Matrix transformations

Matrix Algebra

Matrix transformations in two dimensions: shears.	Candidates will be expected to recognise the matrix for a shear parallel to the x or y axis. Where the line of invariant points is not the x or y axis candidates will be informed that the matrix represents a shear. The combination of a shear with a matrix transformation from MFP1 is included.
Rotations, reflections and enlargements in three dimensions, and combinations of these.	Rotations about the coordinate axes only. Reflections in the planes $x = 0$, $y = 0$, $z = 0$, $x = y$, $x = z$, $y = z$ only.

Pre-requisite: Further pure 1 - matrix transformations

Transformations of points in the $x - y$ plane represented by 2×2 matrices.	Transformations will be restricted to rotations about the origin, reflections in a line through the origin, stretches parallel to the $x-$ and $y-$axes, and enlargements with centre the origin. Use of the standard transformation matrices given in the formulae booklet. Combinations of these transformations

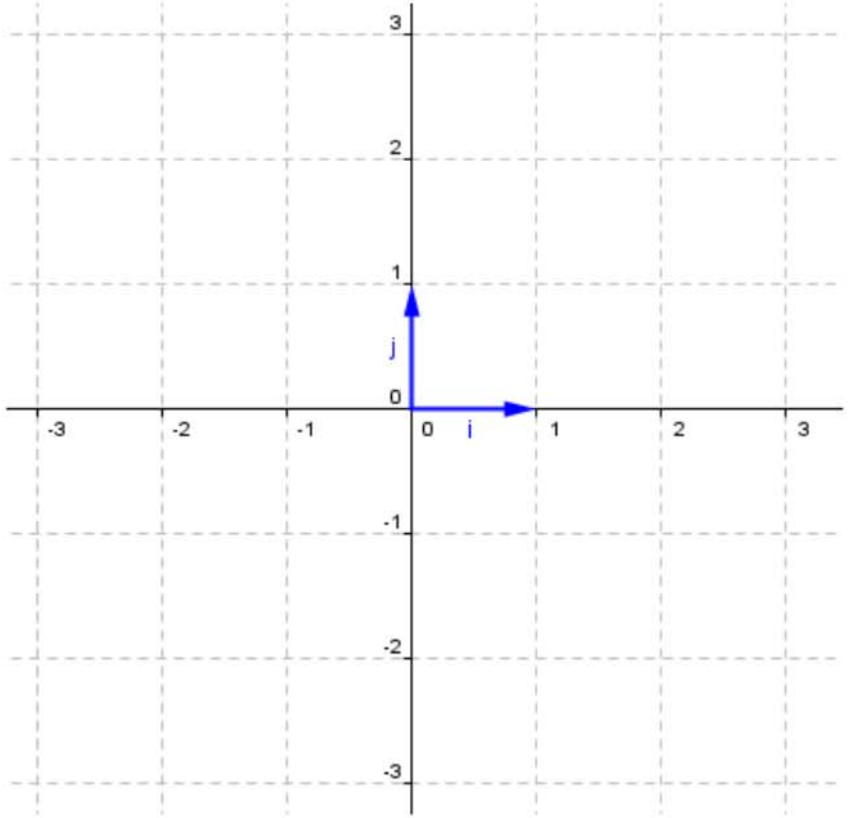
Transformations in 2D

Consider the matrix M= $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$

The base of the set of axes are the vectors

$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ *and* $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

- The image of $\mathbf{i}$ through the transformation is $\mathbf{M} \times \mathbf{i} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$
- The image of $\mathbf{j}$ through the transformation is $\mathbf{M} \times \mathbf{j} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$
- The image of any point $P(x, y)$ *with* position vector $\mathbf{p} = \begin{pmatrix} x \\ y \end{pmatrix}$ *is* $\mathbf{M} \times \mathbf{p}$



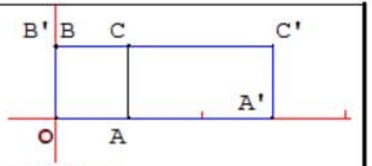
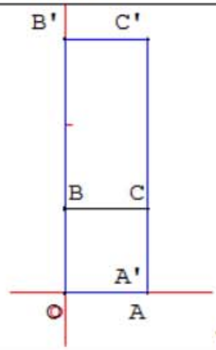
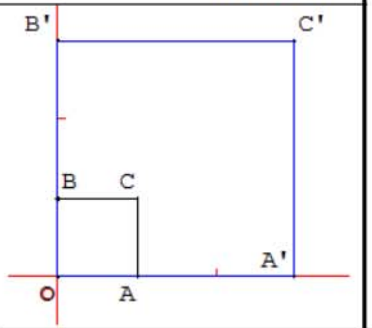
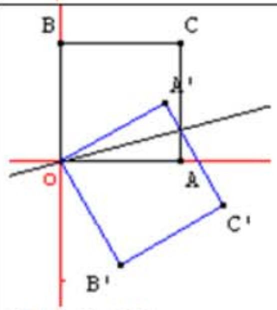
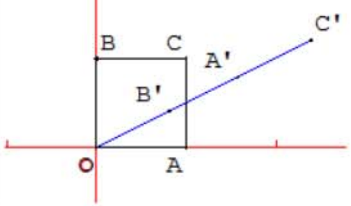
Example:

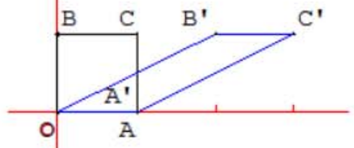
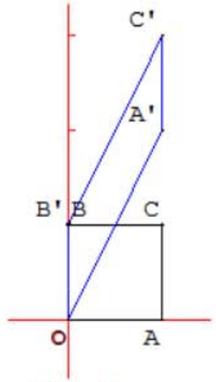
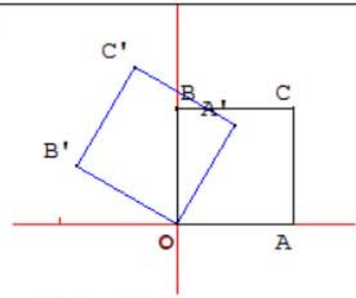
The transformation T is represented by the matrix M= $\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$

Work out the position vector image of the following:

$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\mathbf{p} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\mathbf{q} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$

Usual transformations

Stretch in the x-direction $M = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$	 with k=3
Stretch in the y-direction $M = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$	 with k=3
Enlargement centre O(0,0) by a scale factor k $M = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$	
Reflection in the line $y = (Tan\theta)x$ $M = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$	 With θ=15°
Example of a matrix transformation with determinant = 0 $M = \begin{pmatrix} 1.6 & 0.8 \\ 0.8 & 0.4 \end{pmatrix}$	

Shear in the x-direction $M = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$	 with k=2
Shear in the y-direction $M = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$	 with k=2
Rotation centre O(0,0) and angle θ anti-clockwise $M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$	 with θ=60°

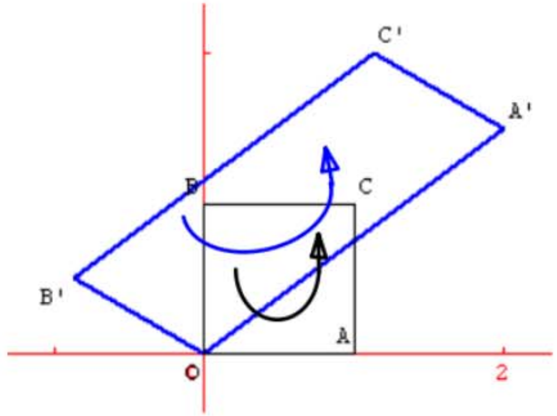
Meaning of the determinant

$$\mathbf{M} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \text{represents a tranformation T.}$$

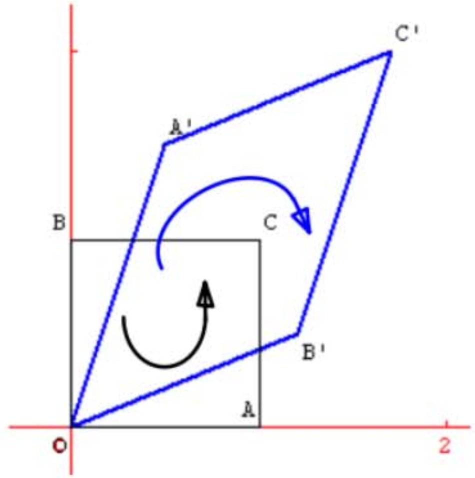
- $\det(\mathbf{M}) = a_1 b_2 - b_1 a_2$

$|\det(\mathbf{M})|$ is the AREA SCALE FACTOR of the transformation

Meaning of the sign of the determinant



If the determinant is positive (det(M)>0),
the sense in which the perimeter is traced is unchanged



If the determinant is negative (det(M)<0),
the sense in which the perimeter is traced is reversed

(If the determinant is negative, a reflection is "involved" is the transformation)

"Preservation" of the area

The 2×2 matrix $\mathbf{M}$ represents the plane transformation T. Write down the value of $\det \mathbf{M}$ in each of the following cases:

(a) T is a rotation;

(b) T is a reflection;

(c) T is a shear;

(d) T is an enlargement with scale factor 3. *(4 marks)*

Composing transformations

The transformation T_1 is represented by the matrix $\mathbf{M}_1$
and the transformation T_2 is represented by the matrix $\mathbf{M}_2$

The transformation T_1 *followed by T_2* is represented by the matrix $\mathbf{M}_2 \times \mathbf{M}_1$

Exercises:

1. A rotation of 90° anticlockwise about O is represented by $\mathbf{M}=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$.

   (a) Find $\mathbf{M}^2$. What transformation is represented by $\mathbf{M}^2$?

   (b) Find $\mathbf{M}^3$. What transformation is represented by $\mathbf{M}^3$?

2. Reflections in the x-axis, the y-axis and in the line $y = x$ are given, respectively, by

$$\mathbf{L}=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},\ \mathbf{M}=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix} \text{ and } \mathbf{N}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}.$$

   Find $\mathbf{L}^2$, $\mathbf{M}^2$ and $\mathbf{N}^2$ and explain your results.

3. Describe the geometrical transformations represented by the matrices

   (a) $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, (b) $\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$, (c) $\begin{bmatrix}3 & -4\\ 4 & 3\end{bmatrix}$.

4. Show that the transformation represented by the matrix $\mathbf{M}=\begin{bmatrix}a^2 & ab\\ ab & b^2\end{bmatrix}$, where a and b are constants, transforms all points onto a line. Find the equation of this line.

5. Write down matrices which represent the following transformations:

   (a) reflection in $y = x\sqrt{3}$,

   (b) anticlockwise rotation of 30° about the origin,

   (c) reflection in $y = -x$.

Formulae

Matrix transformations

Anticlockwise rotation through θ about O: $\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$

Reflection in the line $y = (\tan\theta)x$: $\begin{bmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{bmatrix}$

Answers:

1. (a) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$; rotation of 180° about O

   (b) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$; rotation of 90° clockwise about O

2. Each squared matrix is the identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. The result of performing any reflection twice is to return all points to their original positions

3. (a) Identity – all points stay fixed
   (b) Zero – all points map on to the origin
   (c) A rotation of $\text{Atn}\left(\frac{4}{3}\right)$ and an enlargement of $\times 5$, both about the origin

4. $x' = a^2x + aby, \quad y' = abx + b^2y \quad \Rightarrow y' = \frac{b}{a}x'$.

5. (a) $\frac{1}{2}\begin{bmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$ (b) $\frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$ (c) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

Transformations in 3D

The matrix **M** representing a given linear transformation has columns given by the images of **i**, **j**, and **k**.

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Image of **i** — Image of **j** — Image of **k**

Meaning of the determinant

If **M** is the matrix representing a transformation T

$|\det(\mathbf{M})|$ is the VOLUME SCALE FACTOR of the transformation

if $\det(\mathbf{M}) = 0$, the image of all points belong to a unique line.

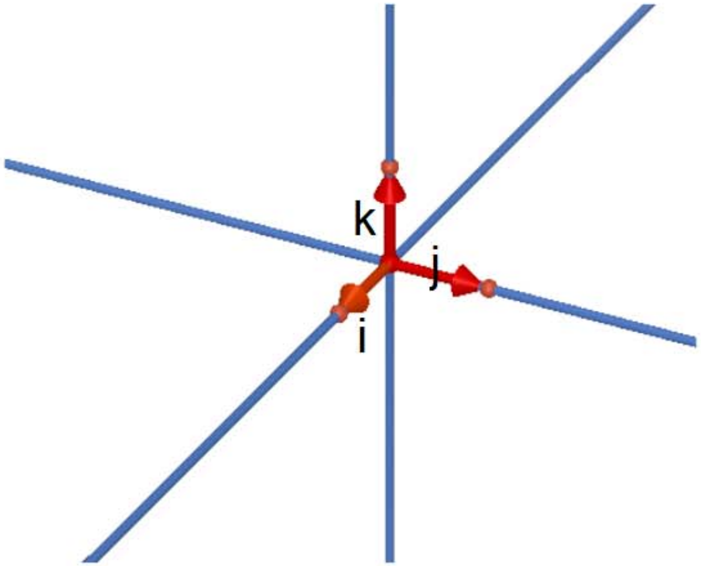
if $\det(\mathbf{M}) = 1$, the volume is "conserved" through the transformation (for example: rotations)

if $\det(\mathbf{M}) = -1$, the volume is "conserved" and the transformation T "involves" a reflection.

You should know the following transformation matrices:

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	Identity
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$, $\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$, $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$	Rotations of $\theta°$ about the x-, y- and z-axes, respectively
$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$	Enlargement, scale factor λ
$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	Reflections in the planes $x = 0$, $y = 0$ and $z = 0$ respectively
$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	Reflections in the planes $x = y$, $y = z$ and $x = z$ respectively

The matrix **BA** represents the transformation **A** followed by the transformation **B**



Write down the 3×3 matrix that represents a rotation of 90° about the x-axis in the direction of y to z as shown in the diagram.

To work out the matrix, map the vectors i, j and k

Through this transformation

$\mathbf{i}\rightarrow$

$\mathbf{j}\rightarrow$ $\qquad$ $so\ \mathbf{M}=\begin{bmatrix} \_ & \_ & \_ \\ \_ & \_ & \_ \\ \_ & \_ & \_ \end{bmatrix}$

$\mathbf{k}\rightarrow$

Questions:

Find the matrices representing each of the following transformations.

(i) Rotation of 90° about the x-axis.
(ii) Reflection in the plane $y = 0$.
(iii) Rotation of 180° about the y-axis
(iv) Reflection in the plane $x = z$
(v) Enlargement, centre the origin, scale factor 3.
(vi) Rotation of 25° about the z-axis.

(i) Describe fully the transformations represented by

$$\mathbf{A}=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\qquad \mathbf{B}=\begin{pmatrix}0&0&1\\0&1&0\\-1&0&0\end{pmatrix}\qquad \mathbf{C}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}$$

(ii) Find the matrix which represents **C** followed by **A**, and describe this transformation fully.

(iii) Find the matrix which represents **B** followed by **C**, and describe this transformation fully.

Exercise:

(a) A transformation, T_1, of three dimensional space is given by $\mathbf{r}' = \mathbf{M}\mathbf{r}$, where

$$\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{r}' = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \quad \text{and} \quad \mathbf{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Describe the transformation geometrically.

(b) Two other transformations are defined as follows: T_2 is a reflection in the x–z plane, and T_3 is a rotation through 180° about the line $x = 0,\ y + z = 0$. By considering the image under each transformation of the points with position vectors **i**, **j**, **k**, or otherwise, find the matrix for each of T_2 and T_3.

(c) Determine the matrices for the combined transformations $T_3\,T_1$ and $T_1\,T_3$ and describe each of these transformations geometrically.

(a) T_1: a rotation through 90° about the x-axis such that the positive y-axis maps on to the positive z-axis

(b) T_2: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; T_3: $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$

(c) T_3T_1: $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; T_1T_3: $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

T_3T_1 causes a rotation of 180° about the z-axis; T_1T_3 causes a rotation of 180° about the y-axis