Series and limits

Specifications

Series and Limits

Maclaurin series

Expansions of e^x, $\ln(1+x)$, $\cos x$ and $\sin x$, and $(1+x)^n$ for rational values of *n*.	Use of the range of values of x for which these expansions are valid, as given in the formulae booklet, is expected to determine the range of values for which expansions of related functions are valid; e.g. $\ln\left(\frac{1+x}{1-x}\right)$; $(1-2x)^{\frac{1}{2}} e^x$.
Knowledge and use, for $k > 0$, of $\lim x^k e^{-x}$ as x tends to infinity and $\lim x^k \ln x$ as x tends to zero.	
Improper integrals.	E.g. $\int_0^e x \ln x \, dx$, $\int_0^\infty x e^{-x} \, dx$. Candidates will be expected to show the limiting processes used.
Use of series expansion to find limits.	E.g. $\lim_{x\to 0} \frac{e^x - 1}{x}$; $\lim_{x\to 0} \frac{\sin 3x}{x}$; $\lim_{x\to 0} \frac{x^2 e^x}{\cos 2x - 1}$; $\lim_{x\to 0} \frac{\sqrt{2+x} - \sqrt{2}}{x}$

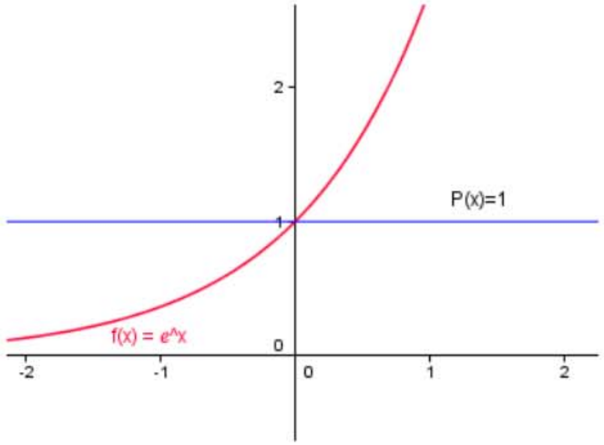
Maclaurin's series expansion

The principle:

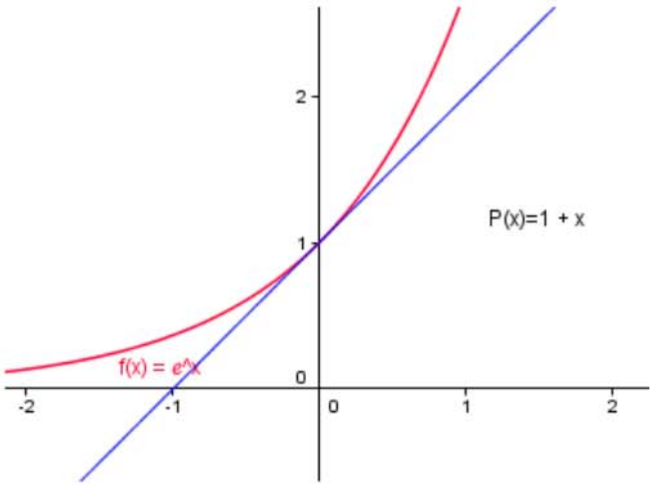
Consider a function f defined for $x = 0$ which can be differentiated several times.

We are going to create a polynomial P which approximate the function around 0.

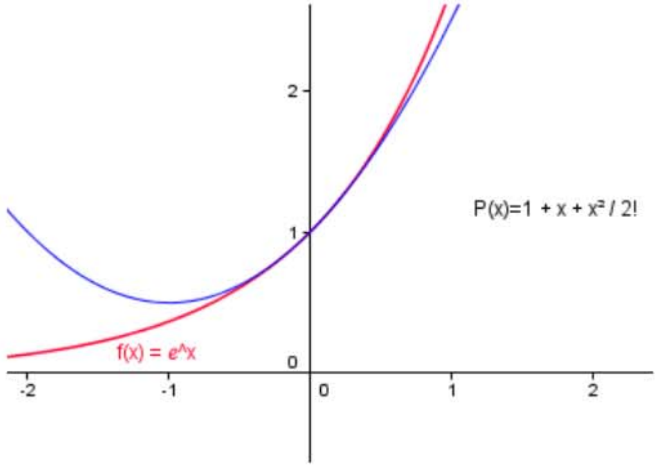
First approximation: f(0)=P(0)



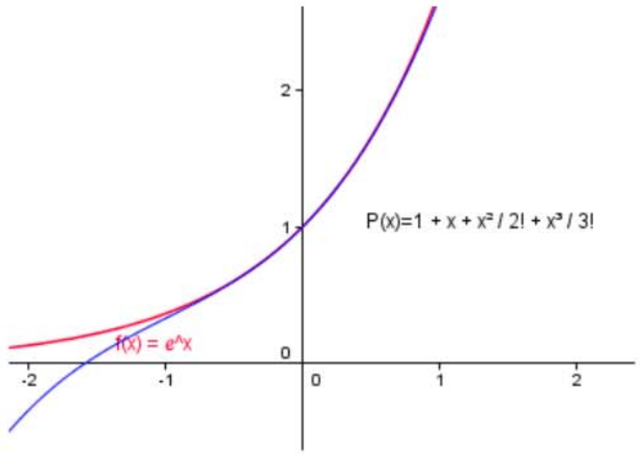
Second approx: f(0)=P(0) and f '(0) = P'(0)



Third approx: f(0)=P(0) , f '(0) = P'(0) and f''(0) = P''(0)



Fourth approx: f(0)=P(0) , f '(0) = P'(0) , f''(0) = P''(0) and $f^{(3)}(0) = P^{(3)}(0)$



Building the Maclaurin's series

Let's call f the function we want to find the Maclaurin's series

$$f(x) = P(x)$$

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

- The polynomial and the function are equal at 0

  $so\ P(0) = a_0 = f(0)$

- The polynomial and the function have the same gradient at 0

  so $P'(0) = a_1 = f'(0)$

- The polynomial and the function have the same second derivative(inflection) at 0

  so $P''(0) = 2a_2 = f''(0) \qquad a_2 = \dfrac{f''(0)}{2}$

- The polynomial and the function have the same third derivative at 0

  so $P^{(3)}(0) = 3a_3 = f^{(3)}(0) \qquad a_3 = \dfrac{f^{(3)}(0)}{6}$

Etc... We can establish that the coefficient of the polynomila P are $a_n = \dfrac{f^{(n)}(0)}{n!}$

where $f^{(n)}$ is the n^{th} derivative of f and $n! = n \times (n-1) \times (n-2) \times \dots 3 \times 2 \times 1$

Summary

The function $\mathrm{f}(x)$ and all its derivatives exist at $x=0$

The Maclaurin series for a function $\mathrm{f}(x)$ is given by:

$$\mathrm{f}(x)=\mathrm{f}(0)+\mathrm{f}'(0)x+\frac{\mathrm{f}''(0)}{2!}x^2+\frac{\mathrm{f}'''(0)}{3!}x^3+\ldots+\frac{\mathrm{f}^{(r)}(0)}{r!}x^r+\ldots,$$

where f' , f'' , f''' , … denote the first, second, third, … derivatives of f, respectively, and $\mathrm{f}^{(r)}$ is the general derivative of order r .

Example:

Use Maclaurin's theorem to obtain the expansion of $\ln(1+x)$ as a series in ascending powers of x .

$f(x)=\ln(1+x)$ $\qquad f(0)=0$

$f'(x)=\dfrac{1}{1+x}$ $\qquad f'(0)=1$

$f''(x)=-\dfrac{1}{(1+x)^2}$ $\qquad f''(0)=-1$

$f'''(x)=\dfrac{2}{(1+x)^3}$ $\qquad f'''(0)=2$

$f^{(4)}(x)=-\dfrac{6}{(1+x)^4}$ $\qquad f^{(4)}(0)=-6$

$$\ln(1+x)=0+x-\frac{1}{2!}x^2+\frac{2}{3!}x^3-\frac{6}{4!}x^4+\ldots$$

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots$$

Exercises:

Obtain the Maclaurin's series for the following functions up to and including x^3.

Write down an expression for the general term of the series.

$a)\ f(x) = Sin(x)$

$b)\ f(x) = e^x$

$c)\ f(x) = \frac{1}{1-x}$

$d)\ f(x) = Cosh(x)$

The following are given in the formulae book:

Maclaurin's series

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^r}{r!} f^{(r)}(0) + \ldots$$

$$e^x = \exp(x) = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^r}{r!} + \ldots \quad \text{for all } x$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{r+1} \frac{x^r}{r} + \ldots \quad (-1 < x \leqslant 1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \ldots \quad \text{for all } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^r \frac{x^{2r}}{(2r)!} + \ldots \quad \text{for all } x$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2} x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{1.2\ldots r} x^r + \ldots \quad (|x| < 1, n \in \mathbb{R})$$

Range of validity

Range of validity

The Maclaurin series expansion of a function $f(x)$ is not necessarily valid for all values of x.
A simple example will show this. Consider

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

$$1 + x + x^2 + x^3 + \dots + x^n = \frac{1-x^n}{1-x}$$

When n tends to infinity, *x^n converges only if* $-1 < x < 1$

(a) Expand $\ln(1-2x)$ as a series in ascending powers of x, up to and including the term in x^3.

(b) Determine the range of validity of this series.

Multiplying and composing maclaurin's series

Given that $f(x) = (1 - x)^2 \ln(1 - x)$

a Show that $f''(x) = 3 + 2\ln(1 - x)$.

b Find the values of $f(0)$, $f'(0)$, $f''(0)$, and $f'''(0)$.

c Express $(1 - x)^2 \ln(1 - x)$ in ascending powers of x up to and including the term in x^3.

Write down the first 4 non-zero terms in the series expansion, in ascending powers of x, of $\cos(2x^2)$.

Given that terms in x^n, $n > 4$ may be neglected, use the series for e^x and $\sin x$, to show that

$$e^{\sin x} \approx 1 + x + \frac{x^2}{2} - \frac{x^4}{8}.$$

Exercises:

1 Use the series expansions of e^x, $\ln(1 + x)$ and $\sin x$ to expand the following functions as far as the fourth non-zero term. In each case state the interval in x for which the expansion is valid.

a $\frac{1}{e^x}$ **b** $\frac{e^{2x} \times e^{3x}}{e^x}$

c e^{1+x} **d** $\ln(1 - x)$

e $\sin\left(\frac{x}{2}\right)$ **f** $\ln(2 + 3x)$

Hint for f: write $2 + 3x = 2\left(1 + \frac{3x}{2}\right)$

2 **a** Using the Maclaurin expansion of $\ln(1 + x)$, show that

$$\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right),\ -1 < x < 1.$$

b Deduce the series expansion for $\ln\sqrt{\left(\frac{1+x}{1-x}\right)}$, $-1 < x < 1$.

c By choosing a suitable value of x , and using only the first three terms of the series in **a**, find an approximation for $\ln\left(\frac{2}{3}\right)$, giving your answer to 4 decimal places.

d Show that the first three terms of your series in **b**, with $x = \frac{3}{5}$, gives an approximation for ln2, which is correct to 2 decimal places.

3 Show that for small values of x, $e^{2x} - e^{-x} \approx 3x + \frac{3}{2}x^2$.

4 **a** Show that $3x \sin 2x - \cos 3x = -1 + \frac{21}{2}x^2 - \frac{59}{8}x^4 - \dots$

9 Using the series given on page 112, show that

a $(1 - 3x)\ln(1 + 2x) = 2x - 8x^2 + \frac{26}{3}x^3 - 12x^4 + \dots$

b $e^{2x} \sin x = x + 2x^2 + \frac{11}{6}x^3 + x^4 + \dots$

c $\sqrt{(1 + x^2)}e^{-x} = 1 - x + x^2 - \frac{2}{3}x^3 + \frac{1}{6}x^4 + \dots$

1 **a** $1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \dots$ valid for all values of x

b $1 + 4x + 8x^2 + \frac{32x^3}{3} + \dots$ valid for all values of x

c $e\left\{1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots\right\}$ valid for all values of x

d $-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots$ $-1 \leq x < 1$

e $\frac{x}{2} - \frac{x^3}{48} + \frac{x^5}{3840} - \frac{x^7}{645120} + \dots$ valid for all values of x

f $\ln 2 + \frac{3x}{2} - \frac{9x^2}{8} + \frac{9x^3}{8} + \dots$ $-\frac{2}{3} < x \leq \frac{2}{3}$

2 **b** $\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right)$, $-1 < x < 1$

c -0.4055 (4 d.p.)

4 **b** $\frac{21}{2}$

Notion of limits

Limit when x tends to infinity

To study a function, it is always interesting to understand how it behaves when x becomes bigger and bigger...

Work out the following limits.

$a)\ \lim_{x\to+\infty} \frac{4x+4}{2x-1} =$ $b)\ \lim_{x\to+\infty} \frac{3x+2}{x^2+1} =$ $c)\ \lim_{x\to+\infty} e^{-x} =$

$d)\ \lim_{x\to+\infty} 3x^2+2x+1 =$ $e)\ \lim_{x\to-\infty} \ln(1-x) =$

Limits at a value

Consider the function $f(x)=\frac{x}{1-\sqrt{1-x}}$.

$a)$ For which values of x is the function defined.

$b)$ Complete the table of value:

x	0.2	0.1	0.05	0.01	0.001	0.0001	0.00001	10^{-10}
f(x)								

x	-0.2	-0.1	-0.05	-0.01	-0.001	-0.0001	-0.00001	-10^{-10}
f(x)								

Note: if a function is defined at a value "a", $\lim_{x\to a} f(x) = f(a)$

Using Maclaurin's series to work out limits

$$f(x)=\frac{x}{1-\sqrt{1-x}}$$

If it exists, what is the limit of f when x tends to 0?

- $\sqrt{(1-x)}=(1-x)^{\frac{1}{2}}=1-\frac{1}{2}x-\frac{1}{8}x^2+\dots$

$$1-\sqrt{(1-x)}=\frac{1}{2}x+\frac{1}{8}x^2+\dots$$

$$So\ f(x)=\frac{x}{1-\sqrt{1-x}}=\frac{x}{\frac{1}{2}x+\frac{1}{8}x^2}\ \text{and}$$

dividing numerator and denominator by x,

$$f(x)=\frac{1}{\frac{1}{2}+\frac{1}{8}x+\dots}\xrightarrow[x\to 0]{}\frac{1}{\frac{1}{2}}=2$$

Exercises:

1. Use series expansions to determine the following limits:

   (a) $\lim_{x\to 0}\dfrac{e^x-1}{x}$, (b) $\lim_{x\to 0}\dfrac{\sin 3x}{x}$, (c) $\lim_{x\to 0}\dfrac{x^2}{1-\cos 2x}$, (d) $\lim_{x\to 0}\dfrac{x\ln(1+x)}{1-\cos x}$.

2. (a) Show that $\ln(1+\sin x) = x - \dfrac{1}{2}x^2 + \dfrac{1}{6}x^3 + \dots$.

   (b) Hence find $\lim_{x\to 0}\dfrac{\ln(1+\sin x)-x}{x^2}$.

3. Find $\lim_{x\to 0}\dfrac{\sqrt{2+x}-\sqrt{2}}{x}$.

4. (a) By using the identity $2^x \equiv e^{x\ln 2}$, obtain the first three terms in the expansion of 2^x as a series in ascending powers of x. Give the coefficients of x and x^2 in terms of $\ln 2$.

   (b) Find $\lim_{x\to 0}\dfrac{2^x-1}{3^x-1}$.

5. Find $\lim_{x\to\infty}\left[\left(x^2+3x\right)^{\frac{1}{2}}-x\right]$.

1) a) 1 b) 3 c) $\frac{1}{2}$ d) 2

2) b) $-\frac{1}{2}$

3) $\frac{1}{4}\sqrt{2}$

4) a) $1+(\ln 2)x+\frac{1}{2}(\ln 2)^2x^2$ b) $\frac{\ln 2}{\ln 3}$

5) $\frac{3}{2}$

Limits and improper integrals

You need to know:

when $x \to \infty$, $x^k e^{-x} \to 0$ for any real number k

when $x \to 0$, $x^k \ln x \to 0$ for all $k > 0$

These results can be used without proof

- Show that $\lim_{x\to\infty} \left(x^2 e^{-2x}\right) = 0.$

- Find $\lim_{x\to 0} \left[(1+x)^2 - 1\right] \ln x$, where $x > 0$.

Workings out:

- $x^2 e^{-2x} = (xe^{-x})^2$

$\lim_{x\to\infty} xe^{-x} = 0 \quad so \quad \lim_{x\to\infty} (xe^{-x})^2 = 0$

$(1+x)^2 - 1 = 1 + 2x + x^2 - 1 = 2x + x^2$

$f(x) = (2x + x^2)\ln(x) = 2x\ln(x) + x^2\ln(x)$

$\lim_{x\to 0} x\ln x = 0 \quad and \quad \lim_{x\to 0} x^2 \ln x = 0 \quad so \quad \lim_{x\to 0} f(x) = 0$

Have a go:

Find the following limits.

(a) $\lim\limits_{x\to\infty} \dfrac{1+x}{\mathrm{e}^x}$, (b) $\lim\limits_{x\to\infty} \dfrac{x^3}{\mathrm{e}^{2x}}$, (c) $\lim\limits_{x\to\infty} \left[(1+x)^3 - 1\right]\mathrm{e}^{-x}$, (d) $\lim\limits_{x\to-\infty} x^{10}\mathrm{e}^x$,

(e) $\lim\limits_{x\to 0} x\ln 2x$, (f) $\lim\limits_{x\to 0+} x\ln\left(x+x^2\right)$, (g) $\lim\limits_{x\to 1-} (1-x)\ln(1-x)$.

Improper integrals

The integral $\int_a^b f(x)\,dx$ is said to be improper if

(1) the interval of integration is infinite,

or (2) $f(x)$ is not defined at one or both of the end points $x = a$ and $x = b$,

Examples:

Case 1: The interval is infinite

Replace the symbol "∞" by a letter, "N" for example.
Work out the integral in terms of N, then study the limit of the expression found when N tends to infinity.

$$I_1 = \int_0^{\infty} \frac{1}{1+x^2}\,dx$$

$$\int_0^N \frac{1}{1+x^2}\,dx = \left[ArcTan\right]_0^N = ArcTan(N) - ArcTan(0) = ArcTan(N)$$

When N tends to infinity, Arctan(N) tends to $\frac{\pi}{2}$

(If you are not convinced, type in your calculator $\tan^{-1}(100000)$ *and compare with* $\frac{\pi}{2} = 1.5707$)

$$Conclusion: \int_0^{\infty} \frac{1}{1+x^2}\,dx \text{ exists and } \int_0^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2}$$

Case 2: The function is not defined at the end point(s)

Replace the "problem" value by a letter, "a" for example. Work out the integral in terms of "a" then study the limit of the expression found when x tends to "a".

$$I_2 = \int_0^4 \frac{1}{\sqrt{x}}\,dx$$

Solution:

$$\int_a^4 \frac{1}{\sqrt{x}}\,dx = \int_a^4 x^{-\frac{1}{2}}\,dx = \left[2\sqrt{x}\right]_a^4 = 2\sqrt{4} - 2\sqrt{a} = 4 - 2\sqrt{a}$$

Now study the limit when a tends to 0: $\lim\limits_{x\to 0} 2\sqrt{a} = 2\sqrt{0} = 0$

$so \int_0^4 \frac{1}{\sqrt{x}}\,dx$ *exists and* $\int_0^4 \frac{1}{\sqrt{x}}\,dx = 4$

Exercise:

(a) Explain why $\int_0^{e} x \ln x \, dx$ is an improper integral.

(b) Show that the integral exists and find its value.

Exercises:

1. (a) Show that one of the following integrals exists and that the other does not:

$$\int_1^{\infty} \frac{1}{x^3}\,dx, \qquad \int_0^{1} \frac{1}{x^3}\,dx.$$

(b) Evaluate the one that does exist.

2. Evaluate the following improper integrals, showing in each case the limiting process used.

(a) $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$, (b) $\int_0^{\infty} \frac{1}{(1+x)^2}\,dx$, (c) $\int_0^{\infty} x\,e^{-x}\,dx$,

(d) $\int_{-\infty}^{0} \frac{1}{(4-x)^{\frac{3}{2}}}\,dx$, (e) $\int_0^1 x^2 \ln x\,dx$, (f) $\int_0^{e} \ln x\,dx$.

3. (a) Explain why each of the following integrals is improper:

(i) $\int_0^{\infty} \frac{1}{\sqrt{1+x}}\,dx$, (ii) $\int_0^1 \frac{x}{1-x^2}\,dx$.

(b) Show that neither integral exists.

1. (a) The first integral exists. The second does not. (b) $\frac{1}{2}$

2. (a) $\lim_{a\to 0}\left(\sin^{-1}1-\sin^{-1}a\right)=\frac{\pi}{2}$ (b) $\lim_{a\to\infty}\left(-\frac{1}{1+a}+1\right)=1$

(c) $\lim_{a\to\infty}\left(-ae^{-a}-e^{-a}+1\right)=1$ (d) $\lim_{a\to-\infty}\left(1-\frac{2}{(4-a)^{\frac{1}{2}}}\right)=1$

(e) $\lim_{a\to 0}\left(-\frac{1}{3}a^3\ln a-\frac{1}{9}+\frac{1}{9}a^3\right)=-\frac{1}{9}$ (f) $\lim_{a\to 0}\left(-a\ln a+a\right)=0$

3. (a) (i) The interval of integration is infinite

(ii) $\frac{x}{1-x^2}$ is not defined at $x=1$

Miscellaneous exercises 1

1. Use Maclaurin's theorem to show that

$$\tan\left(x+\frac{\pi}{4}\right)=1+2x+2x^2+\dots\ .$$

2. Explain why $\ln x$ cannot have a Maclaurin expansion.

3. Use Maclaurin's theorem to show that

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\dots\ .$$

4. (a) Expand $(1+2x)^{\frac{1}{2}}\sin x$ as a series in ascending powers of x, up to and including the term in x^3.

   (b) Determine the range of values of x for which the expansion is valid.

5. (a) Obtain the first two non-zero terms in the expansion of

$$e^{3x}+\ln(1-3x)\ .$$

   (b) Determine the range of values of x for which the expansion is valid.

6. (a) Obtain the first three non-zero terms in the expansions in ascending powers of x of (i) x^2e^x, (ii) $\cos 2x$.

   (b) Hence find $\lim\limits_{x\to 0}\dfrac{x^2e^x}{\cos 2x-1}$.

7. By means of the substitution $x=y+3$, or otherwise, evaluate

$$\lim_{x\to 3}\frac{\sqrt{(4-x)}-1}{\sqrt{(1+x)}-2}.$$

8. (a) Find $\lim\limits_{x\to 0}\left[1-\dfrac{1}{(1+x)^2}\right]\ln x$.

   (b) Find $\lim\limits_{x\to\infty}\dfrac{x+e^x}{x-e^x}$.

9. The function f is defined

$$f(x)=\frac{e^x}{1-x},\qquad x\neq 1.$$

   Show that $f(x)\to-\infty$ as $x\to\infty$.

10. (a) Use integration by parts to evaluate

$$\int_a^1 \ln x\,dx,\qquad a>0.$$

   (b) Explain why $\int_0^1 \ln x\,dx$ is an improper integral. Determine whether the integral exists or not, giving a reason for your answer.

[NEAB, 1995]

11. (a) Use the expansion of $\cos x-1$ to obtain the expansion of $e^{\cos x-1}$ in a series in ascending powers of x, up to and including the term in x^4.

   (b) Evaluate $\lim\limits_{x\to 0}\dfrac{e-e^{\cos x}}{x^2}$.

[JMB, 1988]

12 (a) Write down the value of $\lim\limits_{x\to\infty}\dfrac{x}{2x+1}$.

   (b) Evaluate

$$\int_1^\infty\left(\frac{1}{x}-\frac{2}{2x+1}\right)dx$$

   giving your answer in the form $\ln k$, where k is a constant to be determined.
   Explain why this is an improper integral.

Miscellaneous exercises 1 Answers

4. (a) $x+x^2-\frac{1}{3}x^3$ (b) $-1<x\le\frac{1}{2}$

5. (a) $1-\frac{9}{2}x^3$ (b) $-\frac{1}{3}<x\le\frac{1}{3}$

6. (a)(i) $x^2+x^3+\frac{1}{2}x^4$ (ii) $1-2x^2+\frac{2}{3}x^4$ (b) $-\frac{1}{2}$

7. -2

8. (a) 0 (b) -1

10. (a) $a-a\ln a-1$

(b) The integral is improper because $\ln x$ is not defined at $x=0$.
The integral exists because $(a-a\ln a-1)\to -1$ as $a\to 0$.

11. (a) $1-\frac{1}{2}x^2+\frac{1}{6}x^4$ (b) $\frac{1}{2}\mathrm{e}$

12 (a) $\frac{1}{2}$ (b) $\frac{3}{2}$; the integral is improper because the interval of integration is infinite