf{c}$ can be expressed in the form $m\mathbf{a}$, where m is a scalar. *(2 marks)*

(iii) Use these results to describe the geometrical relationship between $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. *(1 mark)*

(c) The points A, B and C have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively relative to an origin O. The points O, A, B and C are four of the eight vertices of a cuboid. Determine the volume of this cuboid. *(2 marks)*

Question 7: Jan08 Q2

It is given that $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + \mathbf{j} - 5\mathbf{k}$ and $\mathbf{c} = \mathbf{i} + 4\mathbf{j} + 28\mathbf{k}$.

(a) Determine:

(i) $\mathbf{a} \cdot \mathbf{b}$; *(1 mark)*

(ii) $\mathbf{a} \times \mathbf{b}$; *(2 marks)*

(iii) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. *(2 marks)*

(b) Describe the geometrical relationship between the vectors:

(i) $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$; *(1 mark)*

(ii) $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. *(1 mark)*

Question 8: Jun07 Q1

Given that $\mathbf{a} \times \mathbf{b} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$ and that $\mathbf{a} \times \mathbf{c} = -\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, determine:

(a) $\mathbf{c} \times \mathbf{a}$; *(1 mark)*

(b) $\mathbf{a} \times (\mathbf{b} + \mathbf{c})$; *(2 marks)*

(c) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{c})$; *(2 marks)*

(d) $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{c})$. *(1 mark)*

Question 9: Jan12 Q1

The vectors $\mathbf{a}$ and $\mathbf{b}$ are such that $\mathbf{a} \cdot \mathbf{b} = 21$, $|\mathbf{a}| = 5\sqrt{2}$ and $|\mathbf{b}| = 3$.

Determine the exact value of $|\mathbf{a} \times \mathbf{b}|$. *(5 marks)*

Question 10: Jan12 Q8

For $n \neq 1$, the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are such that

$$\mathbf{a} = \begin{bmatrix} 1 \\ n \\ n^2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2n \\ 2n^2 + n \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} n-1 \\ n^2 - 1 \\ 1 - n^2 \end{bmatrix}$$

Determine the value of n for which $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are linearly dependent. *(9 marks)*

Vector product and scalar product - Answers

Question 1: Jan07 Q3

(a)(i)	$\mathbf{p}\times\mathbf{q}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&1&4\\-3&4&20\end{vmatrix}=\begin{bmatrix}4\\-32\\7\end{bmatrix}$	M1 A1	2
(ii)	$A=\frac{1}{2}\lvert\mathbf{p}\times\mathbf{q}\rvert$	M1	
	$=\frac{1}{2}\sqrt{4^2+32^2+7^2}$	B1	
	$=\frac{33}{2}$	A1F	3
(b)	$\mathbf{p}\times\mathbf{q}\,.\,\mathbf{r}=\begin{bmatrix}4\\-32\\7\end{bmatrix}.\begin{bmatrix}9\\2\\4\end{bmatrix}$ *or* $\begin{vmatrix}1&1&4\\-3&4&20\\9&2&4\end{vmatrix}$	M1	
	$=36-64+28=0$	A1	
	($\Rightarrow$ Lin Dep) O, P, Q, R **Or** $\mathbf{p}, \mathbf{q}, \mathbf{r}$ co-planar	B1	3
	Total		**8**

Question 2: Jun11 Q3

(a)	Vector product attempted	M1	
	$\mathbf{p}\times\mathbf{q}=\begin{bmatrix}1\\4\\7\end{bmatrix}\times\begin{bmatrix}7\\-2\\4\end{bmatrix}=\begin{bmatrix}30\\45\\-30\end{bmatrix}$	A1	
	$\ldots=15\begin{bmatrix}2\\3\\-2\end{bmatrix}$, so $t=-2$	A1	**3**
(b)	Scalar triple product attempted	M1	
	$\mathbf{p}\times\mathbf{q}\bullet\mathbf{r}=15\begin{bmatrix}2\\3\\-2\end{bmatrix}\bullet\begin{bmatrix}2\\3\\t\end{bmatrix}=15(13-2t)$	A1	
	$\ldots=0$, so $t=6\frac{1}{2}$	A1	**3**
	ALT: $5\mathbf{p}+\mathbf{q}=6\mathbf{r}$	B2,0	
	$\ldots\Rightarrow t=6\frac{1}{2}$	B1	
	Total		**6**

Question 3: Jun08 Q2

(a)(i)	$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&2&3\\2&1&2\end{vmatrix}=\begin{bmatrix}1\\4\\-3\end{bmatrix}$	M1 A1	2
(ii)	$(\mathbf{a}\times\mathbf{b})\bullet\mathbf{c}=\begin{bmatrix}1\\4\\-3\end{bmatrix}\bullet\begin{bmatrix}-2\\t\\6\end{bmatrix}=4t-20$	M1 A1	2
(iii)	$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&4&-3\\-2&t&6\end{vmatrix}=\begin{bmatrix}3t+24\\0\\t+8\end{bmatrix}$	M1 A1	2
(b)	$(\mathbf{a}\times\mathbf{b})\bullet\mathbf{c}=0\ \Rightarrow\ t=5$	M1A1	2
(c)	$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=0$ or $\mathbf{c}=$ mult. of $(\mathbf{a}\times\mathbf{b})$	M1	
	$\Rightarrow t=-8$	A1	2
	Total		**10**

Question 4: Jan09 Q3

(a)(i)	$\mathbf{x}\times\mathbf{y}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&3&2\\5&7&4\end{vmatrix}=\begin{bmatrix}-2\\2\\-1\end{bmatrix}$	M1 A1	2
(ii)	$(\mathbf{x}\times\mathbf{y})\bullet\mathbf{z}=\begin{vmatrix}2&3&2\\5&7&4\\-8&1&a\end{vmatrix}$	M1	
	$=18-a$	A1F	2
(b)(i)	$A=½\lvert\mathbf{x}\times\mathbf{y}\rvert$	M1	
	$=½\sqrt{2^2+2^2+1^2}=1.5$	A1F	2
(ii)	$(\mathbf{x}\times\mathbf{y})\bullet\mathbf{z}=0\ \Rightarrow\ a=18$	M1 A1F	2
	Total		**8**

Question 5: Jun10 Q1

(a)	$\begin{vmatrix}3&4&-1\\-1&2&2\\1&4&1\end{vmatrix}=6+8+4+2-24+4$	M1	
	or $3(2-8)-4(-1-2)-1(-4-2)$ etc or $3(2-8)+1(4+4)+1(8+2)$ etc Correctly shown $=0$	A1	
	Or $3\mathbf{p}+4\mathbf{q}=5\mathbf{r}$	(M1) (A1)	2
(b)	For attempt at 2 of $(\pm)\ \overline{PQ}, \overline{PR}, \overline{QR}$	M1	
	Area $\Delta PQR=\frac{1}{2}\lvert\overline{QP}\times\overline{QR}\rvert$ e.g.	M1	
	$=\frac{1}{2}\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\4&2&-3\\2&0&-2\end{vmatrix}=\frac{1}{2}\lvert\pm(4\mathbf{i}-2\mathbf{j}+4\mathbf{k})\rvert$		
	$=\frac{1}{2}\sqrt{4^2+2^2+4^2}$	M1	
	$=3$	A1	4
	Total		**6**

Question 6: Jan06 Q4

(a)(i)	$\det\mathbf{M}=15+2+4+12-1+10=42$	M1 A1	2
(ii)	Since answer is non-zero, lin. Indt.	B1	1
(b)(i)	$\mathbf{b}\,.\,\mathbf{c}=8-3-5=0$	M1 A1	2
(ii)	$\mathbf{b}\times\mathbf{c}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&3&-1\\4&-1&5\end{vmatrix}=\begin{bmatrix}14\\-14\\-14\end{bmatrix}=14\mathbf{a}$	M1 A1	2
(iii)	$\mathbf{a}, \mathbf{b}, \mathbf{c}$ (mutually) perpendicular	B1	1
(c)	$V=\lvert\det\mathbf{M}\rvert=42$	M1 A1	
	Or		
	$V=\mathbf{a}\,.\,\mathbf{b}\times\mathbf{c}=\begin{bmatrix}14\\-14\\-14\end{bmatrix}.\begin{bmatrix}1\\-1\\-1\end{bmatrix}=42$	M1 A1	
	Or $V=OA.OB.OC=\sqrt{3}\,.\sqrt{14}\,.\sqrt{42}=42$	M1 A1√	2
	Total		**10**

Question 7: Jan08 Q2

(a)(i)	$\mathbf{a}\,.\,\mathbf{b}=0$	B1	1
(ii)	$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&3&1\\1&1&-5\end{vmatrix}=\begin{bmatrix}-16\\11\\-1\end{bmatrix}$	M1 A1	2
(iii)	$\mathbf{a}\,.\,(\mathbf{b}\times\mathbf{c})=\begin{vmatrix}2&3&1\\1&1&-5\\1&4&28\end{vmatrix}=0$	M1 A1	2
(b)(i)	$\mathbf{a},\ \mathbf{b},\ \mathbf{a}\times\mathbf{b}$ mutually perpendicular	B1	1
(ii)	$\mathbf{a},\ \mathbf{b},\ \mathbf{c}$ co-planar	B1	1
	Total		7

Question 8: Jun07 Q1

(a)	$\mathbf{c}\times\mathbf{a}=-\mathbf{a}\times\mathbf{c}=\mathbf{i}+2\mathbf{j}-\mathbf{k}$	B1	1
(b)	$\mathbf{a}\times(\mathbf{b}+\mathbf{c})=(\mathbf{a}\times\mathbf{b})+(\mathbf{a}\times\mathbf{c})$ $=2\mathbf{i}-\mathbf{j}+2\mathbf{k}$	M1 A1	2
(c)	$(\mathbf{a}\times\mathbf{b})\,.\,(\mathbf{a}\times\mathbf{c})=\begin{pmatrix}3\\1\\1\end{pmatrix}\bullet\begin{bmatrix}-1\\-2\\1\end{bmatrix}=-4$	M1 A1	2
(d)	$\mathbf{a}\,.\,(\mathbf{a}\times\mathbf{c})=0$ (since $\mathbf{a}\times\mathbf{c}$ perp^r to $\mathbf{a}$)	B1	1
			6

Question 9: Jan12 Q1

Use of $ab\cos\theta=\mathbf{a}\,.\,\mathbf{b}=21$ $\Rightarrow\ \cos\theta=\dfrac{7}{5\sqrt{2}}$	M1 A1	
$\Rightarrow\ \sin\theta=\dfrac{1}{5\sqrt{2}}$	B1 ft	
Use of $\lvert\,\mathbf{a}\times\mathbf{b}\,\rvert=ab\sin\theta=3$	M1 A1	5
Total		5

Question 10: Jan12 Q8

For considering $\begin{vmatrix}1&2n&n-1\\n&2n^2+n&n^2-1\\n^2&-1&1-n^2\end{vmatrix}$	B1	
$=(n-1)\begin{vmatrix}1&2n&1\\n&2n^2+n&n+1\\n^2&-1&-1-n\end{vmatrix}$	M1A1	
$=(n-1)\begin{vmatrix}1&2n&1\\n&2n^2+n&n+1\\n(n+1)&(n+1)(2n-1)&0\end{vmatrix}$ $R_3{}'=R_3+R_2$	M1	
$=(n-1)(n+1)\begin{vmatrix}1&2n&1\\n&2n^2+n&n+1\\n&2n-1&0\end{vmatrix}$	M1A1	
$=(n-1)(n+1)$ $\{2n^3+2n^2+2n^2-n-2n^3-n^2-2n^2-n+1\}$		
OR $=(n-1)(n+1)\begin{vmatrix}1&2n&1\\0&n&1\\n&2n-1&0\end{vmatrix}$ $R_2{}'=R_2-nR_1\ =$		
$(n-1)(n+1)\{2n^2-n^2-2n+1\}$	M1	
$=(n-1)(n+1)(n-1)^2$	A1	
$n=-1$	B1	9
Total		9