Solving linear equations

Solution of Linear Equations

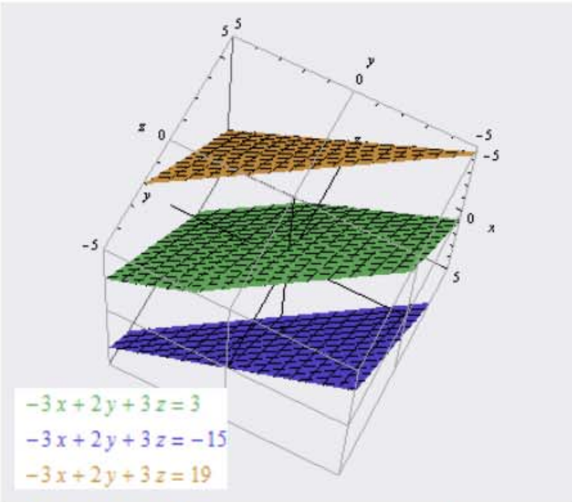
Consideration of up to three linear equations in up to three unknowns. Their geometrical interpretation and solution.	Any method of solution is acceptable.

Reminder

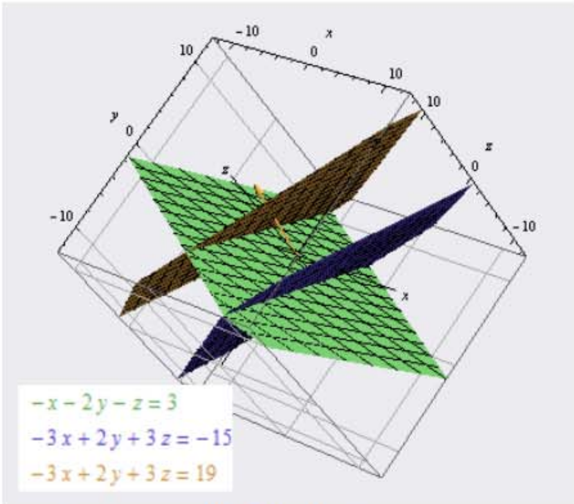
In 3D, the equation of a plan can be written as : $ax + by + cz = d$

In this chapter, we consider a set of three planes/ equations and study their relative position by solving these equations simultaneously.

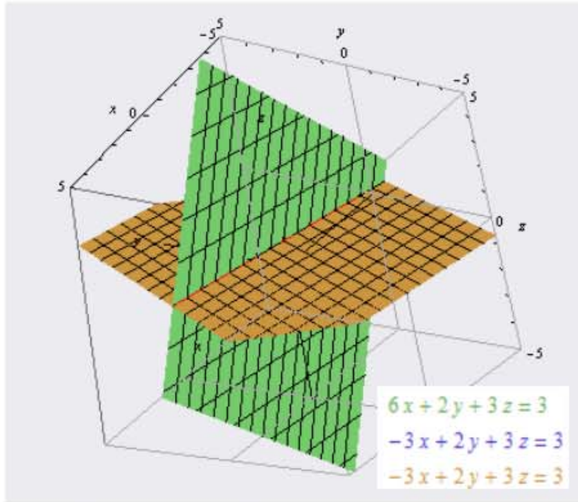
Case 1: there are at least two parallel planes



Three parallel planes: no solutions



Two parallel distinct planes: no solutions



Two parallel equal planes: the solution is a line

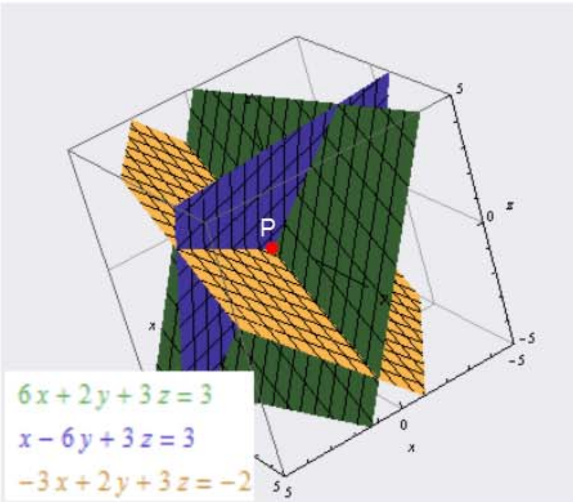
The system of equations is INCONSISTENT

The system of equations is CONSISTENT

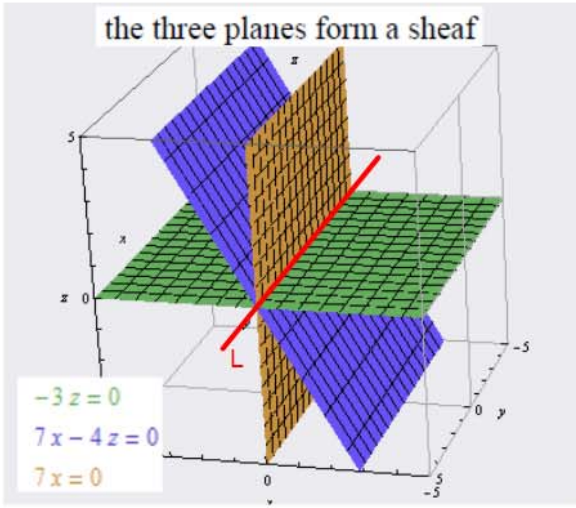
Case 2: the planes are not parallel to each other.

there is a unique solution

there is NOT a unique solution

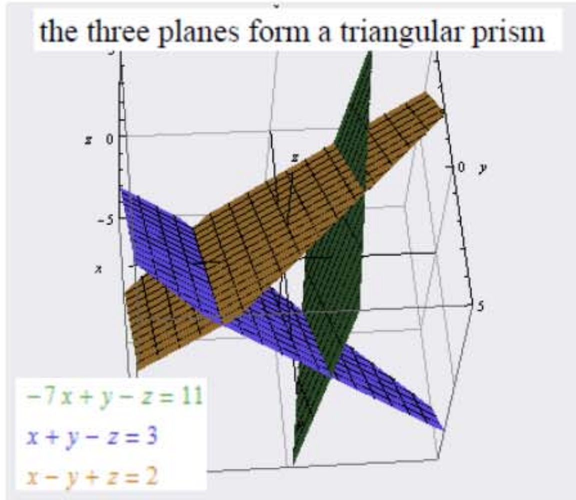


The planes intersect at a unique point



The planes intersect at a line

The system of equations is CONSISTENT



The planes do not intersect

The system of equations is INCONSISTENT

In front of a set of linear equations,
how do we know in which situation we are?

Set of linear equations and matrices

Consider the set of linear equations:

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

If you call **A** the matrix $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $\mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ *and* $\mathbf{B} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

then the set of linear equations is equivalent the the matrix equation:

$$\mathbf{AX=B}$$

Case 1:

A^{-1} exists

⇔A is non-sigular

⇔ $\det(A) \neq 0$

then we can solve the matrix equation:

$$\mathbf{AX=B}$$

$$\mathbf{A^{-1}AX = A^{-1}B}$$

$$\mathbf{X = A^{-1}B}$$

There is a UNIQUE solution.

Case 2:

A^{-1} does not exist

⇔ A is singular

⇔ $\det(A) = 0$

then there is NOT a unique solution

This is the only conclusion you can draw by working out the determinant of A.
If det(A)=0, the system could be consistent or inconsistent
(we will see how to proceed in this case later on)

Exercise:

By considering the equivalent matrix equation "AX=B"

i) Show that the system of equations has a unique solution.

ii) Work out A^{-1} and then find this unique solution.

(a) $\left.\begin{aligned} x+3y-4z&=10 \\ x+4y-2z&=9 \\ 3x+5y+2z&=12 \end{aligned}\right\}$

(b) $\left.\begin{aligned} x-2y+4z&=12 \\ -3x+4y-z&=-1 \\ 2x+5y+2z&=-3 \end{aligned}\right\}$

(c) $\left.\begin{aligned} 3x-2y+4z&=4 \\ 2x+3y-5z&=7 \\ 4x-2y-3z&=6 \end{aligned}\right\}$

(d) $\left.\begin{aligned} y+z&=4 \\ x+3y+z&=5 \\ 2x-2y-z&=3 \end{aligned}\right\}$

(e) $\left.\begin{aligned} y+z&=4 \\ x+y&=5 \\ x+z&=3 \end{aligned}\right\}$

(f) $\left.\begin{aligned} 5x+2y-3z&=9 \\ 4x-5y+5z&=8 \\ 3x-8y+7z&=5 \end{aligned}\right\}$

a)(3,1,−1)
b)(−2,−1,3)
c)(2,1,0)
d)(3,−1,5)
e)(2,3,1)
f)(2,1,1)

The Gaussian elimination/pivot

This method works in any situation, the consistency or inconsistency of the system will "naturally" appear in the process.

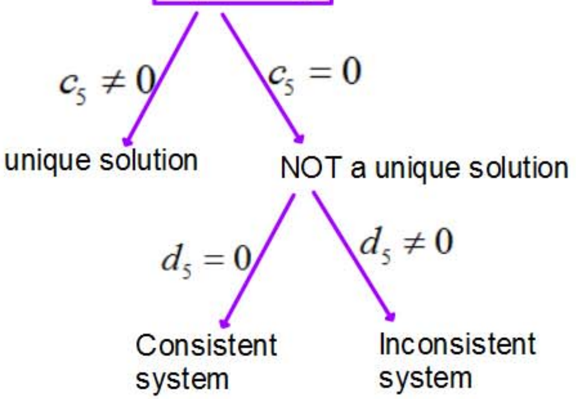
The principal:
To manipulate the equations using linear combinations in order to obtain an equivalent system of which the associated matrix is triangular.

- These manipulations do no change the solutions of a system of equations

> - Equations can be multiplied by any non-zero number
> - Any multiple of a row can be subtracted from or added to another row

$$\begin{cases} a_1x+b_1y+c_1z=d_1 \\ a_2x+b_2y+c_2z=d_2 \\ a_3x+b_3y+c_3z=d_3 \end{cases} \Leftrightarrow \dots \Leftrightarrow \begin{cases} a_1x+b_1y+c_1z=d_1 \\ b_4y+c_4z=d_4 \\ c_5z=d_5 \end{cases}$$

This line will determine in which situation you are:



Work out z, then by sustitution, find y and x.

In practice:

Notation:

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases} \qquad \begin{matrix} row1 = r_1 \\ row2 = r_2 \\ row3 = r_3 \end{matrix}$$

Algorithm:

- In the first row, select the unknown you want to eliminate. This is the pivot
  - *Tips*: choose an unknown with coefficient 1 or -1
  - swap rows, if necessary, if your pivot is in the second or third row
- Replace r_2 by a linear combination of r_1 *and* r_2 so that the coefficent of the pivot is 0

$$r_2' = \alpha r_2 + \beta r_1$$

- Do the same with r_3: $\quad r_3' = \gamma r_3 + \delta r_1$
- "Ignore r_1" *and* repeat the process with the two equations r_2' *and* r_3'
- *Conclude.*

$$\begin{matrix} 3x + y - 2z = 7 \\ 2x + 4y + z = 8 \\ 5x - y + 3z = 9 \end{matrix}$$

$$\begin{matrix} x - 3y + 4z = 4 \\ 2x - y + 3z = 8 \\ 3x + y + 2z = 12 \end{matrix}$$

$$\begin{matrix} x - 3y + 4z = 4 \\ 2x - y + 3z = 8 \\ 3x + y + 2z = 10 \end{matrix}$$

Exercise:

Solve, where possible, each of the following sets of equations

$$a)\begin{cases} x+y-z=4 \\ 2x-y-6z=6 \\ x-2y+3z=-6 \end{cases} \qquad b)\begin{cases} 2x-y+3z=5 \\ x+3y-2z=4 \\ y-z=2 \end{cases} \qquad c)\begin{cases} x+y-2z=5 \\ 2x-y+z=1 \\ x-2y-z=-4 \end{cases}$$

$a)(1,2,-1)$

$b)$ *No solution*

$$c)\mathbf{r}=\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}+t\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

Augmented matrix notation

We can represent the system $\begin{cases} x+y+z=7 \\ x-y+2z=9 \\ 2x+y-z=1 \end{cases}$ *by*

the AUGMENTED matrix $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 1 & -1 & 2 & 9 \\ 2 & 1 & -1 & 1 \end{array}\right)$

We can now apply the Gaussian elimination (it saves you writing x, y *and* z).

Let's solve it:

$$\begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 1 & -1 & 2 & 9 \\ 2 & 1 & -1 & 1 \end{array}\right) \Leftrightarrow \begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 1 & -1 & 2 & 9 \\ 2 & 1 & -1 & 1 \end{array}\right) \Leftrightarrow \begin{matrix} r_1' = r_1 \\ r_2' = r_2 - r_1 \\ r_3' = r_3 - 2r_1 \end{matrix} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & -2 & 1 & 2 \\ 0 & -1 & -3 & -13 \end{array}\right)$$

$$\Leftrightarrow \begin{matrix} r_1' \\ r_2' \\ 2r_3' - r_2' \end{matrix} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & -2 & 1 & 2 \\ 0 & 0 & -7 & -28 \end{array}\right) \Leftrightarrow \begin{matrix} \\ \\ \div 7 \end{matrix} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & -2 & 1 & 2 \\ 0 & 0 & 1 & 4 \end{array}\right)$$

From this point, go back to the equation form:

$$\begin{cases} z=4 \\ -2y+z=2 \\ x+y+z=7 \end{cases} \Leftrightarrow \begin{cases} z=4 \\ y=1 \\ x=2 \end{cases} \qquad (2,1,4)$$

Exercise:

Using the augmented matrix notation, solve the following:

$$a)\begin{cases} 3x-y-z=2 \\ x+y+z=4 \\ 4x-y+z=7 \end{cases} \qquad b)\begin{cases} 4x-y+5z=8 \\ 5x+7y-3z=42 \\ 3x+4y+z=27 \end{cases} \qquad c)\begin{cases} x-3y+z=4 \\ 2x-y+z=2 \\ x+2y \quad\;\; =-2 \end{cases}$$

$a)\left(\frac{3}{2},\frac{3}{4},\frac{7}{4}\right) \quad b)(2,5,1) \quad c)\; Line: \mathbf{r}=\begin{pmatrix}0\\-1\\1\end{pmatrix}+t\begin{pmatrix}-2\\1\\5\end{pmatrix}$

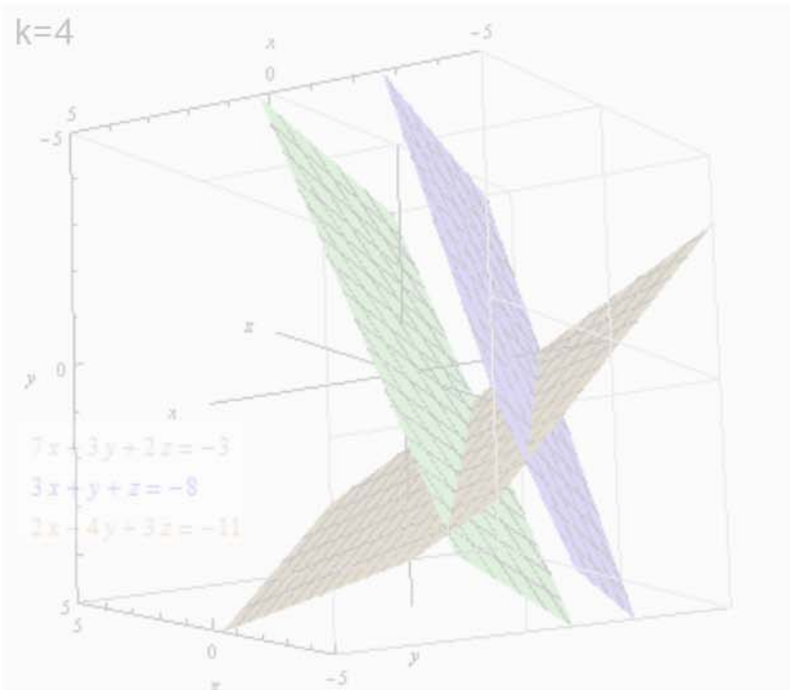
Solving equations with parameters

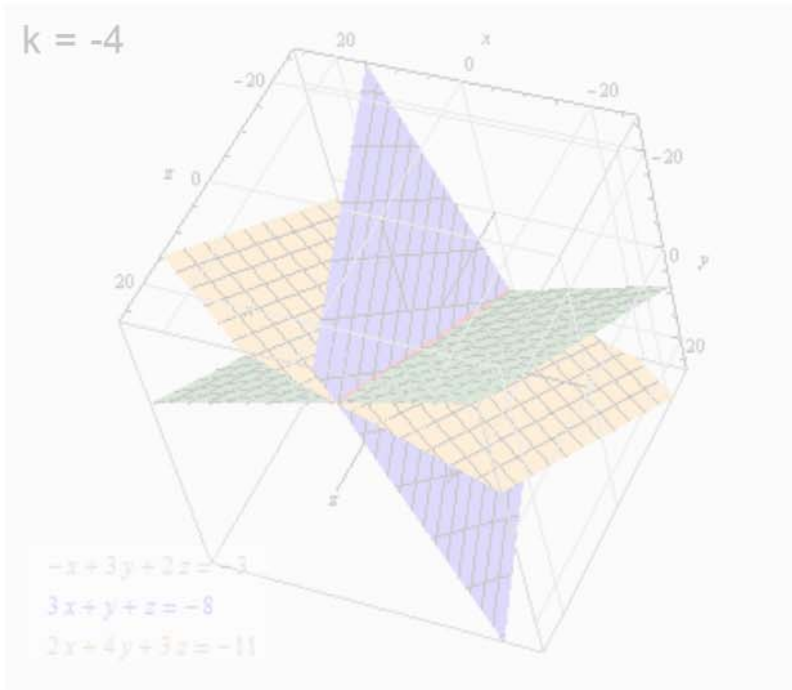
Consider the set of equations $\begin{cases} (k+3)x+3y+2z=-3 \\ 3x+y+z=-8 \\ 2x-ky+3z=-11 \end{cases}$

$a)$ Work out the values of k for which the system has no unique solution.

$b)$ For these values of k, determine if the system is consistent or not.

(in case of consistency, express the set of solutions as a parametric vector equation)





More exercises:

1. (a) Find the value of p for which the equations

$$3x-y+2z=5$$
$$2x+y+3z=0$$
$$x-3y+pz=7$$

do **not** have a unique solution.

2. (a) Prove that the equation

$$\begin{bmatrix}1 & 2 & -3\\ 2 & 6 & -11\\ 1 & -2 & 7\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}a\\ b\\ c\end{bmatrix}$$

is soluble only if $c+2b-5a=0$.

(b) Hence show that the planes

$$x+2y-3z=1$$
$$2x+6y-11z=2$$
$$x-2y+7z=1$$

intersect in a line.

(c) Find, in terms of s, the coordinates of the point in which this line meets the plane $z=s$.

3. (a) Find a vector equation for the line of intersection of the planes

$$2x+y-z=3$$
$$x+2y+4z=0$$

(b) Hence, or otherwise, solve the equations

$$2x+y-z=3$$
$$x+2y+4z=0$$
$$3x+\lambda y+6z=2$$

for **all** real values of λ, interpreting your results geometrically.

4. (a) Show that the simultaneous equations

$$6x-7y+2z=4$$
$$6x-y-z=7$$
$$2x-3y+z=k,$$

where k is a constant, are consistent only when $k=1$.

(b) Give, with reasons, a geometrical interpretation of the three equations in **each** of the cases when $k=1$ and $k\neq 1$.

7. (a) Show that the only real value of λ, for which the simultaneous equations

$$(2+\lambda)x-y+z=0$$
$$x-2\lambda y-z=0$$
$$4x-y-(\lambda-1)z=0$$

have a solution other than $x=y=z=0$, is -1.

(b) Solve the equations in the case when $\lambda=-1$, and interpret your result geometrically.

1. (a) $p=-2$ (b) $\mathbf{a}=\mathbf{b}-2\mathbf{c}$; $z=0$

2. (c) $\left(1-2s,\ \frac{5}{2}s,\ s\right)$

3. (a) $\mathbf{r}=\begin{bmatrix}2\\ -1\\ 0\end{bmatrix}+t\begin{bmatrix}2\\ -3\\ 1\end{bmatrix}$

(b) $\lambda=4$: $x=2+2t,\ y=-1-3t,\ z=t$; a common line (sheaf)

$\lambda\neq 4$: $x=\frac{4}{3},\ y=0,\ z=-\frac{1}{3}$; a unique point

4. (a) $9(2x-3y+z=k)+(6x-y-z=7)-4(6x-7y+2z=4)$

$\Rightarrow 0=9(k-1) \quad \Rightarrow k=1$

(b) $k=1$: there are infinitely many solutions; sheaf

$k\neq 1$: there are no solutions; prism

7. (a) $\begin{vmatrix}2+\lambda & -1 & 1\\ 1 & -2\lambda & -1\\ 4 & -1 & -(\lambda-1)\end{vmatrix}=2(\lambda+1)(\lambda^2+1)$ (b) $\mathbf{r}=t\begin{bmatrix}-1\\ 2\\ 3\end{bmatrix}$: common line; sheaf