Calculus of inverse trig functions

Specifications:

The calculus of inverse trigonometrical functions

Use of the derivatives of $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$ as given in the formulae booklet.
To include the use of the standard integrals.

$\int \frac{1}{a^2+x^2}\,dx$; $\int \frac{1}{\sqrt{a^2-x^2}}\,dx$ given in the formulae booklet

In the formulae book:

Differentiation

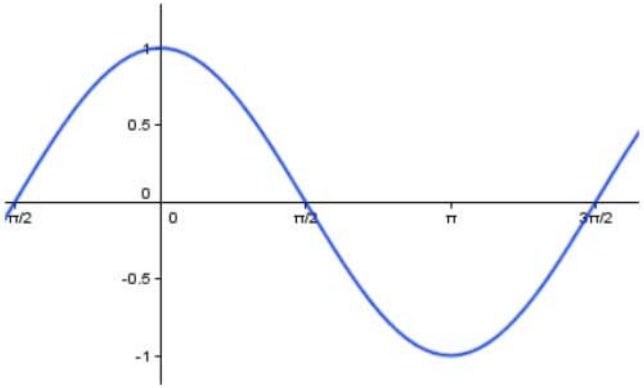
$\mathbf{f}(x)$	$\mathbf{f}'(x)$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$

Integration

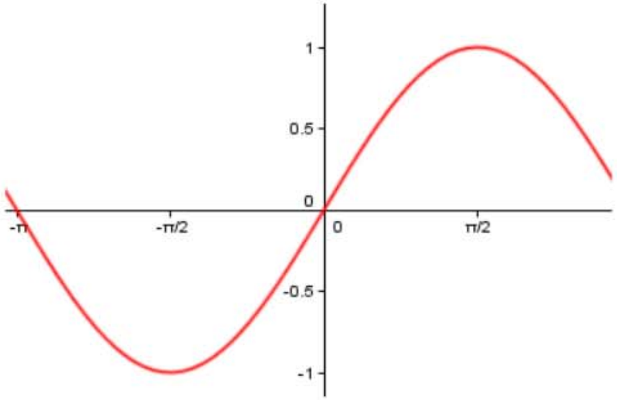
(+ constant; $a > 0$ where relevant)

$\mathbf{f}(x)$	$\int \mathbf{f}(x)\,\mathrm{d}x$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right) \quad (\lvert x\rvert < a)$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

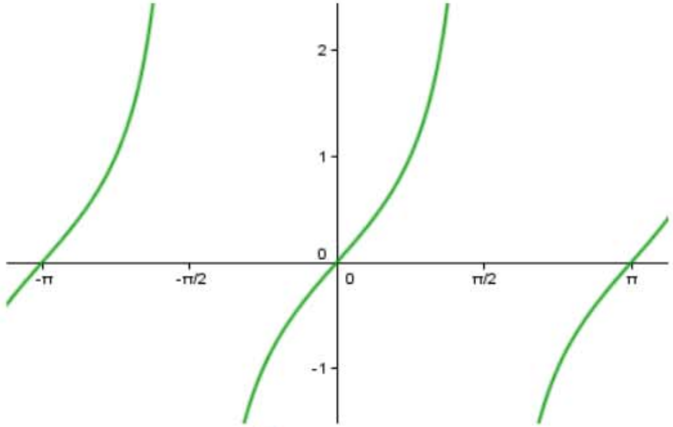
Reminder and introduction



$$Cos: \mathbb{R} \to [-1,1]$$
$$x \to Cos(x)$$



$$Sin: \mathbb{R} \to [-1,1]$$
$$x \to Sin(x)$$



$$Tan: \mathbb{R} - \{\frac{\pi}{2} + k\pi, k \in \mathbb{Z}\} \to \mathbb{R}$$
$$x \to Tan(x) = \frac{Sin(x)}{Cos(x)}$$

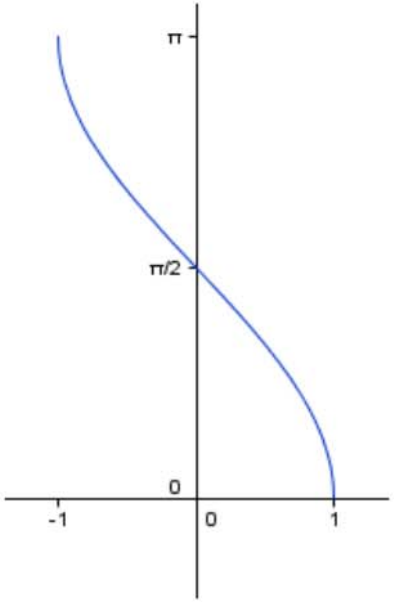
Identities

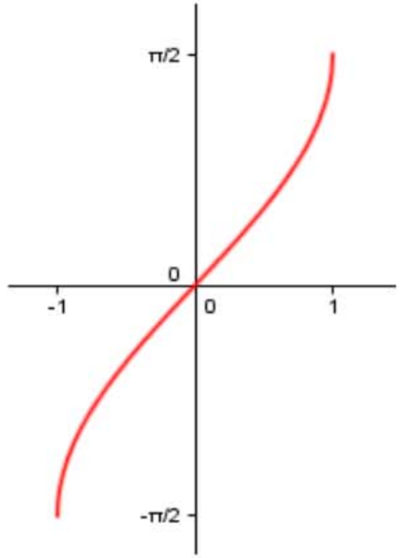
- $\mathrm{Cos}^2(x) + Sin^2(x) \equiv 1$
- $Tan(x) = \dfrac{Sin(x)}{Cos(x)}$

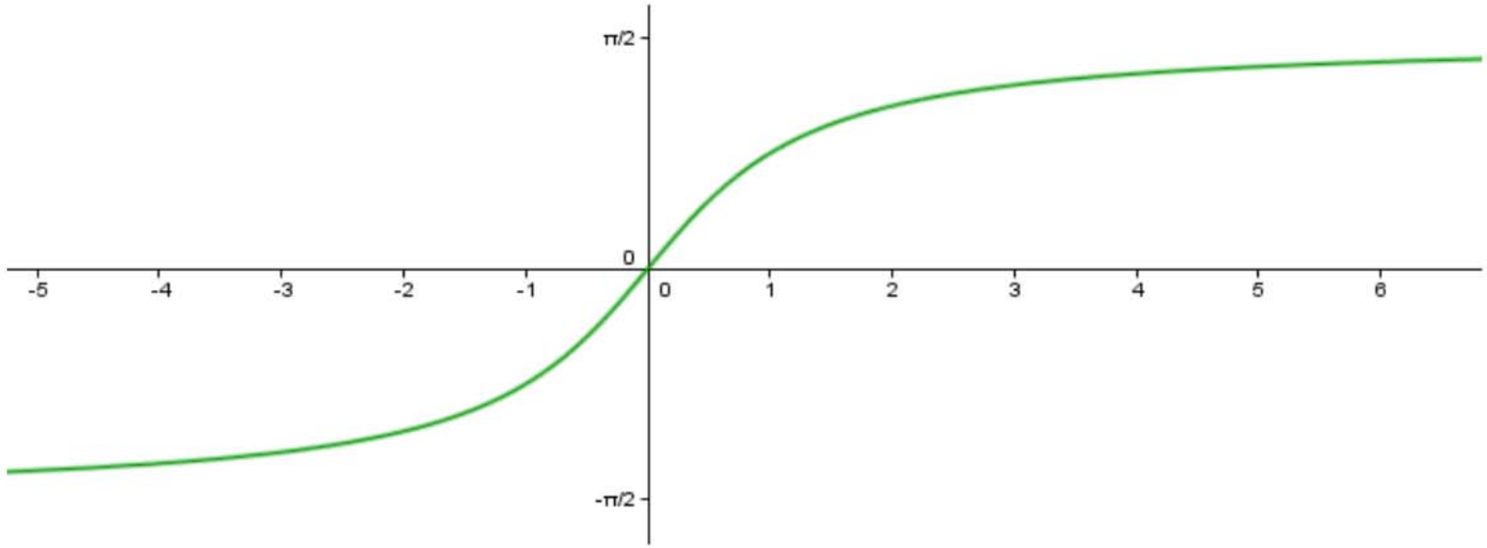
Calculus

- $\dfrac{d}{dx} Cos(x) = -Sin(x)$
- $\dfrac{d}{dx} Sin(x) = Cos(x)$
- $\dfrac{d}{dx} Tan(x) = 1 + Tan^2(x) = Sec^2(x) = \dfrac{1}{Cos^2(x)}$

Graph of Cos^{-1}, Sin^{-1}, Tan^{-1}







Differentiation

The derivative of y = sin^{-1}(x)

$$y = Sin^{-1}(x) \;\; means \;\; Sin(y) = x$$

$$\frac{dx}{dy} = Cos(y) = \sqrt{1 - Sin^2(y)} = \sqrt{1-x^2}$$

$$(Explanation: \text{for all } y, Cos^2(y) + Sin^2(y) = 1$$

$$so\, Cos(y) = \sqrt{1 - Sin^2(y)} \;)$$

$$hence: \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

The derivative of y = cos^{-1}(x)

$$y = Cos^{-1}(x) \;\; means \;\; Cos(y) = x$$

$$\frac{dx}{dy} = -Sin(y) = -\sqrt{1 - Cos^2(y)} = -\sqrt{1-x^2}$$

$$(Explanation: \text{for all } y, Cos^2(y) + Sin^2(y) = 1$$

$$so\, Sin(y) = \sqrt{1 - Cos^2(y)} \;)$$

$$hence: \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

The derivative of y = tan^{-1}(x)

$$y = \tan^{-1}(x)\, means \;\; \tan(y) = x$$

$$\frac{dx}{dy} = 1 + \tan^2(y) = 1 + x^2$$

$$so\, \frac{dy}{dx} = \frac{1}{1+x^2}$$

Summary

- $\dfrac{d}{dx}Sin^{-1}(x)=\dfrac{1}{\sqrt{1-x^2}}$
- $\dfrac{d}{dx}Sin^{-1}(ax+b)=\dfrac{a}{\sqrt{1-(ax+b)^2}}$
- $\dfrac{d}{dx}Cos^{-1}(x)=-\dfrac{1}{\sqrt{1-x^2}}$
- $\dfrac{d}{dx}Cos^{-1}(ax+b)=-\dfrac{a}{\sqrt{1-(ax+b)^2}}$
- $\dfrac{d}{dx}Tan^{-1}(x)=\dfrac{1}{1+x^2}$
- $\dfrac{d}{dx}Tan^{-1}(ax+b)=\dfrac{1}{1+(ax+b)^2}$

Exercise:

Differentiate the following:

1. (a) $\tan^{-1}3x$ (b) $\cos^{-1}(3x-1)$ (c) $\sin^{-1}2x$

2. (a) $x\tan^{-1}x$ (b) $e^x\cos^{-1}2x$ (c) $x^2\sin^{-1}(2x-3)$

3. (a) $\dfrac{\sin^{-1}3x}{x^3}$ (b) $\dfrac{\tan^{-1}(3x^2+1)}{1+x^2}$

4. (a) $\sin^{-1}(ax+b)$ (b) $\tan^{-1}(ax+b)$ where a and b are positive numbers.

1. (a) $\dfrac{3}{1+9x^2}$ (b) $\dfrac{-3}{\sqrt{6x-9x^2}}$ (c) $\dfrac{2}{\sqrt{1-4x^2}}$

2. (a) $\dfrac{x}{1+x^2}+\tan^{-1}x$ (b) $e^x\cos^{-1}2x-\dfrac{2e^x}{\sqrt{1-4x^2}}$ (c) $2x\sin^{-1}(2x-3)+\dfrac{2x^2}{\sqrt{-8+12x-4x^2}}$

3. (a) $\dfrac{3}{x^3\sqrt{1-9x^2}}-\dfrac{3\sin^{-1}3x}{x^4}$ (b) $\dfrac{6x}{(1+x^2)\left(1+(3x^2+1)^2\right)}-\dfrac{2x\tan^{-1}(3x^2+1)}{(1+x^2)^2}$

4. (a) $\dfrac{a}{\sqrt{1-(ax+b)^2}}$ (b) $\dfrac{a}{1+(ax+b)^2}$

Integration

Let's prove that $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}Tan^{-1}\left(\frac{x}{a}\right)+c$

Integrate $\int \frac{1}{a^2+x^2}\,dx$ using *the substitution* $x = a\tan(u)$

Prove that $\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = Sin^{-1}\left(\frac{x}{a}\right) + c$

Integrate $\int \frac{1}{\sqrt{a^2 - x^2}}\,dx$ using *the substitution* $x = a\sin(u)$

Summary:

- $\int \frac{dx}{\sqrt{1-x^2}} = Sin^{-1}(x) + c$
- $\int \frac{dx}{\sqrt{a^2-x^2}} = Sin^{-1}\left(\frac{x}{a}\right) + c$
- $\int \frac{dx}{1+x^2} = Tan^{-1}(x) + c$
- $\int \frac{dx}{a^2+x^2} = \frac{1}{a}Tan^{-1}\left(\frac{x}{a}\right) + c$

Application:

a) Evaluate $\int_0^2 \frac{dx}{4+x^2}$

b) Evaluate $\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9-x^2}}$

a) $\frac{\pi}{8}$ b) $\frac{\pi}{6}$

Exercises:

Integrate the following, leaving your answers in terms of π.

1. $\int_{1}^{\sqrt{3}} \frac{2\,dx}{1+x^2}$

2. $\int_{\frac{1}{2}}^{1} \frac{3\,dx}{\sqrt{1-x^2}}$

3. $\int_{-3}^{4} \frac{dx}{\sqrt{25-x^2}}$

4. $\int_{0}^{1} \frac{dx}{1+x^2}$

5. $\int_{-\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{dx}{x^2+1}$

1) $\frac{\pi}{6}$ 2) π 3) $\frac{\pi}{2}$ 4) $\frac{\pi}{4}$ 5) $\frac{\pi}{2}$

More practice:

4 $\int \frac{dx}{9+x^2}$	12 $\int \frac{dx}{x^2+16}$	16 $\int \frac{dx}{4x^2+9}$	24 $\int \frac{dx}{\sqrt{5-x^2}}$
7 $\int \frac{dx}{\sqrt{16-x^2}}$	13 $\int \frac{dx}{\sqrt{25-9x^2}}$	19 $\int \frac{dx}{9x^2+4}$	25 $\int \frac{dx}{\sqrt{5-4x^2}}$

Q16 Q25) Let 2x be u
Q13 Q19) Let 3x be u

4 $\frac{1}{3}\tan^{-1}\frac{x}{3}+C$	12 $\frac{1}{4}\tan^{-1}\frac{x}{4}+C$	16 $\frac{1}{6}\tan^{-1}\frac{2x}{3}+C$	24 $\sin^{-1}\frac{x}{\sqrt{5}}+C$
7 $\sin^{-1}\frac{x}{4}+C$	13 $\frac{1}{3}\sin^{-1}\frac{3x}{5}+C$	19 $\frac{1}{6}\tan^{-1}\frac{3x}{2}+C$	25 $\frac{1}{2}\sin^{-1}\frac{2x}{\sqrt{5}}+C$

Integrating using the completed square form

To integrate functions of the kind $\frac{1}{ax^2+bx+c}$ or $\frac{1}{\sqrt{ax^2+bx+c}}$,

write ax^2+bx+c in its completed square form $a(x+p)^2+q$.

Then use an appropriate substitution (i.e $u=\sqrt{a}(x+p)$) in order to obtain

an expression of the form u^2+p...

<ins>*Example*</ins>: $\int \frac{1}{x^2+2x+10}\,dx$

$$I=\int \frac{1}{x^2+2x+10}\,dx=\int \frac{1}{(x+1)^2+9}\,dx$$

Let $u=x+1$, *then* $du=dx$ *and* I *becomes*

$$I=\int \frac{1}{u^2+9}\,du=\frac{1}{3}\tan^{-1}\left(\frac{u}{3}\right)+c=\frac{1}{3}\tan^{-1}\left(\frac{x+1}{3}\right)+c$$

<ins>*Example*</ins>: $\int \frac{1}{\sqrt{12-4x-x^2}}\,dx$

$$I=\int \frac{1}{\sqrt{12-4x-x^2}}\,dx=\int \frac{1}{\sqrt{16-(x+2)^2}}\,dx$$

Let $u=x+2$, then $du=dx$ and I becomes

$$I=\int \frac{1}{\sqrt{16-u^2}}\,du=Sin^{-1}\left(\frac{u}{4}\right)+c=Sin^{-1}\left(\frac{x+2}{4}\right)+c$$

Exercises:

1. Integrate

(a) $\dfrac{1}{x^2+4x+5}$ (b) $\dfrac{1}{2x^2-4x+5}$ (c) $\dfrac{1}{x^2-x+2}$

2. Integrate

(a) $\dfrac{2x}{x^2+2x+3}$ (b) $\dfrac{x}{x^2+x+1}$

3. Find

(a) $\displaystyle\int \frac{dx}{\sqrt{7+6x-x^2}}$ (b) $\displaystyle\int \frac{dx}{\sqrt{3+2x-x^2}}$ (c) $\displaystyle\int \frac{dx}{\sqrt{x(1-2x)}}$

4. Find

(a) $\displaystyle\int \frac{x+1}{\sqrt{1-x^2}}\,dx$ (b) $\displaystyle\int \frac{3x-2}{\sqrt{3+2x-x^2}}\,dx$ (c) $\displaystyle\int \frac{(1-x)}{\sqrt{1-x-x^2}}\,dx$

1. (a) $\tan^{-1}(x+2)$ (b) $\frac{1}{\sqrt{6}}\tan^{-1}\sqrt{\frac{2}{3}}(x-1)$ (c) $\frac{2}{\sqrt{7}}\tan^{-1}\frac{2x-1}{\sqrt{7}}$

2. (a) $\ln\left(x^2+2x+3\right)-\sqrt{2}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)$

(b) $\frac{1}{2}\ln\left(x^2+x+1\right)-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$

3. (a) $\sin^{-1}\left(\frac{x-3}{4}\right)$ (b) $\sin^{-1}\left(\frac{x-1}{2}\right)$ (c) $\frac{1}{\sqrt{2}}\sin^{-1}(4x-1)$

4. (a) $\sin^{-1}x-\sqrt{1-x^2}$

(b) $-3\sqrt{3+2x-x^2}+\sin^{-1}\left(\frac{x-1}{2}\right)$

(c) $\sqrt{1-x-x^2}+\frac{3}{2}\sin^{-1}\left(\frac{2x+1}{\sqrt{5}}\right)$