Length of arc – area of surface of revolution

Question 1: June 2006 Q2

A curve has parametric equations

$$x = t - \frac{1}{3}t^3, \quad y = t^2$$

(a) Show that

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (1 + t^2)^2$$ *(3 marks)*

(b) The arc of the curve between $t = 1$ and $t = 2$ is rotated through 2π radians about the x-axis.

Show that S, the surface area generated, is given by $S = k\pi$, where k is a rational number to be found. *(5 marks)*

Question 2: Jan 2006 Q7

(a) Use the definitions

$$\sinh\theta = \frac{1}{2}(e^{\theta} - e^{-\theta}) \quad \text{and} \quad \cosh\theta = \frac{1}{2}(e^{\theta} + e^{-\theta})$$

to show that:

(i) $2\sinh\theta\cosh\theta = \sinh 2\theta$; *(2 marks)*

(ii) $\cosh^2\theta + \sinh^2\theta = \cosh 2\theta$. *(3 marks)*

(b) A curve is given parametrically by

$$x = \cosh^3\theta, \quad y = \sinh^3\theta$$

(i) Show that

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \frac{9}{4}\sinh^2 2\theta\cosh 2\theta$$ *(6 marks)*

(ii) Show that the length of the arc of the curve from the point where $\theta = 0$ to the point where $\theta = 1$ is

$$\frac{1}{2}\left[(\cosh 2)^{\frac{3}{2}} - 1\right]$$ *(6 marks)*

Question 3: June 2007 Q7

A curve has equation $y = 4\sqrt{x}$.

(a) Show that the length of arc s of the curve between the points where $x = 0$ and $x = 1$ is given by

$$s = \int_0^1 \sqrt{\frac{x+4}{x}}\, dx$$ *(4 marks)*

(b) (i) Use the substitution $x = 4\sinh^2\theta$ to show that

$$\int \sqrt{\frac{x+4}{x}}\, dx = \int 8\cosh^2\theta\, d\theta$$ *(5 marks)*

(ii) Hence show that

$$s = 4\sinh^{-1} 0.5 + \sqrt{5}$$ *(6 marks)*

Question 4: Jan 2007 Q4

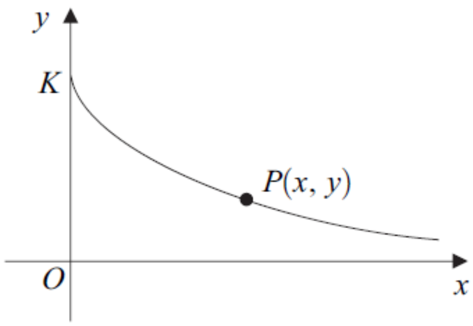
(a) Given that $y = \operatorname{sech} t$, show that:

(i) $\dfrac{dy}{dt} = -\operatorname{sech} t \tanh t$; *(3 marks)*

(ii) $\left(\dfrac{dy}{dt}\right)^2 = \operatorname{sech}^2 t - \operatorname{sech}^4 t$. *(2 marks)*

(b) The diagram shows a sketch of part of the curve given parametrically by

$$x = t - \tanh t \qquad y = \operatorname{sech} t$$



The curve meets the y-axis at the point K, and $P(x, y)$ is a general point on the curve. The arc length KP is denoted by s. Show that:

(i) $\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 = \tanh^2 t$; *(4 marks)*

(ii) $s = \ln \cosh t$; *(3 marks)*

(iii) $y = e^{-s}$. *(2 marks)*

(c) The arc KP is rotated through 2π radians about the x-axis. Show that the surface area generated is

$$2\pi(1 - e^{-s})$$ *(4 marks)*

Question 5: June 2008 Q5

(a) Use the definition $\cosh x = \frac{1}{2}(e^x + e^{-x})$ to show that $\cosh 2x = 2\cosh^2 x - 1$.
(2 marks)

(b) (i) The arc of the curve $y = \cosh x$ between $x = 0$ and $x = \ln a$ is rotated through 2π radians about the x-axis. Show that S, the surface area generated, is given by

$$S = 2\pi \int_0^{\ln a} \cosh^2 x \, dx$$ *(3 marks)*

(ii) Hence show that

$$S = \pi\left(\ln a + \frac{a^4 - 1}{4a^2}\right)$$ *(5 marks)*

Question 6: Jan 2008 Q7

(a) Given that $y = \ln \tanh \frac{x}{2}$, where $x > 0$, show that

$$\frac{dy}{dx} = \operatorname{cosech} x \qquad \textit{(6 marks)}$$

(b) A curve has equation $y = \ln \tanh \frac{x}{2}$, where $x > 0$. The length of the arc of the curve between the points where $x = 1$ and $x = 2$ is denoted by s.

(i) Show that

$$s = \int_1^2 \coth x \, dx \qquad \textit{(2 marks)}$$

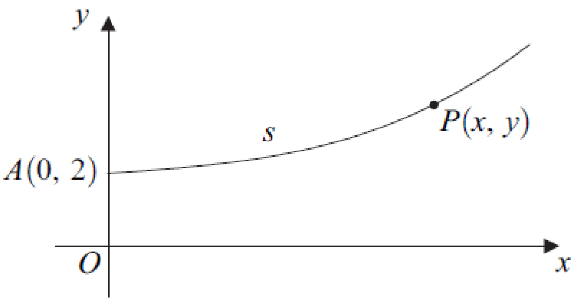
(ii) Hence show that $s = \ln(2 \cosh 1)$. *(4 marks)*

Question 7: June 2009 Q7

The diagram shows a curve which starts from the point A with coordinates (0, 2). The curve is such that, at every point P on the curve,

$$\frac{dy}{dx} = \frac{1}{2}s$$

where s is the length of the arc AP.



(a) (i) Show that

$$\frac{ds}{dx} = \frac{1}{2}\sqrt{4 + s^2} \qquad \textit{(3 marks)}$$

(ii) Hence show that

$$s = 2 \sinh \frac{x}{2} \qquad \textit{(4 marks)}$$

(iii) Hence find the cartesian equation of the curve. *(3 marks)*

(b) Show that

$$y^2 = 4 + s^2 \qquad \textit{(2 marks)}$$

Question 8: Jan 2009 Q7

(a) Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\cosh^{-1}\frac{1}{x}\right) = \frac{-1}{x\sqrt{1-x^2}}$$ *(3 marks)*

(b) A curve has equation

$$y = \sqrt{1-x^2} - \cosh^{-1}\frac{1}{x} \qquad (0 < x < 1)$$

Show that:

(i) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sqrt{1-x^2}}{x}$; *(4 marks)*

(ii) the length of the arc of the curve from the point where $x = \frac{1}{4}$ to the point where $x = \frac{3}{4}$ is $\ln 3$. *(5 marks)*

Question 9: June 2010 Q5

(a) Using the identities

$$\cosh^2 t - \sinh^2 t = 1, \quad \tanh t = \frac{\sinh t}{\cosh t} \quad \text{and} \quad \operatorname{sech} t = \frac{1}{\cosh t}$$

show that:

(i) $\tanh^2 t + \operatorname{sech}^2 t = 1$; *(2 marks)*

(ii) $\frac{\mathrm{d}}{\mathrm{d}t}(\tanh t) = \operatorname{sech}^2 t$; *(3 marks)*

(iii) $\frac{\mathrm{d}}{\mathrm{d}t}(\operatorname{sech} t) = -\operatorname{sech} t \tanh t$. *(3 marks)*

(b) A curve C is given parametrically by

$$x = \operatorname{sech} t, \quad y = 4 - \tanh t$$

(i) Show that the arc length, s, of C between the points where $t = 0$ and $t = \frac{1}{2}\ln 3$ is given by

$$s = \int_0^{\frac{1}{2}\ln 3} \operatorname{sech} t \,\mathrm{d}t$$ *(4 marks)*

(ii) Using the substitution $u = \mathrm{e}^t$, find the exact value of s. *(6 marks)*

Question 10: Jan 2010 Q4

A curve C is given parametrically by the equations

$$x = \frac{1}{2}\cosh 2t, \qquad y = 2\sinh t$$

(a) Express

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2$$

in terms of $\cosh t$. *(6 marks)*

(b) The arc of C from $t = 0$ to $t = 1$ is rotated through 2π radians about the x-axis.

(i) Show that S, the area of the curved surface generated, is given by

$$S = 8\pi \int_0^1 \sinh t \cosh^2 t \,\mathrm{d}t$$ *(2 marks)*

(ii) Find the exact value of S. *(2 marks)*

Length of arc – area of surface of revolution

Question 1: June 2006 Q2

$$a)\left(\frac{dx}{dt}\right)^2+\left(\frac{dx}{dt}\right)^2=\left(1-t^2\right)^2+\left(2t\right)^2=1-2t^2+t^4+4t^2=1+2t^2+t^4=\left(1+t^2\right)^2$$

$$b)\text{ From the formula book: } S=2\pi\int_1^2 y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

$$S=2\pi\int_1^2 y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt=2\pi\int_1^2 t^2\times(1+t^2)dt$$

$$S=2\pi\int_1^2 t^2+t^4dt=2\pi\left[\frac{1}{3}t^3+\frac{1}{5}t^5\right]_1^2=2\pi\left(\frac{8}{3}+\frac{32}{5}-\frac{1}{3}-\frac{1}{5}\right)\qquad S=\frac{256}{15}\pi$$

Question 2: Jan 2006 Q7

$$a)i)\ Sinh\theta=\frac{1}{2}(e^{\theta}-e^{-\theta})\ \text{ and }\ Cosh\theta=\frac{1}{2}(e^{\theta}+e^{-\theta})$$

$$2\,Sinh\theta\,Cosh\theta=2\times\frac{1}{2}(e^{\theta}-e^{-\theta})\times\frac{1}{2}(e^{\theta}+e^{-\theta})=\frac{1}{2}\left(e^{2\theta}+e^{0}-e^{0}-e^{-2\theta}\right)$$

$$=\frac{1}{2}(e^{2\theta}-e^{-2\theta})=Sinh(2\theta)$$

$$ii)\ Cosh^2\theta+Sinh^2\theta=\left(\frac{1}{2}(e^{\theta}-e^{-\theta})\right)^2+\left(\frac{1}{2}(e^{\theta}+e^{-\theta})\right)^2=\frac{1}{4}\left(e^{2\theta}-2e^{0}+e^{-2\theta}\right)+\frac{1}{4}\left(e^{2\theta}+2e^{0}+e^{-2\theta}\right)$$

$$=\frac{1}{4}\left(2e^{2\theta}+2e^{2\theta}\right)=\frac{1}{2}(e^{2\theta}+e^{-2\theta})=Cosh(2\theta)$$

$$b)\ x=Cosh^3\theta\ ,\quad y=Sinh^3\theta$$

$$i)\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2=\left(3Sinh\theta Cosh^2\theta\right)^2+\left(3Cosh\theta Sinh^2\theta\right)^2$$

$$=9Sinh^2\theta Cosh^4\theta+9Cosh^2\theta Sinh^4\theta$$

$$=9Sinh^2\theta Cosh^2\theta\left(Cosh^2\theta+Sinh^2\theta\right)=9\left(\frac{1}{2}Sinh2\theta\right)^2(Cosh2\theta)$$

$$\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2=\frac{9}{4}Sinh^2 2\theta Cosh2\theta$$

$$ii)\ S=\int_0^1\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta=\int_0^1\sqrt{\frac{9}{4}Sinh^2 2\theta Cosh2\theta}\,d\theta$$

$$S=\int_0^1\frac{3}{2}Sinh2\theta\sqrt{Cosh2\theta}\,d\theta=\frac{3}{2}\int_0^1 Sinh2\theta\sqrt{Cosh2\theta}\,d\theta$$

$$S=\frac{3}{4}\int_0^1 2Sinh2\theta\times Cosh^{\frac{1}{2}}2\theta\,d\theta.$$

This is an integral of the form $\int f'\times f^n=\frac{1}{n+1}f^{n+1}$

$$S=\frac{3}{4}\left[\frac{2}{3}\times Cosh^{\frac{3}{2}}2\theta\right]_0^1=\frac{1}{2}\left(Cosh^{\frac{3}{2}}2-Cosh^{\frac{3}{2}}0\right)\qquad S=\frac{1}{2}\left((Cosh2)^{\frac{3}{2}}-1\right)$$

Question 3: June 2007 Q7

$y = 4\sqrt{x}$

$a)\, s = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$ *from the formulae booklet*

$\frac{dy}{dx} = 4 \times \frac{1}{2\sqrt{x}} = \frac{2}{\sqrt{x}} \qquad \left(\frac{dy}{dx}\right)^2 = \left(\frac{2}{\sqrt{x}}\right)^2 = \frac{4}{x}$

$s = \int_0^1 \sqrt{1 + \frac{4}{x}}\, dx = \int_0^1 \sqrt{\frac{x+4}{x}}\, dx$

$b)\, i)\, x = 4Sinh^2\theta \qquad \frac{dx}{d\theta} = 4 \times 2 \times Cosh\theta \times Sinh\theta$

$dx = 8Cosh\theta\, Sinh\theta\, d\theta$

$when\ x = 1, Sinh\theta = \frac{1}{2}\, so\, \theta = Sinh^{-1}0.5 \quad and \quad when\ x = 0, Sinh\theta = 0$

$s = \int_0^1 \sqrt{\frac{x+4}{x}}\, dx = \int_o^{Sinh^{-1}0.5} \sqrt{\frac{4Sinh^2\theta + 4}{4Sinh^2\theta}} \times 8Cosh\theta\, Sinh\theta\, d\theta$

$s = \int_0^{Sinh^{-1}0.5} \sqrt{\frac{4Cosh^2\theta}{4Sinh^2\theta}} \times 8Cosh\theta\, Sinh\theta\, d\theta = \int_0^{Sinh^{-1}0.5} \frac{Cos\theta}{Sinh\theta} \times 8Cosh\theta\, Sinh\theta\, d\theta$

$s = \int_0^{Sinh^{-1}0.5} 8Cosh^2\theta\, d\theta$

$b)\, ii)\, Cosh2\theta = 2Cosh^2\theta - 1\ so\ Cosh^2\theta = \frac{1}{2} + \frac{1}{2}Cosh2\theta$

$so\ \ s = \int_0^{Sinh^{-1}0.5} 8(\frac{1}{2} + \frac{1}{2}Cosh2\theta)d\theta = \int_0^{Sinh^{-1}0.5} 4 + 4Cosh2\theta d\theta$

$s = [4\theta + 2Sinh2\theta]_0^{Sinh^{-1}0.5} = 4Sinh^{-1}0.5 + 2Sinh(2Sinh^{-1}0.5)$

$Sinh2\theta = 2Sinh\theta Cos\theta = 2 \times Sinh\theta \times \sqrt{1 + Sinh^2\theta}$

$with\ \ \theta = Sinh^{-1}0.5\ \ we\ have\, Sinh(2Sinh^{-1}0.5) = 2 \times \frac{1}{2}\sqrt{1 + \frac{1}{4}} = \frac{\sqrt{5}}{2}$

$s = 4Sinh^{-1}0.5 + 2Sinh(2Sinh^{-1}0.5) = 4Sinh^{-1}0.5 + 2\frac{\sqrt{5}}{2} = 4Sinh^{-1}0.5 + \sqrt{5}$

Question 4: Jan 2007 Q4

$$a)\ y = Sech\,t = \frac{1}{Cosh\,t}$$

$$i)\frac{dy}{dx} = -\frac{Sech\,t}{Cosh^2 t} \qquad (if\ \ y = \frac{1}{f}\ then\ \frac{dy}{dx} = -\frac{f'}{f^2})$$

$$\frac{dy}{dx} = -\frac{Sech\,t}{Cosh\,t} \times \frac{1}{Cosh\,t} = -Sech\,t \times Tanh\,t$$

$$ii)\left(\frac{dy}{dx}\right)^2 = (-Sech\,t \times Tanh\,t)^2 = Sech^2 t \times Tanh^2 t$$

Using $\boxed{Tanh^2 t = 1 - Sech^2 t}$

$$\left(\frac{dy}{dx}\right)^2 = Sech^2 t(1 - Sech^2 t) = Sech^2 t - Sech^4 t$$

$$b)\ x = t - Tanh\,t \quad and \quad y = Sech\,t$$

$$i)\frac{dx}{dt} = 1 - Sech^2 t \quad and \quad \left(\frac{dx}{dt}\right)^2 = 1 - 2Sech^2 t + Sech^4 t$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 1 - 2Sech^2 t + Sech^4 t + Sech^2 t - Sech^4 t$$

$$= 1 - Sech^2 t = Tanh^2 t$$

$$ii)\ s = \int_0^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^t Tanh(t)dt = \left[\ln(Cosh\,t)\right]_0^t = \ln(Cosh\,t)$$

$$iii)\ e^s = Cosh\,t \qquad so\ \ e^{-s} = \frac{1}{Cosh\,t} = Sech\,t = y$$

$$c)\ S_x = 2\pi\int_0^t y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = 2\pi\int_0^t Sech\,t \times Tanh\,t\,dt$$

$$S_x = 2\pi\int_0^t \frac{Sinh\,t}{Cosh^2 t}\,dt = 2\pi\int_0^t Sinh\,t \times Cosh^{-2} t\,dt = 2\pi\left[-Cosh^{-1} t\right]_0^t$$

$$S_x = 2\pi\left(-Sech\,t + 1\right) = 2\pi(1 + e^{-s}) \quad because\ \ Sech\,t = e^{-s}\ (Qb)iii))$$

Question 5: June 2008 Q5

$$a)\cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right)$$

$$\cosh^2 x = \frac{1}{4}\left(e^{x} + e^{-x}\right)^2 = \frac{1}{4}\left(e^{2x} + e^{-2x} + 2\right)$$

$$\cosh^2 x = \frac{1}{2}\times\frac{1}{2}(e^{2x} + e^{-2x}) + \frac{1}{2}$$

$$\cosh^2 x = \frac{1}{2}\cosh 2x + \frac{1}{2}$$

$$\cosh 2x = 2\cosh^2 x - 1$$

$$b)i)\ S = 2\pi\int_0^{\ln a} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 2\pi\int_0^{\ln a}\cosh x \times \sqrt{1+\operatorname{sech}^2 x}\,dx$$

$$S = 2\pi\int_0^{\ln a}\cosh x \times \cosh x\,dx = 2\pi\int_0^{\ln a}\cosh^2 x\ dx$$

$$ii)\ S = 2\pi\int_0^{\ln a}\frac{1}{2}\cosh 2x + \frac{1}{2}dx = 2\pi\left[\frac{1}{4}\sinh 2x + \frac{1}{2}x\right]_0^{\ln a}$$

$$= 2\pi\left(\frac{1}{4}\sinh(2\ln a) + \frac{1}{2}\ln a - 0\right)$$

$$S = \pi\left(\frac{1}{2}\sinh(\ln a^2) + \ln a\right) = \pi\left(\frac{1}{2}\times\frac{1}{2}\left(e^{\ln a^2} - e^{-\ln a^2}\right) + \ln a\right)$$

$$S = \pi\left(\frac{1}{4}\left(a^2 - \frac{1}{a^2}\right) + \ln a\right) = \pi\left(\frac{a^4-1}{4a^2} + \ln a\right)$$

Question 6: Jan 2008 Q7

$a)\ y = \ln\left(Tanh\frac{x}{2}\right) \quad x > 0$

$$\frac{dy}{dx} = \frac{\frac{1}{2}\left(\sec h^2\frac{x}{2}\right)}{Tanh\frac{x}{2}} = \frac{1}{2}\times\frac{1}{Cosh^2\frac{x}{2}}\times\frac{Cosh\frac{x}{2}}{Sinh\frac{x}{2}} = \frac{1}{2Cosh\frac{x}{2}Sinh\frac{x}{2}} = \frac{1}{Sinh(2\times\frac{x}{2})} = \frac{1}{Sinhx}$$

$$\frac{dy}{dx} = \text{Cosech}\,x$$

$$b)\,i)\ s = \int_1^2\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_1^2\sqrt{1+\text{Cosech}^2x}\,dx$$

$$1+\text{Cosech}^2x = 1+\frac{1}{Sinh^2x} = \frac{Sinh^2x+1}{Sinh^2x} = \frac{Cosh^2x}{Sinh^2x} = Coth^2x$$

$$s = \int_1^2 Coth\,x\,dx$$

$$ii)\ s = \int_1^2 Coth\,x\,dx = \int_1^2\frac{Cosh\,x}{Sinh\,x}dx = \left[\ln\left(Sinh\,x\right)\right]_1^2 = \ln(\sinh 2) - \ln(\sinh 1)$$

$$s = \ln\left(\frac{Sinh2}{Sinh1}\right)$$

Using $Sinh2x = 2\sinh x\cosh x$, *we have*

$$\sinh 2 = 2\sinh 1\cosh 1$$

$$s = \ln\left(\frac{2\sinh 1\cosh 1}{\sinh 1}\right) \qquad s = \ln\left(2\cosh 1\right)$$

Question 7: June 2009 Q7

$$a)\,i)\ \frac{ds}{dx}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{1+\left(\frac{s}{2}\right)^2}=\sqrt{\frac{4+s^2}{4}}=\frac{1}{2}\sqrt{4+s^2}$$

$$ii)\ \frac{1}{\sqrt{4+s^2}}\frac{ds}{dx}=\frac{1}{2}$$

$$\int\frac{1}{\sqrt{4+s^2}}ds=\int\frac{1}{2}dx$$

$$\sinh^{-1}\left(\frac{s}{2}\right)=\frac{1}{2}x+c$$

When $x=0, s=0$ *so* $c=0$

$$\frac{s}{2}=\sinh\frac{x}{2}\qquad\qquad s=2\sinh\frac{x}{2}$$

$$iii)\ \frac{dy}{dx}=\frac{1}{2}s=\sinh\frac{x}{2}$$

$$y=2\cosh\frac{x}{2}+k$$

$A(0,2)$ belongs to the curve so

$$2=2\cosh 0+k\qquad k=0$$

$$y=2\cosh\frac{x}{2}$$

$$b)\ y^2=4\cosh^2\frac{x}{2}=4(1+\sinh^2\frac{x}{2})$$

from a)ii) we know that $\sinh\frac{x}{2}=\frac{s}{2}$

$$y^2=4(1+\frac{s^2}{4})$$

$$y^2=4+s^2$$

Question 8: Jan 2009 Q7

$$a)\ \frac{d}{dx}\left(\cosh^{-1}\frac{1}{x}\right)=-\frac{1}{x^2}\times\frac{1}{\sqrt{\left(\frac{1}{x}\right)^2-1}}=-\frac{1}{x^2\sqrt{\frac{1-x^2}{x^2}}}\times\frac{\sqrt{x^2}}{\sqrt{x^2}}$$

$$=-\frac{x}{x^2\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\cosh^{-1}\frac{1}{x}\right)=-\frac{1}{x\sqrt{1-x^2}}$$

$$b)\ y=\sqrt{1-x^2}-\cosh^{-1}\frac{1}{x}\qquad(0<x<1)$$

$$i)\ \frac{dy}{dx}=\frac{1}{2}\times-2x\times\frac{1}{\sqrt{1-x^2}}+\frac{1}{x\sqrt{1-x^2}}=\frac{-x}{\sqrt{1-x^2}}+\frac{1}{x\sqrt{1-x^2}}$$

$$\frac{dy}{dx}=\frac{1-x^2}{x\sqrt{1-x^2}}=\frac{\sqrt{1-x^2}}{x}$$

$$ii)\ s=\int_{\frac{1}{4}}^{\frac{3}{4}}\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx=\int_{\frac{1}{4}}^{\frac{3}{4}}\sqrt{1+\frac{1-x^2}{x^2}}dx=\int_{\frac{1}{4}}^{\frac{3}{4}}\sqrt{\frac{1}{x^2}}$$

$$s=\int_{\frac{1}{4}}^{\frac{3}{4}}\frac{1}{x}dx=[\ln x]_{\frac{1}{4}}^{\frac{3}{4}}=\ln\left(\frac{3}{4}\right)-\ln\left(\frac{1}{4}\right)=\ln(3)$$

Question 9: June 2010 Q5

$$i)\tanh^2 t+\text{sech}^2 t=\frac{\sinh^2 t}{\cosh^2 t}+\frac{1}{\cosh^2 t}=\frac{\sinh^2 t+1}{\cosh^2 t}=\frac{\cosh^2 t}{\cosh^2 t}=1$$

$$ii)\frac{d}{dt}(\tanh t)=\frac{d}{dt}\left(\frac{\sinh t}{\cosh t}=\frac{u}{v}\right)=\left(\frac{u'v-uv'}{v^2}\right)=\frac{\cosh t\times\cosh t-\sinh t\times\sinh t}{\cosh^2 t}$$

$$=\frac{\cosh^2 t-\sinh^2 t}{\cosh^2 t}=\frac{1}{\cosh^2 t}=\text{sech}^2 t$$

$$iii)\frac{d}{dt}(\text{sech t})=\frac{d}{dt}\left(\frac{1}{\cosh t}=\frac{1}{u}\right)=\left(-\frac{u'}{u^2}\right)=-\frac{\sinh t}{\cosh^2 t}=-\frac{\sinh t}{\cosh t}\times\frac{1}{\cosh t}=-\text{sech}\, t\tanh t$$

$b)\ x=\text{sech}\, t\ \ and\ \ y=4-\tanh t$

$$i)\frac{dx}{dt}=-\text{sech}\, t\times\tanh t\ \ and\ \ \frac{dy}{dx}=-\text{sech}^2 t$$

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dx}\right)^2=\text{sech}^2 t\times\tanh^2 t+\text{sech}^4 t=\text{sech}^2 t\left(\tanh^2 t+\sec h^2 t\right)=\text{sech}^2 t$$

$$s=\int_0^{\frac{1}{2}\ln 3}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dx}\right)^2}\,dt=\int_0^{\frac{1}{2}\ln 3}\text{sech}\, t\, dt$$

$$ii)u=e^t\qquad \frac{du}{dt}=e^t=u\qquad \frac{du}{u}=dt$$

$when\ t=0,\ u=1$

$$t=\frac{1}{2}\ln 3=\ln\sqrt{3},\quad u=\sqrt{3}$$

$$s=\int_0^{\frac{1}{2}\ln 3}\text{sech}\, t\, dt=\int_1^{\sqrt{3}}\frac{2}{e^x+e^{-x}}dx=\int_1^{\sqrt{3}}\frac{2}{u+\frac{1}{u}}\times\frac{du}{u}$$

$$s=\int_1^{\sqrt{3}}\frac{2}{u^2+1}du=\left[2\tan^{-1}u\right]_1^{\sqrt{3}}=2\tan^{-1}\sqrt{3}-2\tan^{-1}1$$

$$s=2\times\frac{\pi}{3}-2\times\frac{\pi}{4}=\frac{2\pi}{3}-\frac{\pi}{2}=\frac{\pi}{6}$$

Question 10: Jan 2010 Q4

$$x=\frac{1}{2}\cosh 2t\ \ and\ \ y=2\sinh t$$

$$a)\frac{dx}{dt}=\sinh 2t\ \ and\ \ \frac{dy}{dt}=2\cosh t$$

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=(\sinh^2 2t+4\cosh^2 t)=\cosh^2 2t-1+4\cosh^2 t$$

Using the identity $\cosh t=2\cosh^2 t-1,\ we\ have$

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=\left(2\cosh^2 t-1\right)^2-1+4\cosh^2 t=4\cosh^4 t-4\cosh^2 t+1-1+4\cosh^2 t=4\cosh^4 t$$

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=4\cosh^4 t$$

$$b)i)\ S=2\pi\int_0^1 y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt=2\pi\int_0^1 2\sinh t\times(2\cosh^2 t)dt=8\pi\int_0^1\sinh t\times\cosh^2 t\, dt$$

$$ii)\ S=8\pi\left[\frac{1}{3}\cosh^3 t\right]_0^1=\frac{8\pi}{3}\left(\cosh^3 1-1\right)$$