Calculus of inverse trig functions – exam questions

Question 1: June 2008

By using the substitution $u = x - 2$, or otherwise, find the exact value of

$$\int_{-1}^{5} \frac{\mathrm{d}x}{\sqrt{32 + 4x - x^2}}$$ *(5 marks)*

Question 2: Jan 2010

(a) Show that the substitution $t = \tan\theta$ transforms the integral

$$\int \frac{\mathrm{d}\theta}{9\cos^2\theta + \sin^2\theta}$$

into $$\int \frac{\mathrm{d}t}{9 + t^2}$$ *(3 marks)*

(b) Hence show that

$$\int_0^{\frac{\pi}{3}} \frac{\mathrm{d}\theta}{9\cos^2\theta + \sin^2\theta} = \frac{\pi}{18}$$ *(3 marks)*

Question 3: June 2007

(a) Differentiate $x\tan^{-1}x$ with respect to x. *(2 marks)*

(b) Show that

$$\int_0^1 \tan^{-1}x \,\mathrm{d}x = \frac{\pi}{4} - \ln\sqrt{2}$$ *(5 marks)*

Calculus of inverse trig functions – exam questions - answers

Question 1: June 2008

$$I=\int_{-1}^{5}\frac{dx}{\sqrt{32+4x-x^2}}$$

$$32+4x-x^2=-(x-2)^2+4+32$$
$$=36-(x-2)^2$$

$$I=\int_{-1}^{5}\frac{dx}{\sqrt{32+4x-x^2}}=\int_{-1}^{5}\frac{dx}{\sqrt{36-(x-2)^2}}\qquad u=x-2\ \ and\ \ dx=du$$

$$when\ x=-1, u=-3$$
$$when\ x=5, u=3$$

$$I=\int_{-3}^{3}\frac{du}{\sqrt{36-u^2}}=\left[Sin^{-1}\left(\frac{u}{6}\right)\right]_{-3}^{3}=Sin^{-1}\left(\frac{1}{2}\right)-Sin^{-1}\left(-\frac{1}{2}\right)=\frac{\pi}{6}+\frac{\pi}{6}=\frac{\pi}{3}$$

Question 2: Jan 2010

$$a)\ t=\tan\theta\qquad so\ \frac{dt}{d\theta}=1+\tan^2\theta=1+t^2\qquad \frac{dt}{1+t^2}=d\theta$$

$$\bullet\cos^2\theta=\frac{1}{1+\tan^2\theta}=\frac{1}{1+t^2}$$

$$\bullet\sin^2\theta=1-\cos^2\theta=1-\frac{1}{1+t^2}=\frac{t^2}{1+t^2}$$

$$\int\frac{d\theta}{9\cos^2\theta+\sin^2\theta}=\int\frac{1}{\frac{9}{1+t^2}+\frac{t^2}{1+t^2}}\times\frac{dt}{1+t^2}=\int\frac{dt}{9+t^2}$$

$$b)\ when\ \theta=0, t=\tan 0=0\quad and\quad when\ \ \theta=\frac{\pi}{3}, t=\tan\frac{\pi}{3}=\sqrt{3}$$

$$I=\int_{0}^{\frac{\pi}{3}}\frac{d\theta}{9\cos^2\theta+\sin^2\theta}=\int_{0}^{\sqrt{3}}\frac{dt}{9+t^2}=\left[\frac{1}{3}\tan^{-1}\left(\frac{t}{3}\right)\right]_{0}^{\sqrt{3}}$$

$$I=\frac{1}{3}\times\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)-\frac{1}{3}\tan^{-1}(0)=\frac{1}{3}\times\frac{\pi}{6}=\frac{\pi}{18}$$

Question 3: June 2007

$$a)\ y=xTan^{-1}x\qquad \frac{dy}{dx}=1\times Tan^{-1}x+x\times\frac{1}{1+x^2}$$

$$\frac{dy}{dx}=Tan^{-1}x+\frac{x}{1+x^2}$$

$$b)\int_{0}^{1}Tan^{-1}x\,dx=\int_{0}^{1}(Tan^{-1}x+\frac{x}{1+x^2})-\frac{x}{1+x^2}dx$$

$$=\left[xTan^{-1}x\right]_{0}^{1}-\int_{0}^{1}\frac{x}{1+x^2}dx$$

$$=Tan^{-1}1-\left[\frac{1}{2}\ln(1+x^2)\right]_{0}^{1}$$

$$=\frac{\pi}{4}-\frac{1}{2}\ln 2=\frac{\pi}{4}-\ln\sqrt{2}$$