Limits and Improper integrals – exam questions

Question 1: Jan 2011

(a) Write $\frac{4}{4x+1} - \frac{3}{3x+2}$ in the form $\frac{C}{(4x+1)(3x+2)}$, where C is a constant. *(1 mark)*

(b) Evaluate the improper integral

$$\int_1^{\infty} \frac{10}{(4x+1)(3x+2)}\,dx$$

showing the limiting process used and giving your answer in the form $\ln k$, where k is a constant. *(6 marks)*

Question 2: June 2011

(a) Find $\int x^2 \ln x\,dx$. *(3 marks)*

(b) Explain why $\int_0^{e} x^2 \ln x\,dx$ is an improper integral. *(1 mark)*

(c) Evaluate $\int_0^{e} x^2 \ln x\,dx$, showing the limiting process used. *(3 marks)*

Question 3: Jan 2009

(a) Use integration by parts to show that $\int \ln x\,dx = x\ln x - x + c$, where c is an arbitrary constant. *(2 marks)*

(b) Hence evaluate $\int_0^{1} \ln x\,dx$, showing the limiting process used. *(4 marks)*

Question 4: June2006

(a) Show that $\lim_{a\to\infty}\left(\frac{3a+2}{2a+3}\right) = \frac{3}{2}$. *(2 marks)*

(b) Evaluate $\int_1^{\infty}\left(\frac{3}{3x+2} - \frac{2}{2x+3}\right)dx$, giving your answer in the form $\ln k$, where k is a rational number. *(5 marks)*

Question 5: June 2010

(a) Explain why $\int_1^{\infty} 4xe^{-4x}\,dx$ is an improper integral. *(1 mark)*

(b) Find $\int 4xe^{-4x}\,dx$. *(3 marks)*

(c) Hence evaluate $\int_1^{\infty} 4xe^{-4x}\,dx$, showing the limiting process used. *(3 marks)*

Question 6: Jan 2007

(a) Explain why $\int_0^{e} \frac{\ln x}{\sqrt{x}}\, dx$ is an improper integral. *(1 mark)*

(b) Use integration by parts to find $\int x^{-\frac{1}{2}} \ln x \, dx$. *(3 marks)*

(c) Show that $\int_0^{e} \frac{\ln x}{\sqrt{x}}\, dx$ exists and find its value. *(4 marks)*

Question 7: June 2007

(a) Write down the value of

$$\lim_{x \to \infty} x e^{-x}$$

(1 mark)

(b) Use the substitution $u = xe^{-x} + 1$ to find $\int \frac{e^{-x}(1-x)}{xe^{-x}+1}\, dx$. *(2 marks)*

(c) Hence evaluate $\int_1^{\infty} \frac{1-x}{x+e^{x}}\, dx$, showing the limiting process used. *(4 marks)*

Question 8: June 2009

Evaluate the improper integral

$$\int_1^{\infty} \left(\frac{1}{x} - \frac{4}{4x+1} \right) dx$$

showing the limiting process used and giving your answer in the form $\ln k$, where k is a constant to be found. *(5 marks)*

Limits and Improper integrals – exam questions - answers

Question 1: Jan 2011

(a)	$\frac{12x+8-12x-3}{(4x+1)(3x+2)}=\frac{5}{(4x+1)(3x+2)}$	B1	1
(b)	$\int\frac{10}{(4x+1)(3x+2)}\mathrm{d}x=2\int\left(\frac{4}{4x+1}-\frac{3}{3x+2}\right)\mathrm{d}x$	M1	
	$=2[\ln(4x+1)-\ln(3x+2)]\ (+c)$	A1	
	$\mathrm{I}=\lim_{a\to\infty}\int_1^a\left(\frac{10}{(4x+1)(3x+2)}\right)\mathrm{d}x$	M1	
	$=2\lim_{a\to\infty}[\ln(4a+1)-\ln(3a+2)]-(\ln 5-\ln 5)$		
	$=2\lim_{a\to\infty}\left[\ln\left(\frac{4a+1}{3a+2}\right)\right]=2\lim_{a\to\infty}\left[\ln\left(\frac{4+\frac{1}{a}}{3+\frac{2}{a}}\right)\right]$	m1,m1	
	$=2\ln\frac{4}{3}=\ln\frac{16}{9}$	A1	6
	Total		7

Question 2: June 2011

(a)	$\int x^2\ln x\,\mathrm{d}x=\frac{x^3}{3}\ln x-\int\frac{x^3}{3}\left(\frac{1}{x}\right)\mathrm{d}x$	M1	
		A1	
	$\ldots\ldots=\frac{x^3}{3}\ln x-\frac{x^3}{9}\ (+c)$	A1	3
(b)	Integrand is not defined at $x=0$	E1	1
(c)	$\int_0^e x^2\ln x\,\mathrm{d}x=\left\{\lim_{a\to 0}\int_a^e x^2\ln x\,\mathrm{d}x\right\}$		
	$=\left(\frac{e^3}{3}\ln e-\frac{e^3}{9}\right)-\lim_{a\to 0}\left[\frac{a^3}{3}\ln a-\frac{a^3}{9}\right]$	M1	
	But $\lim_{a\to 0}a^3\ln a=0$	E1	
	So $\int_0^e x^2\ln x\,\mathrm{d}x=\frac{2e^3}{9}$	A1	3
	Total		7

Question 3: Jan 2009

(a)	$\int\ln x\,\mathrm{d}x=x\ln x-\int x\left(\frac{1}{x}\right)\mathrm{d}x$	M1	
	$=x\ln x-x+c$	A1	2
(b)	$\int_0^1\ln x\,\mathrm{d}x=\lim_{a\to 0}\int_a^1\ln x\,\mathrm{d}x$	M1	
	$=\lim_{a\to 0}\{0-1-[a\ln a-a]\}$	M1	
	But $\lim_{a\to 0}a\ln a=0$	E1	
	So $\int_0^1\ln x\,\mathrm{d}x=-1$	A1	4
	Total		6

Question 4: June2006

(a)	$\Rightarrow\lim_{a\to\infty}\left(\frac{3+\frac{2}{a}}{2+\frac{3}{a}}\right)=\frac{3+0}{2+0}=\frac{3}{2}$	M1 A1	2
(b)	$\int_1^\infty\frac{3}{(3x+2)}-\frac{2}{2x+3}\,\mathrm{d}x$		
	$=[\ln(3x+2)-\ln(2x+3)]_1^\infty$	M1 A1	
	$=\left[\ln\left(\frac{3x+2}{2x+3}\right)\right]_1^\infty$	m1	
	$=\ln\left\{\lim_{a\to\infty}\left(\frac{3a+2}{2a+3}\right)\right\}-\ln 1$	M1	
	$=\ln\frac{3}{2}-\ln 1=\ln\frac{3}{2}$	A1	5
	Total		7

Question 5: June 2010

(a)	The interval of integration is infinite	E1	1
(b)	$\int 4xe^{-4x}\mathrm{d}x=-xe^{-4x}-\int -e^{-4x}\mathrm{d}x$	M1 A1	
	$=-xe^{-4x}-\frac{1}{4}e^{-4x}\ \{+c\}$	A1F	3
(c)	$\mathrm{I}=\int_1^\infty 4xe^{-4x}\,\mathrm{d}x=\lim_{a\to\infty}\int_1^a 4xe^{-4x}\,\mathrm{d}x$		
	$\lim_{a\to\infty}\{-ae^{-4a}-\frac{1}{4}e^{-4a}\}-\left[-\frac{5}{4}e^{-4}\right]$	M1	
	$\lim_{a\to\infty}a\,e^{-4a}=0$	M1	
	$\mathrm{I}=\frac{5}{4}e^{-4}$	A1	3
	Total		7

Question 6: Jan 2007

(a)	Integrand is not defined at $x=0$	E1	1
(b)	$\int x^{-\frac{1}{2}}\ln x\,\mathrm{d}x=2x^{\frac{1}{2}}\ln x-\int 2x^{\frac{1}{2}}\left(\frac{1}{x}\right)\mathrm{d}x$	M1	
		A1	
	$\ldots\ldots=2x^{\frac{1}{2}}\ln x-4x^{\frac{1}{2}}\ (+c)$	A1	3
(c)	$\int_0^e\frac{\ln x}{\sqrt{x}}\,\mathrm{d}x=\lim_{a\to 0}\int_a^e\frac{\ln x}{\sqrt{x}}\,\mathrm{d}x$	M1	
	$=-2e^{\frac{1}{2}}-\lim_{a\to 0}\left[2a^{\frac{1}{2}}\ln a-4a^{\frac{1}{2}}\right]$	M1	
	But $\lim_{a\to 0}a^{\frac{1}{2}}\ln a=0$	B1	
	So $\int_0^e\frac{\ln x}{\sqrt{x}}\,\mathrm{d}x$ exists and $=-2e^{\frac{1}{2}}$	A1	4
	Total		8

Question 7: June 2007

(a)	0	B1	1
(b)	$u = xe^{-x} + 1 \Rightarrow du = \left(e^{-x} - xe^{-x}\right)dx$	M1	
	$\int \frac{e^{-x}(1-x)}{xe^{-x}+1}\,dx = \int \frac{1}{u}\,du = \ln u + c$		
	$= \ln\left(xe^{-x}+1\right)\{+c\}$	A1	2
(c)	$\int \frac{1-x}{x+e^{x}}\,dx = \int \frac{e^{-x}(1-x)}{xe^{-x}+1}\,dx$	B1	
	$\int_1^{\infty} \frac{1-x}{x+e^{x}}\,dx = \lim_{a\to\infty} \left[\ln(xe^{-x}+1)\right]_1^a$		
	$= \lim_{a\to\infty}\left\{\ln\left(ae^{-a}+1\right)\right\} - \ln\left(e^{-1}+1\right)$	M1	
	$= \ln\left\{\lim_{a\to\infty}\left(ae^{-a}+1\right)\right\} - \ln\left(e^{-1}+1\right)$		
	$= \ln 1 - \ln\left(e^{-1}+1\right) = -\ln\left(e^{-1}+1\right)$	M1 A1	4
	Total		**7**

Question 8: June 2009

$\int\left(\frac{1}{x} - \frac{4}{4x+1}\right)dx = \ln x - \ln(4x+1)\{+c\}$	B1	
$I = \lim_{a\to\infty} \int_1^a \left(\frac{1}{x} - \frac{4}{4x+1}\right)dx$	M1	
$= \lim_{a\to\infty} \left[\ln x - \ln(4x+1)\right]_1^a$		
$= \lim_{a\to\infty} \left[\ln\left(\frac{a}{4a+1}\right) - \ln\frac{1}{5}\right]$	m1	
$= \lim_{a\to\infty} \left[\ln\left(\frac{1}{4+\frac{1}{a}}\right) - \ln\frac{1}{5}\right]$	m1	
$= \ln\frac{1}{4} - \ln\frac{1}{5} = \ln\frac{5}{4}$	A1	5
Total		**5**

Limits and McLaurin series – exam questions

Question 1: Jan 2011

(a) Write down the expansions in ascending powers of x up to and including the term in x^3 of:

(i) $\cos x + \sin x$; *(1 mark)*

(ii) $\ln(1+3x)$. *(1 mark)*

(b) It is given that $y = e^{\tan x}$.

(i) Find $\frac{dy}{dx}$ and show that $\frac{d^2y}{dx^2} = (1+\tan x)^2 \frac{dy}{dx}$. *(5 marks)*

(ii) Find the value of $\frac{d^3y}{dx^3}$ when $x = 0$. *(2 marks)*

(iii) Hence, by using Maclaurin's theorem, show that the first four terms in the expansion, in ascending powers of x, of $e^{\tan x}$ are

$$1 + x + \frac{1}{2}x^2 + \frac{1}{2}x^3$$ *(2 marks)*

(c) Find

$$\lim_{x \to 0} \left[\frac{e^{\tan x} - (\cos x + \sin x)}{x \ln(1+3x)} \right]$$ *(3 marks)*

Question 2: June 2011

(a) Given that $y = \ln(1 + 2\tan x)$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

(You may leave your expression for $\frac{d^2y}{dx^2}$ unsimplified.) *(4 marks)*

(b) Hence, using Maclaurin's theorem, find the first two non-zero terms in the expansion, in ascending powers of x, of $\ln(1 + 2\tan x)$. *(2 marks)*

(c) Find

$$\lim_{x \to 0} \left[\frac{\ln(1 + 2\tan x)}{\ln(1-x)} \right]$$ *(4 marks)*

Question 3: Jan 2009

The function f is defined by $f(x) = e^{2x}(1+3x)^{-\frac{2}{3}}$.

(a) (i) Use the series expansion for e^x to write down the first four terms in the series expansion of e^{2x}. *(2 marks)*

(ii) Use the binomial series expansion of $(1+3x)^{-\frac{2}{3}}$ and your answer to part (a)(i) to show that the first three non-zero terms in the series expansion of $f(x)$ are $1 + 3x^2 - 6x^3$. *(5 marks)*

(b) (i) Given that $y = \ln(1 + 2\sin x)$, find $\dfrac{d^2y}{dx^2}$. *(4 marks)*

(ii) By using Maclaurin's theorem, show that, for small values of x,

$$\ln(1 + 2\sin x) \approx 2x - 2x^2$$ *(2 marks)*

(c) Find

$$\lim_{x \to 0} \frac{1 - f(x)}{x \ln(1 + 2\sin x)}$$ *(3 marks)*

Question 4: June 2006

(a) (i) Write down the first three terms of the binomial expansion of $(1+y)^{-1}$, in ascending powers of y. *(1 mark)*

(ii) By using the expansion

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

and your answer to part (a)(i), or otherwise, show that the first three non-zero terms in the expansion, in ascending powers of x, of $\sec x$ are

$$1 + \frac{x^2}{2} + \frac{5x^4}{24}$$ *(5 marks)*

(b) By using Maclaurin's theorem, or otherwise, show that the first two non-zero terms in the expansion, in ascending powers of x, of $\tan x$ are

$$x + \frac{x^3}{3}$$ *(3 marks)*

(c) Hence find $\lim_{x \to 0} \left(\dfrac{x \tan 2x}{\sec x - 1} \right)$. *(4 marks)*

Question 5: June 2010

(a) Write down the expansion of $\cos 4x$ in ascending powers of x up to and including the term in x^4. Give your answer in its simplest form. *(2 marks)*

(b) (i) Given that $y = \ln(2 - e^x)$, find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$.

(You may leave your expression for $\frac{d^3y}{dx^3}$ unsimplified.) *(6 marks)*

(ii) Hence, by using Maclaurin's theorem, show that the first three non-zero terms in the expansion, in ascending powers of x, of $\ln(2 - e^x)$ are

$$-x - x^2 - x^3$$

(2 marks)

(c) Find

$$\lim_{x \to 0} \left[\frac{x \ln(2 - e^x)}{1 - \cos 4x} \right]$$

(3 marks)

Question 6: Jan 2007

The function f is defined by $f(x) = (1 + 2x)^{\frac{1}{2}}$.

(a) (i) Find $f'''(x)$. *(4 marks)*

(ii) Using Maclaurin's theorem, show that, for small values of x,

$$f(x) \approx 1 + x - \tfrac{1}{2}x^2 + \tfrac{1}{2}x^3$$

(4 marks)

(b) Use the expansion of e^x together with the result in part (a)(ii) to show that, for small values of x,

$$e^x(1 + 2x)^{\frac{1}{2}} \approx 1 + 2x + x^2 + kx^3$$

where k is a rational number to be found. *(3 marks)*

(c) Write down the first four terms in the expansion, in ascending powers of x, of e^{2x}. *(1 mark)*

(d) Find

$$\lim_{x \to 0} \frac{e^x(1 + 2x)^{\frac{1}{2}} - e^{2x}}{1 - \cos x}$$

(4 marks)

Question 7: June 2009

The function f is defined by

$$f(x) = (9 + \tan x)^{\frac{1}{2}}$$

(a) (i) Find $f''(x)$. *(4 marks)*

(ii) By using Maclaurin's theorem, show that, for small values of x,

$$(9 + \tan x)^{\frac{1}{2}} \approx 3 + \frac{x}{6} - \frac{x^2}{216}$$ *(3 marks)*

(b) Find

$$\lim_{x \to 0} \left[\frac{f(x) - 3}{\sin 3x}\right]$$ *(3 marks)*

Limits and McLaurin series – exam questions - answers

Question 1: Jan 2011

(a)(i)	$\cos x+\sin x=1+x-\frac{1}{2}x^2-\frac{1}{6}x^3$	B1	1
(ii)	$\ln(1+3x)=3x-\frac{1}{2}(3x)^2+\frac{1}{3}(3x)^3=3x-\frac{9}{2}x^2+9x^3$	B1	1
(b)(i)	$y=e^{\tan x}$, $\frac{dy}{dx}=\sec^2 x\, e^{\tan x}$	M1 A1	
	$\frac{d^2y}{dx^2}=2\sec^2 x\tan x\, e^{\tan x}+\sec^4 x\, e^{\tan x}$	m1 A1	
	$=\sec^2 x\, e^{\tan x}(2\tan x+\sec^2 x)$		
	$=\frac{dy}{dx}(2\tan x+1+\tan^2 x)$		
	$\frac{d^2y}{dx^2}=(1+\tan x)^2\frac{dy}{dx}$	A1	5
(ii)	$\frac{d^3y}{dx^3}=2(1+\tan x)\sec^2 x\frac{dy}{dx}+(1+\tan x)^2\frac{d^2y}{dx^2}$	M1	
	When $x=0$, $\frac{d^3y}{dx^3}=2(1)(1)(1)+(1)(1)=3$	A1	2
(iii)	$y(0)=1;\ y'(0)=1;\ y''(0)=1;\ y'''(0)=3;$		
	$y(x)\approx y(0)+x\,y'(0)+\frac{1}{2}x^2y''(0)+\frac{1}{3!}x^3y'''(0)$	M1	
	$e^{\tan x}\approx 1+x+\frac{1}{2}x^2+\frac{1}{2}x^3$	A1	2
(c)	$\lim_{x\to0}\left[\frac{e^{\tan x}-(\cos x+\sin x)}{x\ln(1+3x)}\right]$		
	$=\lim_{x\to0}\frac{1+x+\frac{x^2}{2}+\frac{x^3}{2}-1-x+\frac{x^2}{2}+\frac{x^3}{6}}{x\left(3x-\frac{9}{2}x^2+\ldots\right)}$	M1	
	$=\lim_{x\to0}\left[\frac{x^2+\frac{2}{3}x^3+..}{3x^2-\frac{9}{2}x^3...}\right]=\lim_{x\to0}\left[\frac{1+\frac{2}{3}x+..}{3-\frac{9}{2}x...}\right]$	m1	
	$=\frac{1}{3}$	A1	3
	Total		**14**

Question 2: June 2011

(a)	$\frac{dy}{dx}=\frac{2\sec^2 x}{1+2\tan x}$	M1 A1	
	$\frac{d^2y}{dx^2}=\frac{(1+2\tan x)(4\sec^2 x\tan x)-2\sec^2 x(2\sec^2 x)}{(1+2\tan x)^2}$	M1	
		A1	4
(b)	McC. Thm: $y(0)+x\,y'(0)+\frac{x^2}{2}y''(0)$		
	$(y(0)=0);\ y'(0)=2;\ y''(0)=-4$	M1	
	$\ln(1+2\tan x)\approx 2x-2x^2$	A1	2
(c)	$\ln(1-x)=-x-\frac{1}{2}x^2....$	B1	
	$\left[\frac{\ln(1+2\tan x)}{\ln(1-x)}\right]\approx\frac{2x-2x^2...}{-x-\frac{1}{2}x^2...}$	M1	
	$=\frac{2-2x...}{-1-\frac{1}{2}x...}$	m1	
	So $\lim_{x\to0}\left[\frac{\ln(1+2\tan x)}{\ln(1-x)}\right]=\frac{2}{-1}=-2$	A1F	4
	Total		**10**

Question 3: Jan 2009

(a)(i)	$e^{2x}=1+2x+2x^2+\frac{4}{3}x^3+...$	M1	
		A1	2
(ii)	$\{f(x)\}=e^{2x}(1+3x)^{-\frac{2}{3}}$		
	$(1+3x)^{-\frac{2}{3}}=1+\left(-\frac{2}{3}\right)(3x)+\frac{\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)(3x)^2}{2}-\frac{40}{3}x^3$	M1	
	$=1-2x+5x^2-\frac{40}{3}x^3$	A1	
	$\{f(x)\approx\}$ $1+2x+2x^2+\frac{4x^3}{3}-2x-4x^2-4x^3+5x^2+10x^3-\frac{40x^3}{3}$	m1 A1ft	
	$=1+3x^2-6x^3$	A1	5
b)(i)	$y=\ln(1+2\sin x)\Rightarrow\frac{dy}{dx}=\frac{1}{1+2\sin x}\times2\cos x$	M1 A1	
	$\frac{d^2y}{dx^2}=\frac{(1+2\sin x)(-2\sin x)-2\cos x(2\cos x)}{(1+2\sin x)^2}=\frac{-2(\sin x+2)}{(1+2\sin x)^2}$	M1	
		A1	4
(ii)	$y(0)=0,\ \ y'(0)=2,\ \ y''(0)=-4$	M1	
	McL Thm.: $\{\ln(1+2\sin x)\}\approx 0+2x-4\left(\frac{x^2}{2}\right)+..\approx 2x-2x^2$	A1	2
(c)	$\lim_{x\to0}\frac{1-f(x)}{x\ln(1+2\sin x)}=\lim_{x\to0}\frac{-3x^2+6x^3}{2x^2-2x^3}$	M1	
	$=\lim_{x\to0}\frac{-3+6x}{2-2x}$	m1	
	$=-\frac{3}{2}$	A1	3
	Total		**16**

Question 4: June 2006

(a)(i)	$(1+y)^{-1}=1-y+y^2......$	B1	1
(ii)	$\sec x\approx\frac{1}{1-\frac{x^2}{2}+\frac{x^4}{24}....}$	B1	
	$=\left[1-\frac{x^2}{2}+\frac{x^4}{24}....\right]^{-1}=$	M1	
	$\left\{1-\left(-\frac{x^2}{2}+\frac{x^4}{24}\right)+\left(-\frac{x^2}{2}+\frac{x^4}{24}\right)^2\right\}$	M1	
	$=\left\{1+\frac{x^2}{2}-\frac{x^4}{24}+\frac{x^4}{4}+...\right\}$		
	$=1+\frac{x^2}{2};+\frac{5x^4}{24}$	A1;A1	5
	Alternative: Those using Maclaurin		
	$f(x)=\sec x$		
	$f(0)=1$; <u>$f'(x)=\sec x\tan x$</u>; $\{f'(0)=0\}$	(B1)	
	$f''(x)=\sec x\tan^2 x+\sec^3 x$; $f''(0)=1$	(M1)	
	$f'''(x)=\sec x\tan^3 x+5\tan x\sec^3 x$;	(m1)	
	$f^{(iv)}(x)=\sec x\tan^4 x+18\tan^2 x\sec^3 x$ … $+5\sec^5 x\Rightarrow f^{(iv)}(0)=5$		
	$\sec x\approx$ printed result	(A2)	
(b)	$f(x)=\tan x$;		
	$f(0)=0$; $f'(x)=\sec^2 x$; $\{f'(0)=1\}$	B1	
	$f''(x)=2\sec x(\sec x\tan x)$; $f''(0)=0$		
	$f'''(x)=4\sec x\tan x(\sec x\tan x)+2\sec^4 x$	M1	
	$f'''(0)=2$		
	$\tan x=0+1x+0x^2+\frac{2}{3!}x^3\ldots=x+\frac{1}{3}x^3$	A1	3
	Alternative: Those using otherwise		
	$..=\frac{\sin x}{\cos x}\approx\left(x-\frac{x^3}{6}...\right)\left(1+\frac{x^2}{2}...\right)$	(M1) (A1)	
	$=x+\frac{x^3}{2}-\frac{x^3}{6}...=x+\frac{1}{3}x^3....$	(A1)	

(c)	$\left(\dfrac{x\tan 2x}{\sec x-1}\right)=\dfrac{x\left(2x+o(x^3)\right)}{\dfrac{x^2}{2}+o(x^4)}$	B1 M1	
	$=\dfrac{2+o(x^2)}{\dfrac{1}{2}+o(x^2)}$	M1	
	$\lim\limits_{x\to 0}\left(\dfrac{x\tan 2x}{\sec x-1}\right)=4$	A1√	4
	Total		**13**

Question 5: June 2010

5(a)	$\cos 4x\approx 1-\dfrac{(4x)^2}{2}+\dfrac{(4x)^4}{4!}\ldots$	M1	
	$\approx 1-8x^2+\dfrac{32}{3}x^4\ldots$	A1	2
b)(i)	$\dfrac{dy}{dx}=\dfrac{1}{2-e^x}\times(-e^x)$	M1 A1	
	$\dfrac{d^2y}{dx^2}=\dfrac{(2-e^x)(-e^x)-(-e^x)(-e^x)}{(2-e^x)^2}$	M1 A1	
	$=\dfrac{-2e^x}{(2-e^x)^2}$		
	$\dfrac{d^3y}{dx^3}=\dfrac{(2-e^x)^2(-2e^x)-(-2e^x)2(2-e^x)(-e^x)}{(2-e^x)^4}$	m1	
		A1	6
(ii)	$y(0)=0;\ y'(0)=-1;\ y''(0)=-2;\ y'''(0)=-6$	M1	
	$\text{Ln}(2-e^x)\approx y(0)+xy'(0)+\dfrac{x^2}{2}y''(0)+\dfrac{x^3}{6}y'''(0)\ldots$		
	$\ldots\approx -x-x^2-x^3\ldots$	A1	2
(c)	$\left[\dfrac{x\ln(2-e^x)}{1-\cos 4x}\right]\approx\dfrac{-x^2-x^3-x^4\ldots}{8x^2-\dfrac{32}{3}x^4}$	M1	
	Limit $=\lim\limits_{x\to 0}\dfrac{-x^2-o(x^3)}{8x^2-o(x^4)}$		
	$\ldots = \lim\limits_{x\to 0}\dfrac{-1-o(x)}{8-o(x^2)}$	m1	
	$\ldots = -\dfrac{1}{8}$	A1	3
	Total		**13**

Question 6: Jan 2007

(a)(i)	$f'(x)=\frac{1}{2}(1+2x)^{-\frac{1}{2}}(2)=(1+2x)^{-\frac{1}{2}}$	M1A1	
	$f''(x)=-(1+2x)^{-\frac{3}{2}}$	A1F	
	$f'''(x)=3(1+2x)^{-\frac{5}{2}}$	A1	4
(ii)	$f(x)=(1+2x)^{\frac{1}{2}}\Rightarrow f(0)=1;$	B1	
	$f'(0)=1;\ f''(0)=-1;\ f'''(0)=3$	M1	
		A1F	
	$f(x)=f(0)+xf'(0)+\dfrac{x^2}{2}f''(0)+\dfrac{x^3}{6}f'''(0)$		
	$\ldots\approx 1+x-\dfrac{x^2}{2}+\dfrac{x^3}{2}$	A1	4
(b)	$e^x(1+2x)^{\frac{1}{2}}\approx$		
	$\left(1+x+\dfrac{x^2}{2}+\dfrac{x^3}{6}\right)\left(1+x-\dfrac{x^2}{2}+\dfrac{x^3}{2}\right)$	M1	
	$\approx 1+x(1+1)+x^2(-0.5+1+0.5)$	A1	
	$+x^3\left(\dfrac{1}{2}-\dfrac{1}{2}+\dfrac{1}{2}+\dfrac{1}{6}\right)$		
	$\approx 1+2x+x^2+\dfrac{2}{3}x^3$	A1	3
(c)	$e^{2x}=1+2x+\dfrac{(2x)^2}{2}+\dfrac{(2x)^3}{6}+\ldots$	B1	1
	$=1+2x+2x^2+\dfrac{4}{3}x^3+\ldots$		
(d)	$1-\cos x=\dfrac{1}{2}x^2+\{o(x^4)\}$	B1	
	$\dfrac{e^x(1+2x)^{\frac{1}{2}}-e^{2x}}{1-\cos x}=$		
	$\dfrac{1+2x+x^2+\frac{2}{3}x^3-\left[1+2x+2x^2+\frac{4}{3}x^3\right]}{\frac{1}{2}x^2+\{o(x^4)\}}$	M1	
	$\lim\limits_{x\to 0}\ldots=\lim\limits_{x\to 0}\dfrac{-x^2+\{o(x^3)\}}{\frac{1}{2}x^2+\{o(x^4)\}}=$	A1F	
	$\lim\limits_{x\to 0}\dfrac{-1+o(x)}{\frac{1}{2}+o(x^2)}=-2$	A1F	4
	Total		**16**

Question 7: June 2009

(a)(i)	$f(x)=(9+\tan x)^{\frac{1}{2}}$		
	so $f'(x)=\frac{1}{2}(9+\tan x)^{-\frac{1}{2}}\sec^2 x$	M1 A1	
	$f''(x)=-\frac{1}{4}(9+\tan x)^{-\frac{3}{2}}\sec^4 x$	M1	
	$+\frac{1}{2}(9+\tan x)^{-\frac{1}{2}}(2\sec^2 x\tan x)$	A1	4
a)(ii)	$f(0)=3$	B1	
	$f'(0)=\frac{1}{2}(9)^{-\frac{1}{2}}=\frac{1}{6};$	M1	
	$f''(0)=-\frac{1}{4}(9)^{-\frac{3}{2}}=-\frac{1}{108}$		
	$f(x)\approx f(0)+xf'(0)+\frac{1}{2}x^2f''(0)$		
	$(9+\tan x)^{\frac{1}{2}}\approx 3+\dfrac{x}{6}-\dfrac{x^2}{216}$	A1	3
(b)	$\dfrac{f(x)-3}{\sin 3x}\approx\dfrac{\frac{x}{6}-\frac{x^2}{216}\ldots}{3x-\frac{(3x)^3}{3!}\ldots}$	M1	
	$\approx\dfrac{\frac{1}{6}-\frac{x}{216}\ldots}{3-\ldots}$	m1	
	$\lim\limits_{x\to 0}\left[\dfrac{f(x)-3}{\sin 3x}\right]=\dfrac{1}{18}$	A1	3
	Total		**10**