Roots of polynomials – exam questions

Question 1: Jan 2006

The cubic equation

$$x^3 + px^2 + qx + r = 0$$

where p, q and r are real, has roots α, β and γ.

(a) Given that

$$\alpha + \beta + \gamma = 4 \qquad \text{and} \qquad \alpha^2 + \beta^2 + \gamma^2 = 20$$

find the values of p and q. *(5 marks)*

(b) Given further that one root is $3 + \mathrm{i}$, find the value of r. *(5 marks)*

Question 2: Jan 2007

The cubic equation

$$z^3 + 2(1 - \mathrm{i})z^2 + 32(1 + \mathrm{i}) = 0$$

has roots α, β and γ.

(a) It is given that α is of the form $k\mathrm{i}$, where k is real. By substituting $z = k\mathrm{i}$ into the equation, show that $k = 4$. *(5 marks)*

(b) Given that $\beta = -4$, find the value of γ. *(2 marks)*

Question 3: Jan 2008

The cubic equation

$$z^3 + \mathrm{i}z^2 + 3z - (1 + \mathrm{i}) = 0$$

has roots α, β and γ.

(a) Write down the value of:

(i) $\alpha + \beta + \gamma$; *(1 mark)*

(ii) $\alpha\beta + \beta\gamma + \gamma\alpha$; *(1 mark)*

(iii) $\alpha\beta\gamma$. *(1 mark)*

(b) Find the value of:

(i) $\alpha^2 + \beta^2 + \gamma^2$; *(3 marks)*

(ii) $\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2$; *(4 marks)*

(iii) $\alpha^2\beta^2\gamma^2$. *(2 marks)*

(c) Hence write down a cubic equation whose roots are α^2, β^2 and γ^2. *(2 marks)*

Question 4: Jan 2009

It is given that α, β and γ satisfy the equations

$$\begin{aligned}\alpha + \beta + \gamma &= 1\\ \alpha^2 + \beta^2 + \gamma^2 &= -5\\ \alpha^3 + \beta^3 + \gamma^3 &= -23\end{aligned}$$

(a) Show that $\alpha\beta + \beta\gamma + \gamma\alpha = 3$. *(3 marks)*

(b) Use the identity

$$(\alpha + \beta + \gamma)(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha) = \alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma$$

to find the value of $\alpha\beta\gamma$. *(2 marks)*

(c) Write down a cubic equation, with integer coefficients, whose roots are α, β and γ. *(2 marks)*

(d) Explain why this cubic equation has two non-real roots. *(2 marks)*

(e) Given that α is real, find the values of α, β and γ. *(4 marks)*

Question 5: Jan 2010

The cubic equation

$$2z^3 + pz^2 + qz + 16 = 0$$

where p and q are real, has roots α, β and γ.

It is given that $\alpha = 2 + 2\sqrt{3}\,\mathrm{i}$.

(a) (i) Write down another root, β, of the equation. *(1 mark)*

(ii) Find the third root, γ. *(3 marks)*

(iii) Find the values of p and q. *(3 marks)*

Question 6: June 2007

The cubic equation

$$z^3 + pz^2 + 6z + q = 0$$

has roots α, β and γ.

(a) Write down the value of $\alpha\beta + \beta\gamma + \gamma\alpha$. *(1 mark)*

(b) Given that p and q are real and that $\alpha^2 + \beta^2 + \gamma^2 = -12$:

(i) explain why the cubic equation has two non-real roots and one real root; *(2 marks)*

(ii) find the value of p. *(4 marks)*

(c) One root of the cubic equation is $-1 + 3\mathrm{i}$.

Find:

(i) the other two roots; *(3 marks)*

(ii) the value of q. *(2 marks)*

Question 7: June 2006

The cubic equation

$$z^3 - 4iz^2 + qz - (4 - 2i) = 0$$

where q is a complex number, has roots α, β and γ.

(a) Write down the value of:

(i) $\alpha + \beta + \gamma$; *(1 mark)*

(ii) $\alpha\beta\gamma$. *(1 mark)*

(b) Given that $\alpha = \beta + \gamma$, show that:

(i) $\alpha = 2i$; *(1 mark)*

(ii) $\beta\gamma = -(1 + 2i)$; *(2 marks)*

(iii) $q = -(5 + 2i)$. *(3 marks)*

(c) Show that β and γ are the roots of the equation

$$z^2 - 2iz - (1 + 2i) = 0$$ *(2 marks)*

(d) Given that β is real, find β and γ. *(3 marks)*

Question 8: June 2008

The cubic equation

$$z^3 + qz + (18 - 12i) = 0$$

where q is a complex number, has roots α, β and γ.

(a) Write down the value of:

(i) $\alpha\beta\gamma$; *(1 mark)*

(ii) $\alpha + \beta + \gamma$. *(1 mark)*

(b) Given that $\beta + \gamma = 2$, find the value of:

(i) α; *(1 mark)*

(ii) $\beta\gamma$; *(2 marks)*

(iii) q. *(3 marks)*

(c) Given that β is of the form ki, where k is real, find β and γ. *(4 marks)*

Question 9: June 2009

The cubic equation

$$z^3 + pz^2 + 25z + q = 0$$

where p and q are real, has a root $\alpha = 2 - 3\mathrm{i}$.

(a) Write down another non-real root, β, of this equation. *(1 mark)*

(b) Find:

(i) the value of $\alpha\beta$; *(1 mark)*

(ii) the third root, γ, of the equation; *(3 marks)*

(iii) the values of p and q. *(3 marks)*

Roots of polynomials – exam questions - answers

Question 1: Jan 2006

$a)\ x^3 + px^2 + qx + r = 0 \quad has\ three\ roots\ \alpha, \beta\ and\ \gamma.$

$\alpha + \beta + \gamma = 4 \quad so\ p = -4$

$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$20 = (4)^2 - 2q$

$q = -2$

$b)\ p, q\ and\ r\ are\ REAL\ numbers\ so\ if\ 3+i\ is\ a\ root, then\ its\ conjugate\ is\ also\ a\ root$

$\alpha = 3+i,\ \beta = 3-i$

$\alpha + \beta + \gamma = 4\ \ gives\ \ 6 + \gamma = 4 \quad and\ \ \gamma = -2$

$r = -\alpha\beta\gamma = -(3+i)(3-i)(-2) = (9 - i^2) \times 2 = 20 \qquad r = 20$

Question 2: Jan 2007

$a)\ z^3 + 2(1-i)z^2 + 32(1+i) = 0\ \ has\ roots\ \alpha, \beta, \gamma$

$\alpha = ki\ so$

$(ki)^3 + 2(1-i)(ki)^2 + 32 + 32i = 0$

$-ik^3 - 2k^2 + 2ik^2 + 32 + 32i = 0$

$(-2k^2 + 32) + i(-k^3 + 2k^2 + 32) = 0$

$This\ gives$

$-2k^2 + 32 = 0 \quad and \quad -k^3 + 2k^2 + 32 = 0$

The first equation gives $k = 4\ or\ k = -4$

$-(-4)^3 + 2 \times (-4)^2 + 32 = 64 + 32 + 32 = 128 \quad k \neq -4$

$-(4)^3 + 2 \times (4)^2 + 32 = -64 + 32 + 32 = 0 \quad k = 4$

$b)\ \alpha = 4i, \beta = -4\ and\ we\ know\ that$

$\alpha + \beta + \gamma = -2(1-i)$

$4i - 4 + \gamma = -2 + 2i$

$\gamma = 2 - 2i$

Question 3: Jan 2008

$z^3 + iz^2 + 3z - (1+i) = 0\ has\ roots\ \alpha, \beta, \gamma.$

$a)\ i)\ \alpha + \beta + \gamma = -i \qquad ii)\ \alpha\beta + \alpha\gamma + \beta\gamma = 3 \qquad iii)\ \alpha\beta\gamma = 1 + i$

$b)\ i)\ \alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$= (-i)^2 - 2 \times 3$

$\alpha^2 + \beta^2 + \gamma^2 = -7$

$ii)\ \alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2 = (\alpha\beta + \alpha\gamma + \beta\gamma)^2 - 2(\alpha^2\beta\gamma + \beta^2\alpha\gamma + \gamma^2\alpha\beta)$

$= (\alpha\beta + \alpha\gamma + \beta\gamma)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma)$

$= (3)^2 - 2 \times (1+i)(-i) = 9 + 2i + 2i^2$

$\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2 = 7 + 2i$

$iii)\ \alpha^2\beta^2\gamma^2 = (\alpha\beta\gamma)^2 = (1+i)^2 = \qquad \alpha^2\beta^2\gamma^2 = 2i$

$c)\ z^3 - (-7)z^2 + (7+2i)z - 2i = 0 \qquad z^3 + 7z^3 + (7+2i)z - 2i = 0$

Question 4: Jan 2009

$a)\left(\alpha+\beta+\gamma\right)^2=\alpha^2+\beta^2+\gamma^2+2\left(\alpha\beta+\alpha\gamma+\gamma\beta\right)$

$1^2=-5+2\left(\alpha\beta+\alpha\gamma+\gamma\beta\right)$

$so\ \ \alpha\beta+\alpha\gamma+\gamma\beta=3$

$b)\left(\alpha+\beta+\gamma\right)\left(\alpha^2+\beta^2+\gamma^2-\alpha\beta-\alpha\gamma-\gamma\beta\right)=\alpha^3+\beta^3+\gamma^3-3\alpha\gamma\beta$

$1\times\left(-5-3\right)=-23-3\alpha\beta\gamma$

$so\ \ \alpha\beta\gamma=-5$

$c)\, z^3-(\alpha+\beta+\gamma)z^2+\left(\alpha\beta+\alpha\gamma+\gamma\beta\right)z-\alpha\beta\gamma=0$

$z^3-z^2+3z+5=0$

$d)\,\alpha^2+\beta^2+\gamma^2=-5<0$ so at least one of the root is complex;

And because the coefficients of the equation are REAL,

its conjugate is also a root.

$e)\, z^3-z^2+3z+5=0$ *has an "obvious" root* $:\alpha=-1$

indeed $: (-1)^3-(-1)^2+3\times(-1)+5=-1-1-3+5=0$

Factorise the polynomial $(z+1)(z^2-2z+5)=0$

Discriminant of $z^2-2z+5:\quad (-2)^2-4\times1\times5=-16=(4i)^2$

$\beta=\dfrac{2+4i}{2}$ *and* $\gamma=\dfrac{2-4i}{2}$

$\alpha=-1,\ \beta=1+2i\ ,\ \gamma=1-2i$

Question 5: Jan 2010

$2z^3+pz^2+qz+16=0$ *has roots* $\alpha,\beta,\gamma.$

p and q are REAL numbers

$\alpha=2+2i\sqrt{3}$

$a)i)$ Since the coefficients of the equation are real numbers,

α^* *is also a root so* $\beta=2-2i\sqrt{3}$

$ii)\,\alpha\beta\gamma=-\dfrac{16}{2}=-8$

$\alpha\beta\gamma=(2+2i\sqrt{3})(2-2i\sqrt{3})\gamma=(4+12)\gamma=16\gamma$

$so\ \ \gamma=-\dfrac{1}{2}$

$iii)-\dfrac{p}{2}=\alpha+\beta+\gamma=2+2i\sqrt{3}+2-2i\sqrt{3}-\dfrac{1}{2}$

$p=-2(4-\dfrac{1}{2})=\ p=-7$

$\dfrac{q}{2}=\alpha\beta+\alpha\gamma+\beta\gamma=\alpha\beta+\gamma\left(\alpha+\beta\right)=16-\dfrac{1}{2}\times4$

$q=28$

Question 6: June 2007

$a)\ \alpha\beta + \beta\gamma + \alpha\gamma = 6$

$b)\ i)\ \alpha^2 + \beta^2 + \gamma^2 = -12 < 0$

This can only happens if one of the root is not a real number so if α is a complex number,

then $\beta = \alpha^*$ *because p and q are real numbers and* γ *is real*

(because otherwise γ^* would be a root too, making 4 roots instead of the expected 3)

$ii)\ \alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma)$

$-12 = (\alpha + \beta + \gamma)^2 - 2 \times 6$

$-12 = (\alpha + \beta + \gamma)^2 - 12 \qquad \alpha + \beta + \gamma = 0$

$So\ \ p = -(\alpha + \beta + \gamma) = 0 \qquad p = 0$

$c)\ \alpha = -1 + 3i \qquad \beta = \alpha^* = -1 - 3i$

$\alpha + \beta + \gamma = 0 \qquad -1 + 3i - 1 - 3i + \gamma = 0 \qquad \gamma = 2$

$ii)\ q = -\alpha\beta\gamma = -(-1 + 3i)(-1 - 3i)(2) = -2(1 + 9) = -20$

Question 7: June 2006

$z^3 - 4iz^2 + qz - (4 - 2i) = 0\ \ has\ \ roots\ \ \alpha, \beta, \gamma$

$a)\ i)\ \alpha + \beta + \gamma = 4i \qquad ii)\ \alpha\beta\gamma = 4 - 2i$

$b)\ \alpha = \beta + \gamma$

$i)\ \alpha + \beta + \gamma = 4i\ \ becomes \quad \alpha + \alpha = 4i \quad so \quad \alpha = 2i$

$ii)\ \alpha\beta\gamma = 4 - 2i$

$$\beta\gamma = \frac{4 - 2i}{\alpha} = \frac{4 - 2i}{2i} \times \frac{i}{i} = \frac{4i + 2}{-2} \qquad \beta\gamma = -2i - 1 = -(1 + 2i)$$

$iii)\ q = \alpha\beta + \alpha\gamma + \beta\gamma$

$= \alpha(\beta + \gamma) + \beta\gamma$

$= \alpha^2 + \beta\gamma = (2i)^2 - (1 + 2i)$

$q = -4 - 1 - 2i \qquad q = -5 - 2i$

$c)\ \beta + \gamma = 2i \quad and \quad \beta\gamma = -(1 + 2i)$

so β *and* γ are roots of the equations

$$z^2 - 2iz - (1 + 2i) = 0$$

$d)\ \beta = 1\ is\ an\ "obvious"\ root\ \ (1^2 - 2i - (1 + 2i) = 0)$

$$z^2 - 2iz - (1 + 2i) = (z - 1)(z^2 + (1 + 2i)) = 0$$

$roots\ are\ \beta = 1\ \ and\ \ \gamma = 1 + 2i$

Question 8: June 2008

$z^3 + qz + 18 - 12i = 0$ *has roots* α, β, γ

$a)\ i)\ \alpha\beta\gamma = -18 + 12i$ $\qquad ii)\ \alpha + \beta + \gamma = 0 \ (z^3 + 0z^2 + qz + ...)$

$b)\ \beta + \gamma = 2$

$$i)\ \alpha + \beta + \gamma = 0$$
$$\alpha + 2 = 0$$
$$\alpha = -2$$
$$ii)\ \alpha\beta\gamma = -18 + 12i$$
$$-2\beta\gamma = -18 + 12i$$
$$\beta\gamma = 9 - 6i$$
$$iii)\ q = \alpha\beta + \alpha\gamma + \beta\gamma = \alpha(\beta + \gamma) + \beta\gamma$$
$$q = -2 \times (2) + 9 - 6i = 5 - 6i$$

$c)\ \beta = ki$ *and it is a root of* $z^3 + qz + 18 - 12i = 0$

$$so\ \ (ki)^3 + (5 - 6i) \times (ki) + 18 - 12i = 0$$
$$-ik^3 + 5ki + 6k + 18 - 12i = 0$$
$$(6k + 18) + i(-k^3 + 5k - 12) = 0$$
$$6k + 18 = 0 \ \ and \ \ -k^3 + 5k - 12 = 0$$
$$k = -3 \quad and \ -(-3)^3 + 5 \times -3 - 12 = 27 - 15 - 12 = 27 - 27 = 0$$

so $\alpha = -2, \beta = -3i\ ,\ \gamma = 2 - \beta = 2 + 3i$

Question 9: June 2009

$z^3 + pz^2 + 25z + q = 0$ *has roots* α, β, γ

p and q are real numbers.

$a)\ \alpha = 2 - 3i.$

Because the coefficients of the equation are REAL numbers,

α^* is also a root : $\beta = 2 + 3i$

$b)\ i)\ \alpha\beta = (2 - 3i)(2 + 3i) = 4 + 9 = 13$

$$ii)\ \alpha\beta + \alpha\gamma + \beta\gamma = 25$$
$$\alpha\beta + \gamma(\alpha + \beta) = 25$$
$$13 + \gamma \times 4 = 25 \qquad \gamma = 3$$
$$iii)\ \alpha\beta\gamma = -q = 13 \times 3 = 39 \qquad q = -39$$
$$\alpha + \beta + \gamma = -p = 4 + 3 = 7 \qquad p = -7$$