Finite series

Specifications:

Finite Series

Summation of a finite series by any method such as induction, partial fractions or differencing.

E.g. $\sum_{r=1}^{n} r.r! = \sum_{r=1}^{n} \left[(r+1)! - r! \right]$

Introduction:

Start writing in full the following series (substituting r by 1, 2, 3 ,...)

$$\sum_{r=1}^{49} \left(r^2 - (r+1)^2 \right) = 1^2 - 2^2 + \text{.................................}$$

What is the value of $\sum_{r=1}^{49} \left(r^2 - (r+1)^2 \right)$?

Using the same principle, can you work out

$$a)\ \sum_{r=1}^{99} \frac{2}{r} - \frac{2}{r+1}$$

$$b)\sum_{k=2}^{101} \frac{1}{r^2} - \frac{1}{(r-1)^2}$$

$$c)\sum_{t=1}^{199} \ln\left(\frac{t}{t+1}\right)$$

Work out in terms of n

$$a)\ \sum_{r=1}^{n} \frac{1}{3r} - \frac{1}{3r+3}$$

$$b)\sum_{r=1}^{n} (r+1)^2 - r^2$$

Summing using the method of differences

If the term of a series u_r can be written $u_r = f(r) - f(r+1)$ (where f is a given function)

then $$\sum_{r=1}^{n} u_r = \sum_{r=1}^{n} \left(f(r) - f(r+1)\right) = f(1) - f(n+1)$$

Worked example:

(a) Simplify $r(r+1)-(r-1)r$.

(b) Use your result to obtain $\sum_{r=1}^{n} r$.

(a) Use the identity $4r^3 = r^2(r+1)-(r-1)^2 r^2$

to show that $\sum_{r=1}^{n} 4r^3 = n^2(n+1)^2$.

Exercises:

1) **(i)** Show that

$$\frac{1}{r}-\frac{1}{r+1}=\frac{1}{r(r+1)}.$$

(ii) Hence find an expression, in terms of n, for

$$\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\ldots+\frac{1}{n(n+1)}.$$

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2) **(i)** Show that $\frac{1}{r!}-\frac{1}{(r+1)!}=\frac{r}{(r+1)!}$.

(ii) Hence find an expression, in terms of n, for

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\ldots+\frac{n}{(n+1)!}.$$

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3) **(i)** Show that $(r+2)!-(r+1)!=(r+1)^2\times r!$.

(ii) Hence find an expression, in terms of n, for

$$2^2\times 1!\ +\ 3^2\times 2!\ +\ 4^2\times 3!\ +\ \ldots\ +\ (n+1)^2\times n!.$$

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4) **(i)** Show that

$$\frac{r+1}{r+2}-\frac{r}{r+1}=\frac{1}{(r+1)(r+2)}.$$

(ii) Hence find an expression, in terms of n, for

$$\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\ldots+\frac{1}{(n+1)(n+2)}.$$

5) The function f is defined for all non-negative integers r by

$$f(r)=r^2+r-1.$$

(a) Verify that $f(r)-f(r-1)=Ar$ for some integer A, stating the value of A.

(b) Hence, using the method of differences, prove that

$$\sum_{r=1}^{n} r=\frac{1}{2}\left(n^2+n\right).$$

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6) **(i)** Show that $\frac{1}{r}-\frac{1}{r+2}=\frac{2}{r(r+2)}$.

(ii) Hence find an expression, in terms of n, for

$$\frac{2}{1\times 3}+\frac{2}{2\times 4}+\ldots+\frac{2}{n(n+2)}.$$

Partial fractions

Theorem:

Any rational function of the form $P(r)=\dfrac{ar+b}{(cr+d)(er+f)}$ can be written as the sum of two partial fractions $\dfrac{A}{cr+d}+\dfrac{B}{er+f}$ where A and B are real numbers.

Examples:

Method 1: identification

$$P(r)=\frac{1}{r(r+1)}$$

$$\frac{1}{r(r+1)}=\frac{A}{r}+\frac{B}{r+1}=\frac{A(r+1)}{r(r+1)}+\frac{Br}{r(r+1)}$$

$$\frac{1}{r(r+1)}=\frac{0r+1}{r(r+1)}=\frac{(A+B)r+A}{r(r+1)}$$

For these fractions to be equal, we need

$$\begin{cases}A+B=0\\A=1\end{cases}\quad \textit{this gives}\ \ A=1\ \ \textit{and}\ \ B=-1$$

Conclusion:

$$\frac{1}{r(r+1)}=\frac{1}{r}-\frac{1}{r+1}$$

Method 2:

$$P(r)=\frac{3}{(2-r)(r-1)}$$

$$\frac{3}{(2-r)(r-1)}=\frac{A}{2-r}+\frac{B}{r-1}$$

Multiply both sides by the denominator $(2-r)(r-1)$:

$$3=A(r-1)+B(2-r)$$

Substitute r by the value of the roots of the denominator (here 1 and 2)

for $r=1$ $\quad 3=0+B(2-1)\ so \quad B=3$

for $r=2$ $\quad 3=A(2-1)+0\ so \quad A=3$

Conclusion:

$$\frac{3}{(2-r)(r-1)}=\frac{3}{2-r}+\frac{3}{r-1}$$

June 2010

(a) Express $\dfrac{1}{r(r+2)}$ in partial fractions. *(3 marks)*

(b) Use the method of differences to find

$$\sum_{r=1}^{48} \frac{1}{r(r+2)}$$

giving your answer as a rational number. *(5 marks)*

Question 2:

a) express $\dfrac{1}{(2n-1)(2n+3)}$ in partial fractions.

b) Sum the series $\dfrac{1}{1\times 5}+\dfrac{1}{3\times 7}+\dfrac{1}{5\times 9}+\dots+\dfrac{1}{(2n-1)(2n+3)}$

$\dfrac{894}{1225}$

$\dfrac{1}{3}-\left(\dfrac{n+1}{(2n+1)(2n+3)}\right).$

a Express $\dfrac{2}{(r+1)(r+3)}$ in partial fractions.

b Hence prove by method of differences that

$$\sum_{r=1}^{n} \frac{2}{(r+1)(r+3)} = \frac{n(an+b)}{6(n+2)(n+3)}$$

where a and b are constants to be found.

c Find the value of $\displaystyle\sum_{r=21}^{30} \frac{2}{(r+1)(r+3)}$ to 5 decimal places.

$\dfrac{665}{24288} = 0.02738$ to 5 d.p.

Summary of key points

- You can use the method of differences to sum simple finite series.
- If the general term, u_r, of a series can be expressed in the form

  $f(r) - f(r+1)$

  then $\sum_{r=1}^{n} u_r = \sum_{r=1}^{n} (f(r) - f(r+1)) = f(1) - f(n+1)$