Roots of polynomials

Specifications: Roots of Polynomials

The relations between the roots and the coefficients of a polynomial equation;
the occurrence of the non-real roots in conjugate pairs when the coefficients of the polynomial are real.

Warm up

The quadratic equation

$$2x^2 - x + 4 = 0$$

has roots α and β.

(a) Write down the values of $\alpha + \beta$ and $\alpha\beta$. *(2 marks)*

(b) Show that $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{1}{4}$. *(2 marks)*

(c) Find a quadratic equation with integer coefficients such that the roots of the equation are

$$\frac{4}{\alpha} \text{ and } \frac{4}{\beta}$$ *(3 marks)*

Roots of cubic polynomials

The polynomial $P(x) = ax^3 + bx^2 + cx + d$ has roots α, β, γ.
It then can be factorised as $a(x-\alpha)(x-\beta)(x-\gamma)$.

$a)$ Expand the brackets and arrange the polynomial in descending power of x.
$b)$ Identifying the coefficents of the expansion and P,
deduce, in terms of a, b, c and d the following

$i)\ \alpha + \beta + \gamma$

$ii)\ \alpha\beta + \alpha\gamma + \beta\gamma$

$iii)\ \alpha\beta\gamma$

Summary

If the cubic equation $ax^3 + bx^2 + cx + d = 0$ has roots α, β and γ, then

$$\sum \alpha = -\frac{b}{a},$$

$$\sum \alpha\beta = \frac{c}{a},$$

$$\alpha\beta\gamma = -\frac{d}{a}$$

Consequence

The numbers α, β, γ are the roots of the equation

$$x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \alpha\gamma + \beta\gamma)x - \alpha\beta\gamma = 0$$

$$x^3 - \left(\sum \alpha\right)x^2 + \left(\sum \alpha\beta\right)x - \alpha\beta\gamma = 0$$

Be careful: $\sum \alpha$, $\sum \alpha\beta$ are only notations

$\sum \alpha$ means sum of the roots

$\sum \alpha\beta$ means the sum of the pair products

etc...

Exercises

Question 1:

The given equations have roots α,β,γ.

In each case, write $\sum\alpha, \sum\alpha\beta$ *and* $\alpha\beta\gamma$

$a)\, x^3+2x^2-3x+7=0$ $\qquad b)\, 2z^3-4z^2+3z-1=0$

$c)\, 3z^3+z-9=0$ $\qquad d)\, z^3+iz^2-(2+i)z+9=0$

Question 2:

α,β and γ are the roots of a cubic equation.

In each case, write a possible equations.

a) $\alpha=1, \beta=-1,\ \gamma=3$ $\qquad b)\, \alpha=2,\ \beta=i,\ \gamma=-1$

$c)\, \alpha=\frac{1}{2},\ \beta=\frac{5}{6},\ \gamma=-1$ $\qquad d)\, \alpha=1+i, \beta=\alpha^*, \gamma=\frac{1}{\alpha}$

Question 3:

The equation $x^3-2x^2+5x+8=0$ has roots α,β,γ

$a)$ Write $i)\, \alpha+\beta+\gamma$

$ii)\, \alpha\beta+\alpha\gamma+\beta\gamma$

$iii)\, \alpha\beta\gamma$

$b)$ *Work out* $\alpha^2+\beta^2+\gamma^2$

$c)$ Give an equation with roots

$i)\ 2\alpha, 2\beta, 2\gamma$

$ii)\, \frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$

Question1:

	$\sum\alpha$	$\sum\alpha\beta$	$\alpha\beta\gamma$
a)	-2	-3	-7
b)	2	1.5	0.5
c)	0	$\frac{1}{3}$	3
d)	$-i$	$-2-i$	-9

Question2:

$a)\sum\alpha=3 \qquad \sum\alpha\beta=-1+3-3=-1 \qquad \alpha\beta\gamma=-3$

α,β *and* γ are the roots of the eqaution: $x^3-3x^2-x+3=0$

$b)\, x^3-(1+i)x^2+(-2+i)x+2i=0$

$c)\, x^3-\frac{1}{3}x^2-\frac{11}{12}x+\frac{5}{12}=0$ *or* $12x^3-4x^2-11x+5=0$

$d)\, x^3-\frac{1}{2}(5-i)x^2+(3-i)x-(1-i)=0$

Question3:

$a) i)\, 2 \quad ii) 5 \quad iii) -8$

$b)\, \alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)=2^2-2\times5=-6$

$c)$ Let $u=2\alpha$ *then* $\alpha=\frac{u}{2}$ *being a root, we have:*

$\alpha^3-2\alpha^2+5\alpha+8=0$

$\Leftrightarrow \quad \left(\frac{u}{2}\right)^3-2\left(\frac{u}{2}\right)^2+5\left(\frac{u}{2}\right)+8=0$

$\Leftrightarrow \quad \frac{u^3}{8}-\frac{u^2}{2}+\frac{5u}{2}+8=0$

$\Leftrightarrow \quad u^3-4u^2+20u+64=0$

$ii)$ Using $u=\frac{1}{\alpha}$ *or* $\alpha=\frac{1}{u}$, *we have*

$\left(\frac{1}{u}\right)^3-2\left(\frac{1}{u}\right)^2+5\left(\frac{1}{u}\right)+8=0$

$\times u^3: \quad 1-2u+5u^2+8u^3=0$

The cubic equation $x^3 - 3x^2 + 4 = 0$ has roots α, β and γ.
Find the cubic equations with roots $2\alpha, 2\beta$ and 2γ

Method 1

Work out

$\sum\alpha =$ $\sum\alpha\beta =$ $\alpha\beta\gamma =$

Let $u = 2\alpha, v = 2\beta$ *and* $w = 2\gamma$

and work out

$u + v + w =$

$uv + uw + vw =$

$uvw =$

then build the cubc equation using these coefficients

Method 2

Let $u = 2\alpha$.

Then $\alpha = \frac{u}{2}$ being a root of the equation,

you have $\alpha^3 - 3\alpha^2 + 4 = 0$

$$\Leftrightarrow \left(\frac{u}{2}\right)^3 - 3\left(\frac{u}{2}\right)^2 + 4 = 0 \ldots$$

Simplify to obtain the equations

(replace "u" by "x" if you prefer)

$$x^3 - 6x^2 + 32 = 0$$

Apply these methods to this exercise

The cubic equation $x^3 - x^2 - 4x - 7 = 0$ has roots α, β and γ. Using the first method described above, find the cubic equations whose roots are

(a) $3\alpha, 3\beta$ and 3γ, (b) $\alpha + 1, \beta + 1$ and $\gamma + 1$, (c) $\frac{2}{\alpha}, \frac{2}{\beta}$ and $\frac{2}{\gamma}$.

Complex roots and complex coefficients

Property 1: The equation $ax^3 + bx^2 + cx + d = 0$ has roots α, β, γ.

a, b, c, d are all REAL numbers.

Case 1: α, β and γ are three real roots

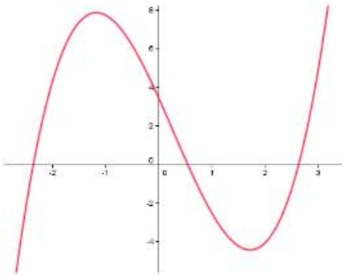
OR

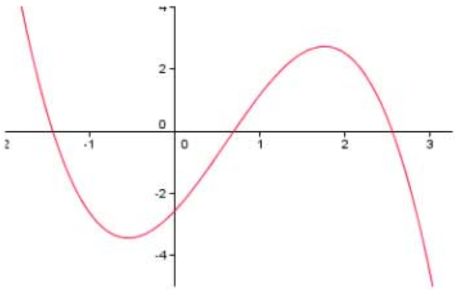
Case 2: α is real and β and γ are complex roots

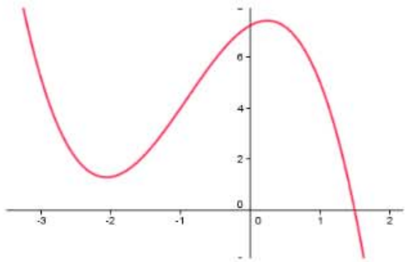
then β and γ are CONJUGATE

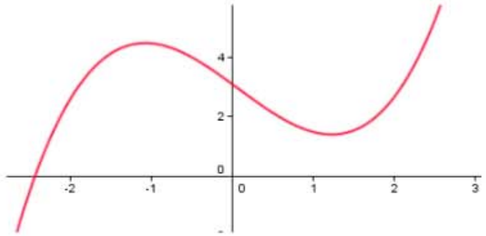
(Consider the graph with equation $y=ax^3+bx^2+cx+d$

How many time does the graph cross the x-axis?)









Be careful, if the coefficents are not all real,
this property does not apply.

Exercises

1) The cubic equation $x^3 - 3x^2 + x + k = 0$, where k is real, has one root equal to $2 - \mathrm{i}$. Find the other two roots and the value of k.

2) The quartic equation $x^4 + 2x^3 + 14x + 15 = 0$ has one root equal to $1 + 2\mathrm{i}$. Find the other three roots.

3) A cubic equation has real coefficients. One root is 2 and another is $1 + \mathrm{i}$. Find the cubic equation in the form $x^3 + ax^2 + bx + c = 0$.

4) The cubic equation $x^3 - 2x^2 + 9x - 18 = 0$ has one root equal to $3\mathrm{i}$. Find the other two roots.

5) The quartic equation $4x^4 - 8x^3 + 9x^2 - 2x + 2 = 0$ has one root equal to $1 - \mathrm{i}$. Find the other three roots.

Answers:

1) One root $\alpha = 2 - i$ and the coefficients of the equation are real

so we know that $\beta = \alpha^* = 2 + 1$

We also know that $\alpha + \beta + \gamma = -\frac{b}{a} = 3$

$$2 - i + 2 + i + \gamma = 3 \qquad \gamma = -1$$

2) $1 + 2i$, $1 - 2i$, -3 , -1

3) $x^3 - 4x^2 + 6x - 4$

4) $3i, -3i, 2$

5) $1 - i$, $1 + i$, $\frac{1}{2}i$, $-\frac{1}{2}i$

Miscellaneous exercises

1. The equation

$$x^3 - 3x^2 + px + 4 = 0,$$

where p is a constant, has roots $\alpha - \beta$, α and $\alpha + \beta$, where $\beta > 0$.

(a) Find the values of α and β.

(b) Find the value of p.

[NEAB June 1998]

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2. The numbers α, β and γ satisfy the equations

$$\alpha^2 + \beta^2 + \gamma^2 = 22$$

and

$$\alpha\beta + \beta\gamma + \gamma\alpha = -11.$$

(a) Show that $\alpha + \beta + \gamma = 0$.

(b) The numbers α, β and γ are also the roots of the equation

$$x^3 + px^2 + qx + r - 0,$$

where p, q and r are real.

(i) Given that $\alpha = 3 + 4\mathrm{i}$ and that γ is real, obtain β and γ.

(ii) Calculate the product of the three roots.

(iii) Write down, or determine, the values of p, q and r.

[AQA June 200]

3. The roots of the cubic equation

$$2x^3 + 3x + 4 = 0$$

are α, β and γ.

(a) Write down the values of $\alpha + \beta + \gamma$, $\alpha\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$.

(b) Find the cubic equation, with integer coefficients, having roots $\alpha\beta$, $\beta\gamma$ and $\gamma\alpha$.

[AQA March 2000]

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4. The roots of the equation

$$7x^3 - 8x^2 + 23x + 30 = 0$$

are α, β and γ.

(a) Write down the value of $\alpha + \beta + \gamma$.

(b) Given that $1 + 2\mathrm{i}$ is a root of the equation, find the other two roots.

[AQA Specimen]

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5. The roots of the cubic equation

$$x^3 + px^2 + qx + r = 0,$$

where p, q and r are real, are α, β and γ.

(a) Given that $\alpha + \beta + \gamma = 3$, write down the value of p.

(b) Given also that

$$\alpha^2 + \beta^2 + \gamma^2 = -5,$$

(i) find the value of q,

(ii) explain why the equation must have two non-real roots and one real root.

(c) One of the two non-real roots of the cubic equation is $3 - 4\mathrm{i}$.

(i) Find the real root.

(ii) Find the value of r.

[AQA March 1999]

Turn over for the answers

Answers:

1. (a) $\alpha = 1,\ \beta = \sqrt{5}$

   (b) -2

2. (b) (i) $\beta = 3 - 4\text{i},\ \gamma = -6$

   (ii) -150

   (iii) $0, -11, 150$

3. (a) $\sum \alpha = 0 \quad \sum \alpha\beta = \frac{3}{2} \quad \alpha\beta\gamma = -2$

   (b) $2x^3 - 3x^2 - 8 = 0$

4. (a) $\frac{8}{7}$

   (b) $1 - 2\text{i}\,,\ \frac{-6}{7}$

5. (a) $p = -3$

   (b) (i) $q = 7$

   (ii) $\sum \alpha^2 < 0$

   (c) (i) -3

   (ii) 75

Key points

A polynomial $ax^3+bx^2+cx+d=0$ has roots α, β, γ

- The sum of the roots: $\sum \alpha = \alpha+\beta+\gamma = -\frac{b}{a}$
- The sum of the double products: $\sum \alpha\beta = \alpha\beta+\alpha\gamma+\beta\gamma = \frac{c}{a}$
- The product of the roots: $\alpha\beta\gamma = -\frac{d}{a}$

If we are given the values of

a) the sums of the roots,

b) the sum of pairs of products and

c) the product of all the roots,

then we can form the corresponding cubic equation :

$x^3-(\text{sum of roots})x^2+(\text{sum of products of pairs})x-(\text{product of roots})=0$

or using the notations

$x^3-\left(\sum\alpha\right)x^2+\left(\sum\alpha\beta\right)x-(\alpha\beta\gamma)=0$

Identities to remember:

- $\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)$

Using the notations:

$$\sum(\alpha^2)=\left(\sum\alpha\right)^2-2\sum\alpha\beta$$

- $\alpha^3+\beta^3+\gamma^3=(\alpha+\beta+\gamma)^3-3(\alpha\beta+\alpha\gamma+\beta\gamma)(\alpha+\beta+\gamma)-3\alpha\beta\gamma$

If all the coefficients of the polynomial (of order 3) are REAL numbers,
there are either:

- 3 real roots
- 1 real root and 2 complex CONJUGATE roots

If the coefficients of the polynomial are complex numbers, there are no rules

Consider a polynomial of order n (degree n):

$a_nz^n+a_{n-1}z^{n-1}+a_{n-2}z^{n-2}+\ldots+a_2z^2+a_1z+a_0=0$

where $a_n, a_{n-1}, .., a_1, a_0$ are numbers

This polynomial has roots $\alpha_1, \alpha_2, \ldots \alpha_n$

- The sum of the roots: $\sum \alpha_i = \alpha_1+\alpha_2+\alpha_3+\ldots+\alpha_n = -\frac{a_{n-1}}{a_n}$
- The sum of the double products: $\sum \alpha\beta = \frac{a_{n-2}}{a_n}$
- The sum of the triple products: $\sum \alpha\beta\gamma = -\frac{a_{n-3}}{a_n}$

....

- The product of the roots: $\alpha_1\alpha_2\alpha_3\ldots\alpha_n = (-1)^n\frac{a_0}{a_n}$

When all the coefficients of the polynomial are REAL numbers,
If a root α is a complex number, then its conjugate α^* is also a root