Series and limits

Series

MacLaurin's series

The function $f(x)$ and all its derivatives exist at $x = 0$

The Maclaurin series for a function $f(x)$ is given by:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + ... + \frac{f^{(r)}(0)}{r!}x^r + ...$$

where $f', f'', f''', ...$ denote the first, second, third,... derivatives of f, respectively.

$f^{(r)}$ is the derivative of order r.

Range of validity

Some series are valid for all values of $x \in \mathbb{R}$,

but some series are valid for only some values of x.

Refer to the formulae book to find the range of validity.

For example: $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ...$ is valid for $-1 < x \le 1$

Multiplying and composing Maclaurin's series

$f(x)$ *and* $g(x)$ are two functions

- The Maclaurin's series of the function $f \times g(x)$ is the product the two maclaurin's series.
- To obtain the Maclaurin's series of the function $f(g(x))$, substitute x in the Maclaurin's series f by the Maclaurin's series of $g(x)$.

Examples: $e^x = 1 + x + \frac{x^2}{2} + ...$ *and* $\sin(x) = x - \frac{x^3}{6} + ...$

- The maclaurin's series of $e^x \sin(x) = (1 + x + \frac{x^2}{2} + ...)(x - \frac{x^3}{6} + ...)$

$$= x - \frac{x^3}{6} + x^2 - \frac{x^4}{6} + \frac{x^3}{2} + ... = x - x^2 + \frac{x^3}{3} - \frac{x^4}{6} + ...$$

- The maclaurin's series of $e^{\sin(x)} = 1 + \left(x - \frac{x^3}{6} + ...\right) + \frac{1}{2}\left(x - \frac{x^3}{6} + ...\right)^2 + ..$

$$= 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + ...$$

Maclaurin's series and limits

If a function f is not defined when $x = 0$, we study the value of the function when x is very close to 0.

If a value exists, it is called the limit of $f(x)$ when x tends to 0.

- When the limit is not obvious, work out the Maclaurin's series of the function and substitute x by 0 in the series (if possible) to obtain the limit.

Example:

$f(x) = \frac{e^x - 1}{x}$ $\lim_{x \to 0}\left(e^x - 1\right) = 0$ *and* $\lim_{x \to 0}(x) = 0$. Not only the function f is not defined at $x = 0$, but also its limit when x tends to 0

can not be determined ("$\frac{0}{0}$"). The Maclaurin's series of f *is* $f(x) = \frac{1}{x}\left(1 + x + \frac{x^2}{2} + ... - 1\right) = 1 + \frac{x}{2} + ...$

So when x tends to 0, $f(x)$ tends to $1 + \frac{0}{2} + ... = 1$ *or* $\lim_{x \to 0} f(x) = 1$