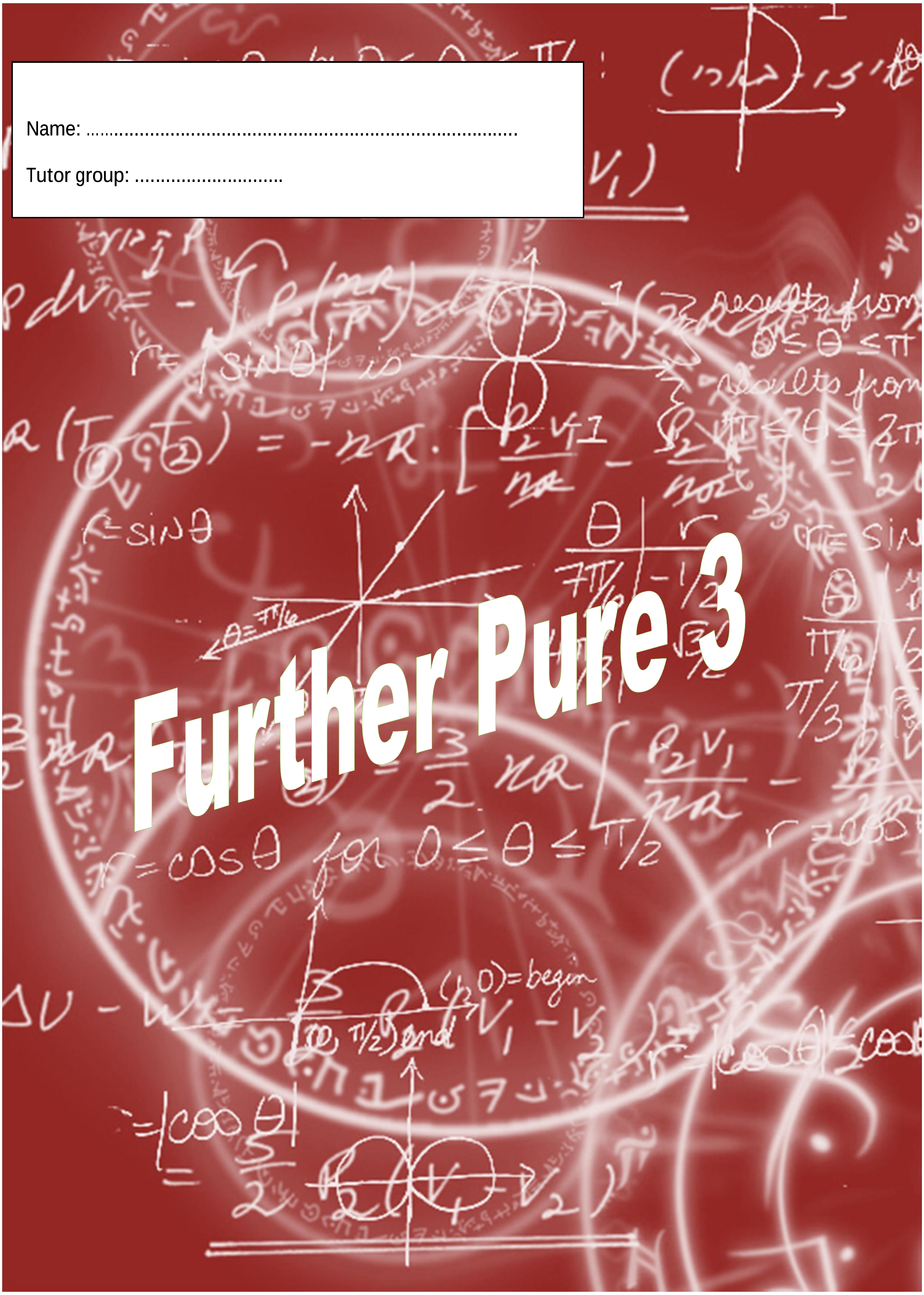
# AQA – Further Pure 3 – Key points and Papers

Name: ……………………………………………………………………………

Tutor group: ………………………….



# Key dates

**Further pure 3 exam:** **10th June 2013 am**

## Term dates:

| | |
|---|---|
| Term 1: Monday 3 September 2012 - Wednesday 24 October 2012 (38 teaching days) | Term 4: Monday 18 February 2013 - Friday 22 March 2013 (25 teaching days) |
| Term 2: Monday 5 November 2012 - Friday 21 December 2012 (35 teaching days) | Term 5: Monday 8 April 2013 - Friday 24 May 2013 (34 teaching days) |
| Term 3: Monday 7 January 2013 - Friday 8 February 2013 (25 teaching days) | Term 6: Monday 3 June 2013 - Wednesday 24 July 2013 (38 teaching days) |

# Scheme of Assessment *Mathematics*
# *Advanced Subsidiary (AS)*
# *Advanced Level (AS + A2)*

The Scheme of Assessment has a modular structure. The A Level award comprises four compulsory Core units, one optional Applied unit from the AS scheme of assessment, and one optional Applied unit either from the AS scheme of assessment or from the A2 scheme of assessment.

For the written papers, each candidate will require a copy of the AQA Booklet of formulae and statistical tables issued for this specification.

| | |
|---|---|
| All the units count for $33\frac{1}{3}$% of the total AS marks<br>$16\frac{2}{3}$% of the total A level marks | Written Paper<br>1hour 30 minutes<br>75 marks |

### Grading System

The AS qualifications will be graded on a five-point scale: A, B, C, D and E. The full A level qualifications will be graded on a six-point scale: A*, A, B, C, D and E.

To be awarded an A* in Further Mathematics, candidates will need to achieve grade A on the full A level qualification and 90% of the maximum uniform mark on the aggregate of the best three of the A2 units which contributed towards Further Mathematics. For all qualifications, candidates who fail to reach the minimum standard for grade E will be recorded as U (unclassified) and will not receive a qualification certificate.

# Further pure 3 subject content

## Series and limits
## Differential equations
## Polar coordinates

# Further pure 3 specifications

Candidates will be expected to be familiar with the knowledge, skills and understanding implicit in the modules Core 1, Core 2, Core 3, Core 4 and Further Pure 1.
Candidates may use relevant formulae included in the formulae booklet without proof.

## Series and limits

| | |
|---|---|
| Maclaurin series | |
| Expansions of $e^x$, $\ln(1+x)$, $\cos x$ and $\sin x$, and $(1+x)^n$ for rational values of $n$. | Use of the range of values of $x$ for which these expansions are valid, as given in the formulae booklet, is expected to determine the range of values for which expansions of related functions are valid; e.g $\ln\left(\frac{1+x}{1-x}\right)$; $(1-2x)^{\frac{1}{2}}e^x$. |
| Knowledge and use, for $k > 0$, of $\lim x^k e^{-x}$ as $x$ tends to infinity and $\lim x^k \ln x$ as $x$ tends to zero. | |
| Improper integrals. | $E.g.\ \int_0^e x\ln x\,dx$ , $\int_0^\infty xe^{-x}dx.$ Candidates will be expected to show the limiting processes used |
| Use of series expansion to find limits. | $E.g.\lim_{x\to 0}\frac{e^x-1}{x}$ ; $\lim_{x\to 0}\frac{Sin(3x)}{x}$ ; $\lim_{x\to 0}\frac{x^2e^x}{\cos(2x)-1}$ ; $\lim_{x\to 0}\frac{\sqrt{2+x}-\sqrt{2}}{x}$ |

## Differential equations

| | |
|---|---|
| The concept of a differential equation and its order. | The relationship of order to the number of arbitrary constants in the general solution will be expected. |
| Boundary values and initial conditions, general solutions and particular solutions. | |

### Differential Equations. First Order

| | |
|---|---|
| Analytical solution of first order linear differential equations of the form $\frac{dy}{dx}+Py=Q$ where P and Q are functions of $x$. | To include use of an integrating factor and solution by complementary function and particular integral. |

### Numerical method

| | |
|---|---|
| Numerical methods for the solution of differential equations of the form $\frac{dy}{dx}=f(x,y)$. | |
| Euler's formula and extensions to second order methods for this first order differential equation. | Formulae to be used will be stated explicitly in questions, but candidates should be familiar with standard notation such as used in Euler's formula $y_{r+1}=y_r+hf(x_r,y_r)$, the formula $y_{r+1}=y_{r-1}+2hf(x_r,y_r)$, and the formula $y_{r+1}=y_r+\frac{1}{2}(k_1+k_2)$ where $k_1=hf(x_r,y_r)$ and $k_2=hf(x_r+h,y_r+k_1)$ |

| Differential equations .Second Order | |
|---|---|
| Solution of differential equations of the form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0,$ where $a, b$ *and* $c$ are integers, by using an auxiliary equation whose roots may be real or complex. | Including repeated roots. |
| Solution of equations of the form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$ where *a*, *b* and *c* are integers by finding the complementary function and a particular integral. | Finding particular integrals will be restricted to cases where f(*x*)is of the form $e^{kx}$ , cos(*kx)* , sin(*kx)* or a polynomial of degree at most 4, or a linear combination of any of the above. |
| Solution of differential equations of the form: $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$ where *P, Q*, and *R* are functions of *x.* A substitution will always be given which reduces the differential equation to a form which can be directly solved using the other analytical methods in this specification or by separating variables. | Level of difficulty as indicated by:-<br>(a) Given $x^2\frac{d^2y}{dx^2}-2y=x$ use the substitution $x = e^t$<br>to show that $\frac{d^2y}{dt^2}-\frac{dy}{dt}-2y=e^t$<br>Hence find *y* in terms of *t*<br>Hence find *y* in terms of *x*<br><br>(b) $(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}=0$ use the substitution $u=\frac{dy}{dx}$<br>to show that $\frac{du}{dx}=\frac{2xu}{1-x^2}$ and hence that $u=\frac{A}{1-x^2}$ where *A* is an arbitrary constant.<br>Hence find *y* in terms of *x.* |
| | |
| **Polar Coordinates** | |
| Relationship between polar and Cartesian coordinates. | The convention *r* > 0 will be used. The sketching of curves given by equations of the form *r* = f(θ ) may be required. Knowledge of the formula $\tan\phi=r\frac{d\theta}{dr}$ is not required. |
| Use of the formula area $=\int_\alpha^\beta \frac{1}{2}r^2 d\theta$ . | |

# The formulae booklet

## Mensuration

Surface area of sphere $= 4\pi r^2$

Area of curved surface of cone $= \pi r \times$ slant height

## Arithmetic series

$$u_n = a + (n-1)d$$

$$S_n = \tfrac{1}{2}n(a+l) = \tfrac{1}{2}n\left[2a + (n-1)d\right]$$

## Geometric series

$$u_n = ar^{n-1}$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_\infty = \frac{a}{1-r} \text{ for } |r| < 1$$

## Summations

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1)$$

$$\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1)$$

$$\sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2$$

## Trigonometry – the Cosine rule

$$a^2 = b^2 + c^2 - 2bc\cos A$$

## Binomial Series

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n \qquad (n \in \mathbb{N})$$

$$\text{where } \binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{1.2\ldots r}x^r + \ldots \quad (|x| < 1, n \in \mathbb{R})$$

## Logarithms and exponentials

$$a^x = e^{x\ln a}$$

## Complex numbers

$$\{r(\cos\theta + i\sin\theta)\}^n = r^n(\cos n\theta + i\sin n\theta)$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

The roots of $z^n = 1$ are given by $z = e^{\frac{2\pi k i}{n}}$, for $k = 0, 1, 2, \ldots, n-1$

## Maclaurin's series

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^r}{r!} f^{(r)}(0) + \ldots$$

$$e^x = \exp(x) = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^r}{r!} + \ldots \quad \text{for all } x$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{r+1} \frac{x^r}{r} + \ldots \quad (-1 < x \leqslant 1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \ldots \quad \text{for all } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^r \frac{x^{2r}}{(2r)!} + \ldots \quad \text{for all } x$$

### Hyperbolic functions

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\cosh^{-1} x = \ln\left\{x + \sqrt{x^2 - 1}\right\} \quad (x \geqslant 1)$$

$$\sinh^{-1} x = \ln\left\{x + \sqrt{x^2 + 1}\right\}$$

$$\tanh^{-1} x = \tfrac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \quad (|x| < 1)$$

### Conics

| | Ellipse | Parabola | Hyperbola | Rectangular hyperbola |
|---|---|---|---|---|
| Standard form | $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ | $y^2 = 4ax$ | $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ | $xy = c^2$ |
| Asymptotes | none | none | $\frac{x}{a} = \pm \frac{y}{b}$ | $x = 0, y = 0$ |

## Trigonometric identities

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \quad \left(A \pm B \neq (k + \tfrac{1}{2})\pi\right)$$

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$$

## Differentiation

| $\mathbf{f}(x)$ | $\mathbf{f}'(x)$ |
|---|---|
| $\sin^{-1} x$ | $\dfrac{1}{\sqrt{1-x^2}}$ |
| $\cos^{-1} x$ | $-\dfrac{1}{\sqrt{1-x^2}}$ |
| $\tan^{-1} x$ | $\dfrac{1}{1+x^2}$ |
| $\tan kx$ | $k\sec^2 kx$ |
| $\operatorname{cosec} x$ | $-\operatorname{cosec} x \cot x$ |
| $\sec x$ | $\sec x \tan x$ |
| $\cot x$ | $-\operatorname{cosec}^2 x$ |
| $\sinh x$ | $\cosh x$ |
| $\cosh x$ | $\sinh x$ |
| $\tanh x$ | $\operatorname{sech}^2 x$ |
| $\sinh^{-1} x$ | $\dfrac{1}{\sqrt{1+x^2}}$ |
| $\cosh^{-1} x$ | $\dfrac{1}{\sqrt{x^2-1}}$ |
| $\tanh^{-1} x$ | $\dfrac{1}{1-x^2}$ |
| $\dfrac{\mathrm{f}(x)}{\mathrm{g}(x)}$ | $\dfrac{\mathrm{f}'(x)\,\mathrm{g}(x)-\mathrm{f}(x)\,\mathrm{g}'(x)}{(\mathrm{g}(x))^2}$ |

## Integration

(+ constant; $a > 0$ where relevant)

| $\mathbf{f}(x)$ | $\int \mathbf{f}(x)\,\mathrm{d}x$ |
|---|---|
| $\tan x$ | $\ln\lvert\sec x\rvert$ |
| $\cot x$ | $\ln\lvert\sin x\rvert$ |
| $\operatorname{cosec} x$ | $-\ln\lvert\operatorname{cosec} x+\cot x\rvert=\ln\lvert\tan(\tfrac{1}{2}x)\rvert$ |
| $\sec x$ | $\ln\lvert\sec x+\tan x\rvert=\ln\lvert\tan(\tfrac{1}{2}x+\tfrac{1}{4}\pi)\rvert$ |
| $\sec^2 kx$ | $\dfrac{1}{k}\tan kx$ |
| $\sinh x$ | $\cosh x$ |
| $\cosh x$ | $\sinh x$ |
| $\tanh x$ | $\ln\cosh x$ |

| | |
|---|---|
| $\dfrac{1}{\sqrt{a^2-x^2}}$ | $\sin^{-1}\left(\dfrac{x}{a}\right) \quad (\lvert x\rvert < a)$ |
| $\dfrac{1}{a^2+x^2}$ | $\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$ |
| $\dfrac{1}{\sqrt{x^2-a^2}}$ | $\cosh^{-1}\left(\dfrac{x}{a}\right)$ or $\ln\{x+\sqrt{x^2-a^2}\} \quad (x>a)$ |
| $\dfrac{1}{\sqrt{a^2+x^2}}$ | $\sinh^{-1}\left(\dfrac{x}{a}\right)$ or $\ln\{x+\sqrt{x^2+a^2}\}$ |
| $\dfrac{1}{a^2-x^2}$ | $\dfrac{1}{2a}\ln\left\lvert\dfrac{a+x}{a-x}\right\rvert=\dfrac{1}{a}\tanh^{-1}\left(\dfrac{x}{a}\right) \quad (\lvert x\rvert<a)$ |
| $\dfrac{1}{x^2-a^2}$ | $\dfrac{1}{2a}\ln\left\lvert\dfrac{x-a}{x+a}\right\rvert$ |

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

## Area of a sector

$$A=\frac{1}{2}\int r^2\,\mathrm{d}\theta \quad \text{(polar coordinates)}$$

## Arc length

$$s=\int\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x \quad \text{(cartesian coordinates)}$$

$$s=\int\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t \quad \text{(parametric form)}$$

## Surface area of revolution

$$S_x = 2\pi\int y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x \quad \text{(cartesian coordinates)}$$

$$S_x = 2\pi\int y\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t \quad \text{(parametric form)}$$

## Numerical integration

The trapezium rule: $\int_a^b y\,\mathrm{d}x \approx \frac{1}{2}h\{(y_0+y_n)+2(y_1+y_2+\ldots+y_{n-1})\}$, where $h=\dfrac{b-a}{n}$

The mid-ordinate rule: $\int_a^b y\,\mathrm{d}x \approx h(y_{\frac{1}{2}}+y_{\frac{3}{2}}+\ldots+y_{n-\frac{3}{2}}+y_{n-\frac{1}{2}})$, where $h=\dfrac{b-a}{n}$

Simpson's rule: $\int_a^b y\,\mathrm{d}x \approx \frac{1}{3}h\{(y_0+y_n)+4(y_1+y_3+\ldots+y_{n-1})+2(y_2+y_4+\ldots+y_{n-2})\}$

where $h=\dfrac{b-a}{n}$ and $n$ is even

# Content

# Series and limits

## Series

### MacLaurin's series

The function $f(x)$ and all its derivatives exist at $x=0$

The Maclaurin series for a function $f(x)$ is given by:

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+...+\frac{f^{(r)}(0)}{r!}x^r+...$$

where $f', f'', f''',...$ denote the first, second, third,... derivatives of $f$, respectively.

$f^{(r)}$ is the derivative of order $r$.

### Range of validity

Some series are valid for all values of $x\in \mathbb{R}$,

but some series are valid for only some values of $x$.

Refer to the formulae book to find the range of validity.

For example: $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-...$ is valid for $-1<x\leq 1$

### Multiplying and composing Maclaurin's series

$f(x)$ *and* $g(x)$ are two functions

- The Maclaurin's series of the function $f\times g(x)$ is the product the two maclaurin's series.
- To obtain the Maclaurin's series of the function $f(g(x))$, substitute $x$ in the Maclaurin's series $f$ by the Maclaurin's series of $g(x)$.

*Examples*: $e^x=1+x+\frac{x^2}{2}+...$ *and* $\sin(x)=x-\frac{x^3}{6}+...$

- The maclaurin's series of $e^x\sin(x)=(1+x+\frac{x^2}{2}+...)(x-\frac{x^3}{6}+...)$

$$=x-\frac{x^3}{6}+x^2-\frac{x^4}{6}+\frac{x^3}{2}+...=x-x^2+\frac{x^3}{3}-\frac{x^4}{6}+...$$

- The maclaurin's series of $e^{\sin(x)}=1+\left(x-\frac{x^3}{6}+...\right)+\frac{1}{2}\left(x-\frac{x^3}{6}+...\right)^2+..$

$$=1+x+\frac{x^2}{2}-\frac{x^4}{8}+...$$

### Maclaurin's series and limits

If a function $f$ is not defined when $x=0$, we study the value of the function when $x$ is very close to 0.

If a value exists, it is called the limit of $f(x)$ when $x$ tends to 0.

- When the limit is not obvious, work out the Maclaurin's series of the function and substitute $x$ by 0 in the series (if possible) to obtain the limit.

Example:

$f(x)=\frac{e^x-1}{x}$ $\lim_{x\to 0}\left(e^x-1\right)=0$ *and* $\lim_{x\to 0}(x)=0$. Not only the function $f$ is not defined at $x=0$,

but also its limit when $x$ tends to 0

can not be determined ("$\frac{0}{0}$"). The Maclaurin's series of $f$ *is* $f(x)=\frac{1}{x}\left(1+x+\frac{x^2}{2}+...-1\right)=1+\frac{x}{2}+...$

So when $x$ tends to 0, $f(x)$ tends to $1+\frac{0}{2}+...=1$ *or* $\lim_{x\to 0}f(x)=1$

# Limits

## Limits you have to know:

You are allowed to use the following results without proof:

- when $x \to \infty$ , $x^k e^{-x} \to 0$ for any real number $k$.

- when $x \to 0$ , $x^k \ln(x) \to 0$ for $k > 0$.

## Improper integrals

The integral $\int_a^b f(x)dx$ is said IMPROPER if

a) the interval of integration is infinite,

or b) $f(x)$ is not defined at one or both of the end points $x = a$ *and* $x = b$.

## Method

To work out if an improper integral has a value or not (exists or not)

1) Replace "∞" or "a", the value where $f$ is not defined, by a letter. "N" for example.
2) Integrate to find an expression in terms of "N".
3) Work out the limit of this expression when "N" tends to "∞" or "a".
4) If the limit exists then the improper integral has a value. If the limit is "∞", the improper integral does not exist.

Example: $\int_0^\infty \frac{1}{1+x^2}dx$ is an improper integral.

Let's work out $\int_0^N \frac{1}{1+x^2}dx = [\text{Arc}\tan(x)]_0^N = \text{Arctan(N)} - \text{Arctan(0)}$

$\text{Arctan}(0) = 0$ and when $\text{N} \to \infty, \text{Arctan(N)} \to \frac{\pi}{2}$.

*conclusion*: $\int_0^\infty \frac{1}{1+x^2}dx$ *exists* and $\int_0^\infty \frac{1}{1+x^2}dx = \frac{\pi}{2}$

# Series and Limits

## Chapter assessment

1. (i) Given that $y = \sin\left(x + \frac{\pi}{6}\right)$, find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$.

   (ii) Hence, by using Maclaurin's theorem, show that the first four terms in the expansion, in ascending powers of $x$, of $\sin\left(x + \frac{\pi}{6}\right)$ are

$$\frac{1}{2} + \frac{\sqrt{3}}{2}x - \frac{1}{4}x^2 - \frac{\sqrt{3}}{12}x^3.$$

2. Find the terms in the series expansion of $\ln(3 + x)$ in ascending powers of $x$, up to and including the term in $x^3$ and state the range of values of $x$ for which this expansion is valid.

3. (i) By using the expansion

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\ldots,$$

   or otherwise, show that the first three non-zero terms in the expansion, in ascending powers of $x$, of $\sec x$ are

$$1 + \frac{x^2}{2} + \frac{5x^4}{24}.$$

   (ii) Hence find $\lim_{x \to 0}\left(\frac{\sec x - \cos x}{x^2}\right)$.

4. (i) Show that $\lim_{a \to \infty}\left(\frac{2a+1}{3a+2}\right) = \frac{2}{3}$.

   (ii) Evaluate $\int_1^{\infty} \frac{1}{(2x+1)(3x+2)}\,dx$.

5. Find $\lim_{x \to -\infty} (2 - x)^2 e^x$.

6. (i) Find the first four non-zero terms in the series expansion of

$$\cos 6x - \ln(1 + 6x)$$

   in ascending powers of $x$.

   (ii) Determine the range of values of $x$ for which the series expansion for

$$\cos 6x - \ln(1 + 6x)$$

   is valid.

7. The function f is defined by $f(x) = \frac{1}{e^x + 1}$.

   (i) Use Maclaurin's theorem to show that when $f(x)$ is expanded in ascending powers of $x$:

   (a) the first two terms are $\frac{1}{2} - \frac{1}{4}x$

   (b) the coefficient of $x^2$ is zero.

   (ii) Find $\lim_{x \to 0} \frac{x^2}{(e^x + 1)(1 - \cos x)}$.

# Series and Limits

## Solutions to Chapter assessment

1. (i) $y = \sin\left(x + \frac{\pi}{6}\right)$

$$\frac{dy}{dx} = \cos\left(x + \frac{\pi}{6}\right)$$

$$\frac{d^2y}{dx^2} = -\sin\left(x + \frac{\pi}{6}\right)$$

$$\frac{d^3y}{dx^3} = -\cos\left(x + \frac{\pi}{6}\right)$$

(ii) Let $y = f(x) = \sin\left(x + \frac{\pi}{6}\right) \Rightarrow$

$$f(0) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$$f'(0) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

$$f''(0) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$$

$$f'''(0) = -\cos\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2}f''(0) + \frac{x^3}{6}f'''(0) + \ldots \quad \text{Maclaurin's series}$$

$$\sin\left(x + \frac{\pi}{6}\right) = \frac{1}{2} + x\left(\frac{\sqrt{3}}{2}\right) + \frac{x^2}{2}\left(-\frac{1}{2}\right) + \frac{x^3}{6}\left(-\frac{\sqrt{3}}{2}\right) + \ldots$$

$$\sin\left(x + \frac{\pi}{6}\right) = \frac{1}{2} + \frac{\sqrt{3}}{2}x - \frac{1}{4}x^2 - \frac{\sqrt{3}}{12}x^3 + \ldots$$

If the question had not stated the method to be used, this series expansion could have been obtained by writing $\sin\left(x + \frac{\pi}{6}\right) = \sin x \cos\frac{\pi}{6} + \cos x \sin\frac{\pi}{6}$ and then using the series expansions for sin *x* and cos *x* that appear in the AQA formulae booklet.

2. $\ln(3 + x) = \ln 3\left(1 + \frac{x}{3}\right)$  (using log law)

$$= \ln 3 + \ln\left(1 + \frac{x}{3}\right)$$

$$= \ln 3 + \frac{x}{3} - \frac{1}{2}\left(\frac{x}{3}\right)^2 + \frac{1}{3}\left(\frac{x}{3}\right)^3 - \ldots$$

(using series for $\ln(1 + X)$ with $X = \frac{x}{3}$)

$$= \ln 3 + \frac{x}{3} - \frac{x^2}{18} + \frac{x^3}{81} - \ldots$$

The expansion is valid for $-1 < \frac{x}{3} \le 1$

i.e. $-3 < x \le 3$

3. (i) $\sec x \approx \dfrac{1}{1 - \frac{x^2}{2} + \frac{x^4}{24} - \ldots}$

$$\approx \left[1 - \frac{x^2}{2} + \frac{x^4}{24} \ldots\right]^{-1}$$

$$\approx \left\{1 - \left(-\frac{x^2}{2} + \frac{x^4}{24}\right) + \left(-\frac{x^2}{2} + \frac{x^4}{24}\right)^2 - \ldots\right\}$$

using the binomial expansion $(1 + y)^{-1} \approx 1 - y + y^2 - \ldots$ with $y = \left(-\frac{x^2}{2} + \frac{x^4}{24}\right)$

$$\approx \left\{1 + \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^4}{4} + \ldots\right\}$$

$$\sec x \approx 1 + \frac{x^2}{2} + \frac{5x^4}{24}$$

Question 3(a) includes the phrase '…or otherwise…'. This means that you could use other methods as an alternative to the series expansion for cos *x*. For example, this series for sec *x* could be found using Maclaurin's theorem.

(ii) $\left(\dfrac{\sec x - \cos x}{x^2}\right) \approx \dfrac{\left(1 + \frac{x^2}{2} + \frac{5x^4}{24} \ldots\right) - \left(1 - \frac{x^2}{2} + \frac{x^4}{24} \ldots\right)}{x^2}$

$$\approx \frac{x^2 + \frac{x^4}{6} + \text{terms in } x^6 \text{ and higher powers}}{x^2}$$

$$\approx 1 + \frac{x^2}{6} + \text{terms in } x^4 \text{ and higher powers}$$

Dividing both the numerator and denominator by the common factor. *(This crucial step must be clearly shown)*

$$\lim_{x\to 0}\left(\frac{\sec x-\cos x}{x^2}\right)=\lim_{x\to 0}\left(1+\frac{x^2}{6}\dots\right)$$

As $x\to 0$, $\frac{x^2}{6}\to 0$ and terms in higher powers of $x$ also $\to 0$

So $\lim_{x\to 0}\left(\frac{\sec x-\cos x}{x^2}\right)=1$

4. (i) $\lim_{a\to\infty}\left(\frac{2a+1}{3a+2}\right)=\lim_{a\to\infty}\left(\frac{2+\frac{1}{a}}{3+\frac{2}{a}}\right)$

$$=\frac{2+0}{3+0}$$

$$=\frac{2}{3}$$

(ii) $\int_1^\infty \frac{1}{(2x+1)(3x+2)}\,dx=\int_1^\infty\left(\frac{2}{2x+1}-\frac{3}{3x+2}\right)dx$

Using partial fractions

$$=\lim_{a\to\infty}\int_1^a\left(\frac{2}{2x+1}-\frac{3}{3x+2}\right)dx$$

$$=\lim_{a\to\infty}\left[\ln(2x+1)-\ln(3x+2)\right]_1^a$$

$$=\lim_{a\to\infty}\left[\ln\left(\frac{2x+1}{3x+2}\right)\right]_1^a$$

$$=\lim_{a\to\infty}\left[\ln\left(\frac{2a+1}{3a+2}\right)-\ln\left(\frac{3}{5}\right)\right]$$

$$=\ln\left[\lim_{a\to\infty}\left(\frac{2a+1}{3a+2}\right)\right]-\ln\left(\frac{3}{5}\right)$$

$$=\ln\left(\frac{2}{3}\right)-\ln\left(\frac{3}{5}\right)$$

$$=\ln\left(\frac{10}{9}\right)$$

The interval of integration is infinite so replace the upper limit ($\infty$) by $a$ and find the limit as $a\to\infty$. This and the next five lines show the limiting process used.

5. Using the substitution $t=-x$ leads to $t\to\infty$ when $x\to-\infty$

so $\lim_{x\to-\infty}(2-x)^2e^x$ becomes $\lim_{t\to\infty}(2+t)^2e^{-t}$

$\lim_{t\to\infty}(2+t)^2e^{-t}=\lim_{t\to\infty}\left(4e^{-t}+4te^{-t}+t^2e^{-t}\right)$

As $t\to\infty$, $e^{-t}\to 0$ and, for $k>0$, $\lim_{t\to\infty}t^ke^{-t}=0$

so $\lim_{x\to-\infty}(2-x)^2e^x=0$

6. (i) $\cos 6x=1-\frac{(6x)^2}{2!}+\frac{(6x)^4}{4!}-\dots$

Replace $x$ by $6x$ in the series expansion for $\cos x$

$$=1-18x^2+54x^4-\dots$$

$\ln(1+6x)=(6x)-\frac{(6x)^2}{2}+\frac{(6x)^3}{3}-\frac{(6x)^4}{4}+\dots$

Replace $x$ by $6x$ in the series expansion for $\ln(1+x)$

$$=6x-18x^2+72x^3-324x^4+\dots$$

$$\cos 6x-\ln(1+6x)=\left(1-18x^2+54x^4-\dots\right)-\left(6x-18x^2+72x^3-324x^4+\dots\right)$$

$$=1-6x-72x^3+378x^4+\dots$$

So $\cos 6x-\ln(1+6x)\approx 1-6x-72x^3+378x^4$

(ii) The series expansion for $\cos 6x$ is valid for all values of $x$

The series expansion for $\ln(1+6x)$ is valid for $-1<6x\le 1$ ie $-\frac{1}{6}<x\le\frac{1}{6}$

The series expansion for $\cos 6x-\ln(1+6x)$ is valid only for those values of $x$ which satisfy both all values of $x$ and $-\frac{1}{6}<x\le\frac{1}{6}$.

So the required range of values is $-\frac{1}{6}<x\le\frac{1}{6}$.

7. (i) $f(x)=\frac{1}{e^x+1}$ $\Rightarrow$ $f(0)=\frac{1}{2}$

$f(x)=(e^x+1)^{-1}$

$f'(x)=-1(e^x+1)^{-2}(e^x)=-e^x(e^x+1)^{-2}$ (chain rule)

$\Rightarrow$ $f'(0)=-\frac{1}{4}$

$f''(x)=-e^x(e^x+1)^{-2}+2e^x(e^x+1)^{-3}(e^x)$ (product rule)

$\Rightarrow$ $f''(0)=-\frac{1}{4}+\frac{2}{8}=0$

$f(x)=f(0)+xf'(0)+\frac{x^2}{2}f''(0)+\dots$ Maclaurin's series

$$f(x) = \frac{1}{2} + x\left(-\frac{1}{4}\right) + \frac{x^2}{2}(0) + \ldots$$

So in the expansion of $f(x)$

(a) the first two terms are $\frac{1}{2} - \frac{1}{4}x$

(b) the coefficient of $x^2$ is zero.

(ii) $$\frac{x^2}{(e^x+1)(1-\cos x)} = \frac{1}{(e^x+1)} \times \frac{x^2}{(1-\cos x)}$$

using (i)

$$\approx \left(\frac{1}{2} - \frac{1}{4}x + \ldots\right) \times \frac{x^2}{1 - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right)}$$

The question does not say 'Hence' so you may prefer to use the expansion of $e^x$ from the formula booklet rather than using the results from part (i).

$$\lim_{x \to 0} \frac{x^2}{(e^x+1)(1-\cos x)} = \lim_{x \to 0} \left( \frac{\left(\frac{1}{2}x^2 - \frac{1}{4}x^3 + \ldots\right)}{\left(\frac{x^2}{2} - \frac{x^4}{24} + \ldots\right)} \right)$$

$$= \lim_{x \to 0} \left( \frac{\frac{1}{2} - \frac{1}{4}x + \ldots}{\frac{1}{2} - \frac{x^2}{24} + \ldots} \right)$$

$$= \frac{\frac{1}{2}}{\frac{1}{2}}$$

$$= 1$$

# Differential equations

## Generalities and definitions

### Definitions

- A differential equation is an equation involving the derivatives of a function.

- The ORDER of a differential equation is the same as the highest order of derivation occuring in the equation.

- A differential equation is linear if it is LINEAR in y and the derivative of y.

  (Any equation containing powers of y and/or its derivative or products of y and/or its derivatives are non-linear)

### Solving differential equations

- To solve a differential equation is to find all the functions satisfying the equation.

  All these solutions constitue a FAMILY of solutions.
- Solutions that involve ARBITRARY constants are called GENERAL SOLUTIONS.
- A solution which contains NO arbitrary CONSTANT is called a PARTICULAR SOLUTION.
- To work out a particular solution, you need initial/boundary conditions: $y(x_0) = y_0$

### Methods to solve *first order* differential equations

- *Method* 1: Direct integration

  This method can be used if the differential equation can be written as

  $\frac{dy}{dx} = f(x)$. By integrating both sides, you obtain

  $$y = \int f(x)dx$$

- *Method* 2: Separating variables

  This method can be used if the differential equation can be written as

  $g(y)\frac{dy}{dx} = f(x)$. By integrating both sides, you obtain

  $$\int g(y)dy = \int f(x)dx$$

- *Method* 3: Recognising the derivative of a product function

  This method can be used if the differential equation can be written as

  $$u\frac{dy}{dx} + \frac{du}{dx}y = f(x), \text{ where } u \text{ is a function of } x.$$

  Re-*write as* $\frac{d}{dx}(u \times y) = f(x)$ *and* integrate both sides:

  $$u \times y = \int f(x)dx \;\; \text{so} \;\; y = \frac{1}{u}\int f(x)dx$$

# First order linear differential equation

## Standard form

A first order linear equation can be re-arrange in the form

$$\frac{dy}{dx} + P(x)y = Q(x) \text{ where } P(x) \text{ and } Q(x) \text{ are two functions.}$$

This form is called the STANDARD form the equation.

## Integrating factors

Considering an equation $\frac{dy}{dx} + P(x)y = Q(x)$,

we want to multiple both sides by a function $I(x)$,

so that the left-hand side of the equation becomes the derivative of a product function.

$$\text{i.e } I \times \frac{dy}{dx} + IP \times y = IQ \qquad \text{with } \frac{dI}{dx} = IP$$

Such a function is called an INTEGRATING FACTOR and $I(x) = e^{\int P(x)dx}$

*Example*:

Find the general solution of the equation $\frac{dy}{dx} - \frac{1}{x}y = x^2$ where $x > 0$

- The integrating factor $I(x) = e^{\int -\frac{1}{x}dx} = e^{-\ln(x)} = e^{\ln\left(\frac{1}{x}\right)} = \frac{1}{x}$
- *Multiplying* the equation by $I(x)$, it becomes

$$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = x \qquad \text{this is} \qquad \frac{d}{dx}\left(\frac{1}{x} \times y\right) = x \text{ and by integrating}$$

$$\frac{1}{x}y = \frac{1}{2}x^2 + c \qquad y = \frac{1}{2}x^3 + cx \quad c \in \mathbb{R}$$

## Substitution

The substitution to use will be given to you in the question.
Use this substitution to transform the given differential equation into one which you can solve using either of the known methods:
(direct integration, separating variables, integrating factors)

Example:

$a)$ Use the substitution $z = \frac{1}{y}$ to transform the diff. eq. $\frac{dy}{dx} + xy = xy^2$ into a diff. eq in $z$ and $x$.

$b)$ Solve the new differential equation.

$c)$ Find $y$ in terms of $x$.

*Solution*: $z = \frac{1}{y}$ so $y = \frac{1}{z}$ and $\frac{dy}{dx} = -\frac{1}{z^2}\frac{dz}{dx}$

*after substitution, we have* $-\frac{1}{z^2}\frac{dz}{dx} + \frac{x}{z} = \frac{x}{z^2}$ $\qquad \frac{dz}{dx} - xz = -x$

An integrating factor is $I(x) = e^{\int -xdx} = e^{-\frac{x^2}{2}}$

$$e^{-\frac{x^2}{2}}\frac{dz}{dx} - xe^{-\frac{x^2}{2}}z = -xe^{-\frac{x^2}{2}} \qquad \frac{d}{dx}\left(e^{-\frac{x^2}{2}}z\right) = -xe^{-\frac{x^2}{2}}$$

$$e^{-\frac{x^2}{2}}z = \int -xe^{-\frac{x^2}{2}}dx = e^{-\frac{x^2}{2}} + c \qquad z = 1 + ce^{\frac{x^2}{2}} \quad \text{this gives } y = \frac{1}{ce^{\frac{x^2}{2}} + 1}$$

# Auxiliary equations method.

## Definitions

$a\dfrac{dy}{dx}+by=f(x)$ $\quad$ *with* $a,b\in \mathbb{R}$

- The REDUCED equation is $a\dfrac{dy}{dx}+by=0.$

  The general solution of the reduced equation is called

  The COMPLEMENTARY FUNCTION

- A PARTICULAR INTEGRAL satisfies the equation $a\dfrac{dy}{dx}+by=f(x)$

- The general solution of $a\dfrac{dy}{dx}+by=f(x)$ is

  the sum of the complementary function and the particular integral

  $y_G=y_P+y_C$

## Solving first order linear differential equations

$a\dfrac{dy}{dx}+by=f(x)$ is a differential equation where $a$ *and* $b$ are real numbers

The reduced eqaution is $a\dfrac{dy}{dx}+by=0$

- The AUXILIARY equation associated with this equation is $a\lambda+b=0$
- The complementary function is : $y=Ce^{\lambda x}$ *where* $\lambda$ is solution to $a\lambda+b=0.$
- Finding the particular integral:

  ⊗if $f(x)$ is a polynomial then $y_P$ is also a polynomial of the same degree

  ⊗if $f(x)=ACos(kx)+BSin(kx)$ then $y_P=aCos(kx)+bSin(kx)$

  *$a$ and $b$ to be worked out.*

  ⊗if $f(x)=Ae^{kx}$ then $y_P=ae^{kx}$ *if* $k\neq\lambda$

  $y_P=axe^{kx}$ *if* $k=\lambda$ $\quad$ *where a is to be worked out*
- The general solution is $y_G=y_P+y_C$

# Introduction to differential equations

## Chapter assessment

1. (a) Show that $xe^{2x}$ is an integrating factor for the first-order differential equation
$$\frac{dy}{dx} + \left(\frac{2x+1}{x}\right)y = e^{-2x}.$$
(b) Hence, find the general solution of $\frac{dy}{dx} + \left(\frac{2x+1}{x}\right)y = e^{-2x}$.

2. (a) Find the complementary function and a particular integral of the differential equation $\frac{dy}{dx} - 2y = 3 - 12x^2$.

(b) Hence, write down the general solution of $\frac{dy}{dx} - 2y = 3 - 12x^2$.

3. By using an integrating factor, find the solution of the differential equation
$$\frac{dy}{dx} + 2y\tan x = \sin^2 x$$
given that $y = 1$ when $x = \frac{\pi}{4}$.

4. (a) Find a particular integral of the differential equation
$$\frac{dy}{dx} + 4y = 12 + 17\cos x$$
(b) **Hence** find the general solution of this first order differential equation.

5. (a) Show that $x^2$ is an integrating factor for the first-order differential equation
$$\frac{dy}{dx} + \frac{2}{x}y = 6(x^3+9)^{\frac{1}{2}}$$
(b) Solve this differential equation, given that $y = 20$ when $x = 3$.

6. (a) Find the value of the constant $k$ for which $kxe^{2x}$ is a particular integral of the differential equation
$$\frac{dy}{dx} - 2y = 4e^{2x}$$
(b) Find the general solution of this differential equation.

7. Solve the differential equation
$$2x\frac{dy}{dx} + 6y = x^2 \ln x \qquad x > 0$$
given that $y = \frac{1}{25}$ when $x = 1$.

# Introduction to differential equations

## Solutions to chapter assessment

1. (a) Integrating factor $I = e^{\int \frac{2x+1}{x}\,dx}$

$= e^{\int 2+\frac{1}{x}\,dx}$

$= e^{2x+\ln x}$

$= e^{2x} \times e^{\ln x}$

$= e^{2x} \times x$

so $xe^{2x}$ is an integrating factor.

(b) $xe^{2x}\frac{dy}{dx} + xe^{2x}\left(\frac{2x+1}{x}\right)y = xe^{2x}e^{-2x}$

$xe^{2x}\frac{dy}{dx} + (e^{2x} + 2xe^{2x})y = x$

$\frac{d}{dx}\left(y(xe^{2x})\right) = x$

$yxe^{2x} = \int x\,dx = \frac{x^2}{2} + A$

$y = \frac{1}{2}xe^{-2x} + \frac{A}{x}e^{-2x}$

2. (a) Reduced equation $\frac{dy}{dx} - 2y = 0$ or $\frac{dy}{dx} = 2y$

The GS of $\frac{dy}{dx} = 2y$ is $y = Ae^{2x}$

so the CF of $\frac{dy}{dx} - 2y = 3 - 12x^2$ is $y = Ae^{2x}$.

Since $f(x) = 3 - 12x^2$ try a PI of the form $y = ax^2 + bx + c$

Substituting this into $\frac{dy}{dx} - 2y = 3 - 12x^2$ gives

$2ax + b - 2(ax^2 + bx + c) = 3 - 12x^2$

Equating coefficients of $x^2$ gives $\quad -2a = -12 \quad \Rightarrow a = 6$
Equating coefficients of $x$ gives $\quad 2a - 2b = 0 \quad \Rightarrow b = 6$
Equating constant terms gives $\quad b - 2c = 3 \quad \Rightarrow c = 1.5$
so a particular integral is $y = 6x^2 + 6x + 1.5$

(b) General solution of $\frac{dy}{dx} - 2y = 3 - 12x^2$ is

$y = Ae^{2x} + 6x^2 + 6x + 1.5$

3. Integrating factor $\quad I = e^{\int 2\tan x\,dx}$

$= e^{2\ln \sec x}$

$= e^{\ln(\sec x)^2}$

$= \sec^2 x$

$\sec^2 x\frac{dy}{dx} + 2y\tan x\sec^2 x = \sin^2 x\sec^2 x$

$\frac{d}{dx}(y\sec^2 x) = \frac{\sin^2 x}{\cos^2 x}$

$y\sec^2 x = \int \tan^2 x\,dx$

$= \int(\sec^2 x - 1)\,dx$

$= \tan x - x + A$

$y = \cos^2 x(\tan x - x + A)$

When $x = \frac{\pi}{4}$, $y = 1 \quad \Rightarrow 1 = \frac{1}{2}\left(1 - \frac{\pi}{4} + A\right)$

$\Rightarrow A = 1 + \frac{\pi}{4}$

So $\quad y = \cos^2 x\left(\tan x - x + 1 + \frac{\pi}{4}\right)$

4. (a) Since $f(x) = 12 + 17\cos x$ try a PI of the form $y = a + b\cos x + c\sin x$

Substituting this into $\frac{dy}{dx} + 4y = 12 + 17\cos x$ gives

$-b\sin x + c\cos x + 4(a + b\cos x + c\sin x) = 12 + 17\cos x$
Equating constant terms gives $\quad 4a = 12 \quad \Rightarrow a = 3$
Equating coefficients of $\cos x$ gives $\quad c + 4b = 17$
Equating coefficients of $\sin x$ gives $\quad 4c - b = 0$

Solving $c + 4b = 17$ and $4c - b = 0$ simultaneously gives $b = 4$, $c = 1$
so a particular integral is $y = 3 + 4\cos x + \sin x$

(b) Reduced equation $\frac{dy}{dx} + 4y = 0$ or $\frac{dy}{dx} = -4y$

The GS of $\frac{dy}{dx} = -4y$ is $y = Ae^{-4x}$

so the CF of $\frac{dy}{dx} + 4y = 12 + 17\cos x$ is $y = Ae^{-4x}$.

General solution is $y = Ae^{-4x} + 3 + 4\cos x + \sin x$

5. (a) Integrating factor $I = e^{\int \frac{2}{x}\,dx}$

$= e^{2\ln x}$

$= e^{\ln x^2}$

$= x^2$

> Since the answer is given in the question all these steps should be shown.

(b) $x^2 \frac{dy}{dx} + x^2 \times \frac{2}{x} y = x^2 \times 6(x^3 + 9)^{\frac{1}{2}}$

$x^2 \frac{dy}{dx} + 2xy = 6x^2(x^3 + 9)^{\frac{1}{2}}$

$\frac{d}{dx}(yx^2) = 6x^2(x^3 + 9)^{\frac{1}{2}}$

$yx^2 = \int 6x^2(x^3 + 9)^{\frac{1}{2}}\,dx$

$= \frac{4}{3}(x^3 + 9)^{\frac{3}{2}} + A$

> Instead of finding the integral by inspection you may prefer to use the substitution $u = (x^3 + 9)$

When $x = 3$, $y = 20$ $\Rightarrow 180 = \Rightarrow 180 = \frac{4}{3} \times 6^3 + A$

$\Rightarrow A = -108$

So $yx^2 = \frac{4}{3}(x^3 + 9)^{\frac{3}{2}} - 108$

$y = \frac{4}{3x^2}(x^3 + 9)^{\frac{3}{2}} - \frac{108}{x^2}$

6. (a) PI $y = kxe^{2x}$

Substituting this into $\frac{dy}{dx} - 2y = 4e^{2x}$ gives

$(ke^{2x} + kx2e^{2x}) - 2kxe^{2x} = 4e^{2x}$ $\Rightarrow ke^{2x} = 4e^{2x}$

$\Rightarrow k = 4$

(b) Reduced equation $\frac{dy}{dx} - 2y = 0$ or $\frac{dy}{dx} = 2y$

The GS of $\frac{dy}{dx} = 2y$ is $y = Ae^{2x}$

so the CF of $\frac{dy}{dx} - 2y = 4e^{2x}$ is $y = Ae^{2x}$

Particular integral of $\frac{dy}{dx} - 2y = 4e^{2x}$ is $y = 4xe^{2x}$

General solution is $y = Ae^{2x} + 4xe^{2x}$

7. Rearranging $2x\frac{dy}{dx} + 6y = x^2 \ln x$ to a form with unitary coefficient of $\frac{dy}{dx}$ gives

$\frac{dy}{dx} + \frac{3}{x}y = \frac{1}{2}x\ln x$

> Since $\frac{3}{x}$ is not a constant we cannot use the complementary function, particular integral method

Integrating factor $I = e^{\int \frac{3}{x}\,dx}$

$= e^{3\ln x}$

$= e^{\ln x^3}$

$= x^3$

$x^3\frac{dy}{dx} + x^3 \times \frac{3}{x}y = x^3 \times \frac{1}{2}x\ln x$

$x^3\frac{dy}{dx} + 3x^2y = \frac{1}{2}x^4\ln x$

$\frac{d}{dx}(yx^3) = \frac{1}{2}x^4 \ln x$

$yx^3 = \int \frac{1}{2}x^4 \ln x\,dx$

> Using integration by parts

$= \frac{1}{5}x^5 \times \frac{1}{2}\ln x - \int \frac{1}{5}x^5 \times \frac{1}{2} \times \frac{1}{x}\,dx$

$= \frac{1}{10}x^5 \ln x - \frac{1}{10}\int x^4\,dx$

$= \frac{1}{10}x^5 \ln x - \frac{1}{50}x^5 + A$

When $x = 1$, $y = \frac{1}{25}$ $\Rightarrow$ $\Rightarrow \frac{1}{25} = 0 - \frac{1}{50} + A$

$\Rightarrow A = \frac{3}{50}$

So $y = \frac{1}{10}x^2 \ln x - \frac{1}{50}x^2 + \frac{3}{50x^3}$

# Second order linear differential equations

## Definitions

$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$ with $a,b,c\in\mathbb{R}$

- The REDUCED equation is $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$.

  The general solution of the reduced equation is called
  The COMPLEMENTARY FUNCTION

- A PARTICULAR INTEGRAL satisfies the equation $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$

- The general solution of $a\frac{dy}{dx}+by=f(x)$ is
  the sum of the complementary function and the particular integral

  $y_G=y_P+y_C$

## Solving second order linear differential equations

$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$ is a differential equation where $a,b$ *and* $c$ are real numbers

The reduced equation is $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$

- The AUXILIARY equation associated with this equation is $a\lambda^2+b\lambda+c=0$

  The auxiliary equation is a quadratic equation, three cases are possible:

*Case* 1: $a\lambda^2+b\lambda+c=0$ has two distinct solutions $\lambda_1$ *and* $\lambda_2$

The complementary function is $y=C_1e^{\lambda_1x}+C_2e^{\lambda_2x}$ $\quad C_1,C_2\in\mathbb{R}$

*Case* 2: $a\lambda^2+b\lambda+c=0$ has equal/repeated root $\lambda_0$

The complementary function is $y=(C_1x+C_2)e^{\lambda_0x}$ $\quad C_1,C_2\in\mathbb{R}$

*Case* 3: $a\lambda^2+b\lambda+c=0$ has two conjugate complex solutions $\lambda_1=p+iq$ *and* $\lambda_2=p-iq$

The complementary function is $y=e^{px}\left(C_1Cos(qx)+C_2Sin(qx)\right)$ $\quad C_1,C_2\in\mathbb{R}$

- Finding the particular integral:

  ⊗ if $f(x)$ is a polynomial then $y_P$ is also a polynomial of the same degree

  ⊗ if $f(x)=ACos(kx)+BSin(kx)$ then $y_P=aCos(kx)+bSin(kx)$
  *a and b to be worked out.*

  ⊗ if $f(x)=Ae^{kx}$ then $y_P=ae^{kx}$ *if* $k\neq\lambda$

  *or* $y_P=axe^{kx}$ *if* $k=\lambda_1$ *or* $\lambda_2$ *where a is to be worked out*

  *or* $y_p=ax^2e^{kx}$ *if* $k=\lambda_0$ (*the repeated root.*)

- The general solution is $y_G=y_P+y_C$

## Substitution

$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=R(x)$ is a differential equation

where P,Q and R are functions of *x*.

Note: this equation is written in its standard form.

These equations are solved using substitution.
The substitution to use will be given in the question.

# Second order differential equations

## Chapter assessment

1. (a) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 0 \qquad [3]$$

(b) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 9x^2 \qquad [5]$$

2. (a) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 4y = 8 - 3\sin x. \qquad [7]$$

(b) Hence find $y$ in terms of $x$ given that $y = 2$ when $x = 0$ and $y = 2+\sqrt{2}$ when $x = \frac{\pi}{4}$. [3]

3. The differential equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 5y = 12e^{-x}$ has general solution

$$y = Ae^{-x} + Be^{5x} + p(x)$$

where $p(x)$ is a particular integral satisfying the given differential equation.

(a) Find $p(x)$. [3]

(b) Given that $y = 14$ and $\frac{dy}{dx} = 2$ when $x = 0$, find $y$ in terms of $x$. [3]

4. A differential equation is given by

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} = 3x^2$$

(a) Show that the substitution $u = \frac{dy}{dx}$ transforms this differential equation into

$$\frac{du}{dx} - \frac{1}{x}u = 3x. \qquad [4]$$

(b) Find the general solution of

$$\frac{du}{dx} - \frac{1}{x}u = 3x$$

giving your answer in the form $u = f(x)$. [4]

(c) Hence find the general solution of the differential equation

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} = 3x^2$$

giving your answer in the form $y = g(x)$. [2]

5. (i) Given that $v = xy$, show that

$$\frac{dy}{dx} = \frac{1}{x}\frac{dv}{dx} - \frac{v}{x^2}$$

and $$x\frac{d^2y}{dx^2} = \frac{d^2v}{dx^2} - \frac{2}{x}\frac{dv}{dx} + \frac{2v}{x^2}. \qquad [5]$$

(ii) Hence show that the substitution $v = xy$ transforms the differential equation

$$x\frac{d^2y}{dx^2} + (2-x)\frac{dy}{dx} - (1+2x)y = e^{2x}$$

into $$\frac{d^2v}{dx^2} - \frac{dv}{dx} - 2v = e^{2x}. \qquad [3]$$

(iii) Hence find the general solution of

$$x\frac{d^2y}{dx^2} + (2-x)\frac{dy}{dx} - (1+2x)y = e^{2x}. \qquad [8]$$

**Total 50 marks**

## Solutions to chapter assessment

1. (a) The auxiliary equation is $k^2 + 4k + 3 = 0$

$(k+3)(k+1) = 0$

$k = -3$ or $k = -1$

The general solution of $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 0$ $\quad y = Ae^{-3x} + Be^{-x}$

(b) From part (a), the complementary function of $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 9x^2$ is

$y_c = Ae^{-3x} + Be^{-x}$

For a particular integral try $y_P = ax^2 + bx + c$

$\frac{dy}{dx} = 2ax + b$

$\frac{d^2y}{dx^2} = 2a$

Substituting into $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 0$

gives $2a + 4(2ax + b) + 3(ax^2 + bx + c) = 9x^2$

Equating coefficients of $x^2$: $3a = 9 \quad \Rightarrow \quad a = 3$

Equating coefficients of $x$: $8a + 3b = 0 \quad \Rightarrow \quad b = -8$

Equating constant terms: $2a + 4b + 3c = 0 \quad \Rightarrow \quad c = \frac{26}{3}$

The particular integral is $y_p = 3x^2 - 8x + \frac{26}{3}$

The general solution is $y = y_c + y_P$

$$y = Ae^{-3x} + Be^{-x} + 3x^2 - 8x + 8\frac{2}{3}$$

2. (a) The auxiliary equation is $k^2 + 4 = 0$

$k = \pm 2i$

The complementary function of $\frac{d^2y}{dx^2} + 4y = 8 - 3\sin x$

is $y_c = A\cos 2x + B\sin 2x$

For a particular integral try $y_P = a + b\cos x + c\sin x$

$\frac{dy}{dx} = -b\sin x + c\cos x$

$\frac{d^2y}{dx^2} = -b\cos x - c\sin x$

Since the complementary function does not contain terms of the form $a$ (constant) or terms of the form $\sin x$ or $\cos x$ the 'obvious' particular integral, $y = a + b\cos x + c\sin x$ is appropriate.

Substitute into the differential equation:

gives $-b\cos x - c\sin x + 4(a + b\cos x + c\sin x) = 8 - 3\sin x$

Equating constant terms: $4a = 8 \quad \Rightarrow \quad a = 2$

Equating coefficients of $\cos x$: $3b = 0 \quad \Rightarrow \quad b = 0$

Equating coefficients of $\sin x$: $3c = -3 \quad \Rightarrow \quad c = -1$

A particular integral is $y_p = 2 - \sin x$

The general solution of $\frac{d^2y}{dx^2} + 4y = 8 - 3\sin x$

is $y = A\cos 2x + B\sin 2x + 2 - \sin x$

(b) $y = A\cos 2x + B\sin 2x + 2 - \sin x$

When $x = 0$, $y = 2 \quad \Rightarrow 2 = A + 2 \quad \Rightarrow A = 0$

When $x = \frac{\pi}{4}$, $y = 2 + \sqrt{2} \quad \Rightarrow 2 + \sqrt{2} = B + 2 - \frac{1}{\sqrt{2}} \quad \Rightarrow B = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$

So $y = \frac{3\sqrt{2}}{2}\sin 2x + 2 - \sin x$

3. (a) Since $Ae^{-x}$ is part of the complementary function, for a particular integral try $y_P = axe^{-x}$

$$\frac{dy_P}{dx} = ae^{-x} - axe^{-x}$$

$$\frac{d^2y_P}{dx^2} = -ae^{-x} - axe^{-x} + axe^{-x} = axe^{-x} - 2ae^{-x}$$

Substitute into $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 5y = 12e^{-x}$

gives $axe^{-x} - 2ae^{-x} - 4(ae^{-x} - axe^{-x}) - 5axe^{-x} = 12e^{-x}$

$-6ae^{-x} = 12e^{-x}$

$a = -2$

So $p(x) = -2xe^{-x}$

(b) General solution is $y = Ae^{-x} + Be^{5x} - 2xe^{-x}$

$$\frac{dy}{dx} = -Ae^{-x} + 5Be^{5x} - 2e^{-x} + 2xe^{-x}$$

When $x = 0$, $y = 14$ $\Rightarrow$ $14 = A + B$

When $x = 0$, $\frac{dy}{dx} = 2$ $\Rightarrow$ $4 = -A + 5B$

$\Rightarrow A = 11,\ B = 3$

So $y = 11e^{-x} + 3e^{5x} - 2xe^{-x}$

4. (a) $u = \frac{dy}{dx} \Rightarrow \frac{du}{dx} = \frac{d^2y}{dx^2}$

So $x\frac{d^2y}{dx^2} - \frac{dy}{dx} = 3x^2$ becomes $x\frac{du}{dx} - u = 3x^2$

or $\frac{du}{dx} - \frac{1}{x}u = 3x$ as required.

(b) An integrating factor for $\frac{du}{dx} - \frac{1}{x}u = 3x$

is $I = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = \frac{1}{x}$

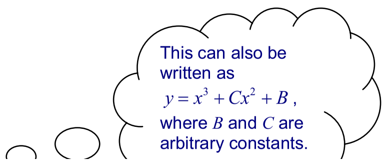
So $\frac{1}{x}\frac{du}{dx} - \frac{1}{x^2}u = 3 \Rightarrow \frac{d}{dx}\left(\frac{u}{x}\right) = 3$

$\Rightarrow \frac{u}{x} = 3x + A$

$\Rightarrow u = 3x^2 + Ax$

(c) Replacing $u$ by $\frac{dy}{dx}$ gives $\frac{dy}{dx} = 3x^2 + Ax$

Integrating gives $y = x^3 + \frac{A}{2}x^2 + B$




5. (i) $v = xy$

$$\frac{dv}{dx} = y + x\frac{dy}{dx}$$

$$\frac{dv}{dx} = \frac{v}{x} + x\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{x}\frac{dv}{dx} - \frac{v}{x^2}$$

$$\frac{dv}{dx} = y + x\frac{dy}{dx}$$

$$\frac{d^2v}{dx^2} = \frac{dy}{dx} + \frac{dy}{dx} + x\frac{d^2y}{dx^2} = 2\frac{dy}{dx} + x\frac{d^2y}{dx^2}$$

$$\frac{d^2v}{dx^2} = 2\left(\frac{1}{x}\frac{dv}{dx} - \frac{v}{x^2}\right) + x\frac{d^2y}{dx^2}$$

$$x\frac{d^2y}{dx^2} = \frac{d^2v}{dx^2} - \frac{2}{x}\frac{dv}{dx} + \frac{2v}{x^2}$$

(ii) $$\frac{d^2v}{dx^2} - \frac{2}{x}\frac{dv}{dx} + \frac{2v}{x^2} + (2-x)\left(\frac{1}{x}\frac{dv}{dx} - \frac{v}{x^2}\right) - (1+2x)\frac{v}{x} = e^{2x}$$

$$\frac{d^2v}{dx^2} - \frac{2}{x}\frac{dv}{dx} + \frac{2v}{x^2} + \frac{2}{x}\frac{dv}{dx} - \frac{2v}{x^2} - \frac{dv}{dx} + \frac{v}{x} - \frac{v}{x} - 2v = e^{2x}$$

$$\frac{d^2v}{dx^2} - \frac{dv}{dx} - 2v = e^{2x}$$

(iii) $\dfrac{d^2v}{dx^2} - \dfrac{dv}{dx} - 2v = e^{2x}$

Auxiliary equation: $k^2 - k - 2 = 0$

$(k-2)(k+1) = 0$

$k = 2$ or $k = -1$

Complementary function: $v = Ae^{2x} + Be^{-x}$

Particular integral has form $v = axe^{2x}$

$$\frac{dv}{dx} = ae^{2x} + 2axe^{2x}$$

$$\frac{d^2v}{dx^2} = 2ae^{2x} + 2ae^{2x} + 4axe^{2x}$$

$$= 4ae^{2x} + 4axe^{2x}$$

Substituting into differential equation

$$4ae^{2x} + 4axe^{2x} - (ae^{2x} + 2axe^{2x}) - 2axe^{2x} = e^{2x}$$

$$3ae^{2x} = e^{2x}$$

$$a = \tfrac{1}{3}$$

Particular integral is $v = \tfrac{1}{3}xe^{2x}$

General solution is $v = Ae^{2x} + Be^{-x} + \tfrac{1}{3}xe^{2x}$

$$v = xy \quad \Rightarrow y = \frac{v}{x}$$

General solution for original equation is $v = \dfrac{Ae^{2x} + Be^{-x}}{x} + \tfrac{1}{3}e^{2x}$.

# Numerical methods to solve first order differential equation

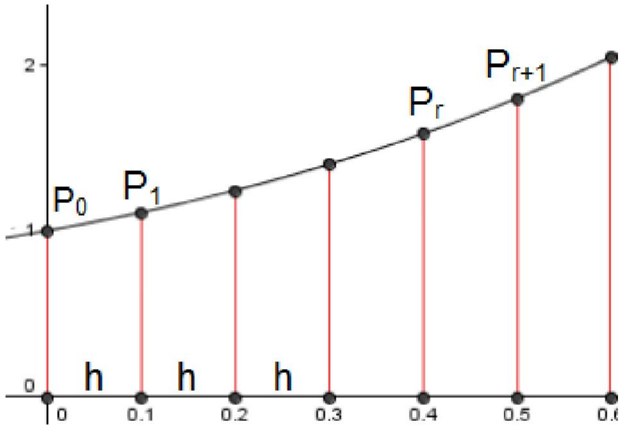
In this chapter, we want to solve equations which can be written

$$\frac{dy}{dx} = f(x,y) \quad and \quad y(x_0) = y_0$$

There are three methods to solve numerically this equation.

Formulae to be used will be stated explicitly in questions.

*Knowing* $P_0(x_0, y_0)$, we work out $P_1$ *then* $P_2$ *then* $P_3$ *etc.*




## Euler's formula

To work out $P_{r+1}$, we consider that

the gradient of the line $P_rP_{r+1}$ is (approx.) equal to the gradient at $P_r$.

This gives: $y_{r+1} = y_r + hf(x_r, y_r)$

## The mid-point formula

We consider that the gradient of the line $P_{r-1}P_{r+1}$ is (approx.) equal to the gradient at $P_r$ :

This gives $y_{r+1} = y_{r-1} + 2hf(x_r, y_r)$

## The improved Euler's formula

We consider that the gradient of the line $P_rP_{r+1}$ is ( approx.) the mean of

the gradient at $P_r$ *and* the gradient at $P_{r+1}$.

This gives : $$y_{r+1} = y_r + \frac{h}{2}\left[ f(x_r, y_r) + f(x_{r+1}, y^*_{r+1}) \right]$$

$$with \ \ y^*_{r+1} = y_r + hf(x_r, y_r)$$

*Or* as it is given in the exam question:

$$y_{r+1} = y_r + \frac{1}{2}(k_1 + k_2)$$

$$where \ \ k_1 = hf(x_r, y_r) \ \ and \ \ k_2 = hf(x_r + h, y_r + k_1)$$

Possible layout for your workings out:

| $r$ | $x_r$ | $y_r$ | $k_1$ | $x_r + h$ | $y_r + k_1$ | $k_2$ | $y_{r+1}$ |
|---|---|---|---|---|---|---|---|
| 0 | | | | | | | |
| 1 | | | | | | | |
| 2 | | | | | | | |

# Numerical methods for the solution of first order differential equations

## Chapter assessment

1. The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = f(x, y)$$

where $f(x, y) = \sin(x^2 + y^2)$
and $y(0) = 1.5$

Show that the Euler formula

$$y_{r+1} = y_r + h\,f(x_r, y_r)$$

with $h = 0.2$, gives $y(0.2) = 1.656$ correct to three decimal places. [4]

2. The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = x^2 + y^2$$

(i) The use of the Euler formula

$$y_{r+1} = y_r + h\,f(x_r, y_r)$$

with $y_0 = y(1) = 1$ gives $y_1 = y(1+h) = 1.1$. Determine the value of $h$ that has been used. [4]

(ii) Show that, with this value of $h$, use of the same Euler formula gives $y_2 = 1.2156$, correct to four decimal places. [4]

(iii) Use the formula

$$y_{r+1} = y_{r-1} + 2h\,f(x_r, y_r)$$

with the same value of $h$ to find $y_3$, giving your answer to three decimal places. [5]

3. The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = f(x, y)$$

where $f(x, y) = \sqrt{x^2 + y^2}$
and $y(1) = 0.5$

(a) Use the Euler formula

$$y_{r+1} = y_r + h\,f(x_r, y_r)$$

with $h = 0.1$ to obtain an approximation to $y(1.1)$ giving your answer to four decimal places. [4]

(b) (i) Use the formula

$$y_{r+1} = y_r + \tfrac{1}{2}(k_1 + k_2)$$

where $k_1 = h\,f(x_r, y_r)$
and $k_2 = h\,f(x_r + h, y_r + k_1)$
with $h = 0.1$ to obtain a further approximation to $y(1.1)$ [5]

(ii) Use the formula given in part (b)(i), together with your value for $y(1.1)$ obtained in part (b)(i), to obtain an approximation to $y(1.2)$, giving your answer to three decimal places. [5]

4. The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = (x + y)^2$$

and also the condition $y(0) = 1$.

(i) Use the formula

$$y_{r+1} = y_{r-1} + 2h\,f(x_r, y_r)$$

with $h = 0.2$ and $y(0.2) = 1.3085$ to find an approximation to $y(0.4)$. [5]

(ii) The general solution of the differential equation $\frac{dy}{dx} = (x + y)^2$ is $y = \tan(x + A) - x$.

Given that $0 < A < \pi$, use the boundary condition $y(0) = 1$ to find the exact value of $A$. [2]

(iii) Hence calculate, to two decimal places, the percentage error in the value of $y(0.4)$ obtained in part (i). [2]

**Total 40 marks**

## Solutions to chapter assessment

1. $x_0 = 0, \quad y_0 = 1.5, \quad h = 0.2, \quad f(x,y) = \sin(x^2 + y^2)$ and $y(0.2) = y_1$

using $\quad y_{r+1} = y_r + h\,f(x_r, y_r)$

$y_1 = y_0 + h\sin(x_0^2 + y_0^2)$

$= 1.5 + 0.2\sin(0^2 + 1.5^2)$

$= 1.6556\ldots$

$y(0.2) = 1.656$ (to 3 d.p.)

2. (i) $x_0 = 1, \quad y_0 = 1, \quad f(x,y) = x^2 + y^2$ and $y_1 = 1.1$

using $\quad y_{r+1} = y_r + h\,f(x_r, y_r)$

$y_1 = y_0 + h\,(x_0^2 + y_0^2)$

$1.1 = 1 + h(1^2 + 1^2)$

$0.1 = 2h$

$h = 0.05$

(ii) $x_1 = 1.05, \quad y_1 = 1.1, \quad h = 0.05$ and $f(x, y) = f(x,y) = x^2 + y^2$

using $\quad y_{r+1} = y_r + h\,f(x_r, y_r)$

with $r = 1$ gives $\quad y_2 = y_1 + h\,(x_1^2 + y_1^2)$

$= 1.1 + 0.05(1.05^2 + 1.1^2)$

$= 1.215625$

$y_2 = 1.2156$ (to 4 d.p.)

(iii) $y_1 = 1.1, \quad y_2 = 1.215625, \quad x_2 = 1.1, \quad h = 0.05$ and $f(x,y) = x^2 + y^2$

using $\quad y_{r+1} = y_{r-1} + 2h\,f(x_r, y_r)$

with $r = 2$ gives $\quad y_3 = y_1 + 2h\,(x_2^2 + y_2^2)$

$= 1.1 + 2 \times 0.05\left(1.1^2 + 1.215625^2\right)$

$= 1.368774\ldots$

$y_3 = 1.369$ (to 3 d.p.)

3. (a) $x_0 = 1, \quad y_0 = 0.5, \quad h = 0.1, \quad f(x,y) = \sqrt{x^2 + y^2}$ and $y(1.1) = y_1$

using $\quad y_{r+1} = y_r + h\,f(x_r, y_r)$

with $r = 0$ gives $\quad y_1 = y_0 + h\sqrt{x_0^2 + y_0^2}$

$= 0.5 + 0.1\sqrt{1^2 + 0.5^2}$

$= 0.611803\ldots$

$y(1.1) = 0.6118$ (to 4 d.p.)

(b) (i) $x_0 = 1, \quad y_0 = 0.5, \quad h = 0.1, \quad f(x,y) = \sqrt{x^2 + y^2}$ and $y(1.1) = y_1$

$k_1 = h\,f(x_0, y_0)$

$= 0.1 \times \sqrt{1^2 + 0.5^2}$

$= 0.111803\ldots$

$k_2 = h\,f(x_0 + h,\ y_0 + k_1)$

$= 0.1 \times f(1.1, 0.611803)$

$= 0.1\sqrt{1.1^2 + 0.611803^2}$

$= 0.125869\ldots$

$y_1 = y_0 + \tfrac{1}{2}(k_1 + k_2)$

$= 0.5 + \tfrac{1}{2}(0.111803 + 0.125869)$

$= 0.618836\ldots$

$y(1.1) = 0.6188$ (to 4 d.p.)

(ii) $x_0 = 1.1, \quad y_0 = 0.618836, \quad h = 0.1, \quad f(x,y) = \sqrt{x^2 + y^2}$

$k_1 = hf(x_0, y_0)$

$= hf(1.1, 0.6188836)$

$= 0.1\sqrt{1.1^2 + 0.618836^2}$

$= 0.126212\ldots$

$k_2 = h\,f(x_0 + h,\ y_0 + k_1)$

$= 0.1\ f(1.2,\ 0.745048)$

$= 0.1\sqrt{1.2^2 + 0.745048^2}$

$= 0.1412479\ldots$

$$y_1 = y_0 + \tfrac{1}{2}(k_1 + k_2)$$
$$= 0.618836 + \tfrac{1}{2}(0.126212 + 0.141248)$$
$$= 0.752565\ldots$$

$y(1.2) = 0.753$ (to 3 d.p.)

4. (i) $x_0 = 0, \quad y_0 = 1, \quad x_1 = 0.2, \quad y_1 = 1.3085$ and $y(0.4) = y_2$

$f(x, y) = (x + y)^2$

Using $\quad y_{r+1} = y_{r-1} + 2h\,f(x_r, y_r)$

with $r = 1$ gives $\quad y_2 = y_0 + 2 \times 0.2 \times (x_1 + y_1)^2$
$$= 1 + 0.4(0.2 + 1.3085)^2$$
$$= 1.9102289$$

$y(0.4) = 1.910$ (to 3 d.p.)

(ii) $y = \tan(x + A) - x$

When $x = 0$, $y = 1$ so $1 = \tan A \Rightarrow A = \dfrac{\pi}{4}$

(iii) Using the exact solution $\quad y(x) = \tan\left(x + \dfrac{\pi}{4}\right) - x$

$$y(0.4) = \tan\left(0.4 + \frac{\pi}{4}\right) - 0.4$$
$$= 2.0649627\ldots$$

$$\text{Percentage error} = \frac{2.0649927\ldots - 1.9102289}{2.0649927\ldots} \times 100$$
$$= 7.4933\ldots$$

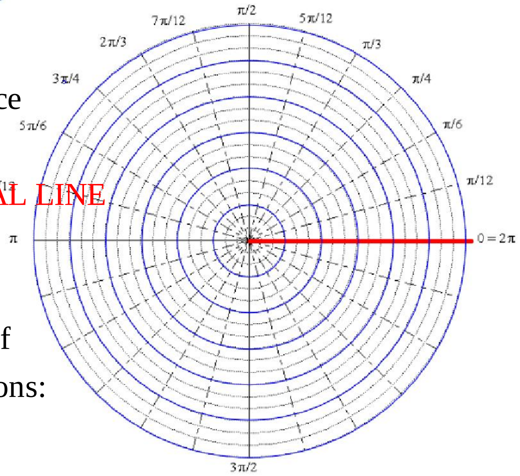
Percentage error $= 7.49$ (to 2 d.p.)

# Polar coordinates

## Definitions

A point M can be placed in a set of two axis using CARTESIAN coordinates $M(x,y)$

This position can also be determied by the distance from the origin O or POLE and the angle made by the line OM with the positive x-axis or INITIAL LINE
The POLAR coordiantes of M(r, $\theta$)



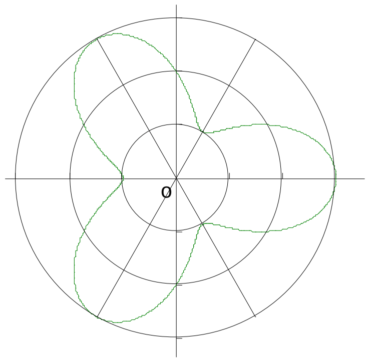

In order to have unicity in the polar coordinates of a given point, we will use the following conventions:

$$r > 0 \ \ and \ \ -\pi < \theta \le \pi \ or \ 0 \le \theta < 2\pi$$

## Curve in Polar coordinates

In cartesian coordinates, an explicit equation of a curve will be given as $y = f(x)$.

In polar coordinates, an explicit equation of a curve will be given as $r = f(\theta)$.




*Examples* : $r = 2Sin\theta$ , $r = e^{-2\theta}$, $r = 3,...$

## Conversions

A point M has cartesian coordinates $M(x,y)$
and polar coordinates M(r, $\theta$)
Using the pythagoras' theorem and trigonometry, we have

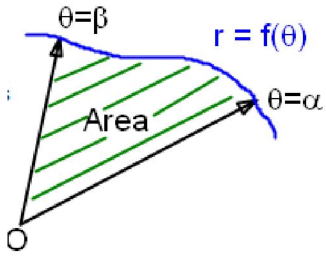
$$\begin{cases} x = rCos\theta \\ y = rSin\theta \end{cases} \quad and \quad \begin{cases} r^2 = x^2 + y^2 \\ Tan\theta = \dfrac{y}{x} \end{cases}$$

*Note* : If $x > 0, \theta = ArcTan(\frac{y}{x})$ but if $x < 0, \theta = ArcTan(\frac{y}{x}) \pm \pi$

## Area bounded by a polar curve

The are bounded by a polar curve and the ray $\theta = \alpha$ and $\theta = \beta$ is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta$$




*IMPORTANT* : This formula is valid only if $r > 0$ for $\alpha \le \theta \le \beta$.

# Polar coordinates

## Chapter assessment

1. A curve has polar equation $r = \dfrac{6}{1+\sin\theta}$.
   Find its cartesian equation in the form $y = f(x)$.

2. The curve whose polar equation is
   $$r = 5 + 2\cos\theta, \quad -\pi < \theta \le \pi$$
   and the line whose polar equation is
   $$r = 3\sec\theta$$
   intersect at the points $A$ and $B$.
   Find the polar coordinates of $A$ and $B$.

3. A straight line through the pole $O$ meets the curve with polar equation
   $$r = \frac{4}{2+\sin\theta}$$
   at the points $A$ and $B$.
   Show that $\dfrac{1}{OA} + \dfrac{1}{OB} = 1$.

4. The polar equation of a curve $C$ is
   $$r = \frac{1}{\theta}, \qquad \frac{\pi}{4} \le \theta \le 2\pi.$$

   (a) Sketch $C$.

   (b) The point $A$ on $C$ is the point where $\theta = \dfrac{\pi}{4}$ and the point $B$ on $C$ is the point where $\theta = 2\pi$. The point $O$ is the pole.

   (i) Given that $P$ is the point on $C$ where $\theta = \alpha$, show that the area of the region bounded by the curve $C$ and the lines $OA$ and $OP$ is
   $$\frac{2}{\pi} - \frac{1}{2\alpha}$$

   (ii) Hence find the value of $\alpha$ for which $OP$ bisects the area between the curve $C$ and the lines $OA$ and $OB$.

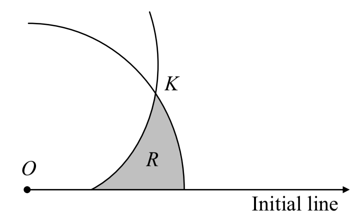
(AQA Jan2003 MAP5)



5. (a) A curve $C$ has polar equation
   $$r = e^{k\theta}$$
   where $k \ne 0$ and $0 \le \theta \le \frac{1}{2}\pi$.

   The points $P$ and $Q$ on $C$ have polar coordinates $(r_1, \theta_1)$ and $(r_2, \theta_2)$ respectively, where $\theta_2 > \theta_1$.
   Show that the area $A$ bounded by $C$ and the lines $OP$ and $OQ$, where $O$ is the pole, is given by
   $$A = \frac{1}{4k}\left(r_2^{\,2} - r_1^{\,2}\right).$$

   (b) The diagram shows a sketch of part of the curve $r = e^{\theta}$ and part of the circle $r = 2$.




   (i) Find the polar coordinates of $K$, the point of intersection of the two curves.

   (ii) Find the area of the shaded region $R$, between the curves and the initial line, giving your answer in the form $p\ln 2 + q$ where $p$ and $q$ are rational numbers.

(AQA Jan2004 MAP5)

# Polar coordinates

## Solutions to chapter assessment

1. $r = \dfrac{6}{1+\sin\theta}$

$r(1+\sin\theta) = 6$

$r + r\sin\theta = 6$

$\sqrt{x^2+y^2} + y = 6$ (using $r^2 = x^2 + y^2$ and $y = r\sin\theta$)

$\sqrt{x^2+y^2} = 6 - y$ (rearranged before squaring)

$x^2 + y^2 = (6-y)^2$

$x^2 + \cancel{y^2} = 36 - 12y + \cancel{y^2}$

$12y = 36 - x^2$

$y = 3 - \dfrac{x^2}{12}$

2. At the points of intersection of $r = 5 + 2\cos\theta$ and $r = 3\sec\theta$

$5 + 2\cos\theta = 3\sec\theta$

$\cos\theta(5 + 2\cos\theta) = 3$

$2\cos^2\theta + 5\cos\theta - 3 = 0$

$(2\cos\theta - 1)(\cos\theta + 3) = 0$

$\Rightarrow \cos\theta = \frac{1}{2}$, since $\cos\theta = -3$ is not possible for real $\theta$.

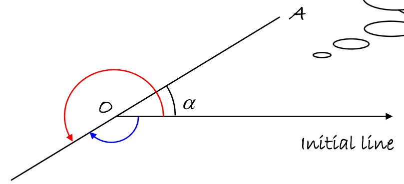
In the interval $-\pi < \theta \le \pi$, when $\cos\theta = \frac{1}{2}$,

$\theta = \pm\dfrac{\pi}{3}$

and $r = 6$ (substituting $\cos\theta = \frac{1}{2}$ into $r = 5 + 2\cos\theta$)

The polar coordinates of the points of intersection $A$ and $B$ are $\left(6, -\dfrac{\pi}{3}\right)$ and $\left(6, \dfrac{\pi}{3}\right)$.

3. If $\theta = \alpha$ at $A$ then $\theta = \pi + \alpha$ at $B$ (or you could use $\theta = -(\pi - \alpha)$ at $B$)




$r = \dfrac{4}{2+\sin\theta}$

When $\theta = \alpha$, $r = OA$ so $OA = \dfrac{4}{2+\sin\alpha} \Rightarrow \dfrac{1}{OA} = \dfrac{2+\sin\alpha}{4}$

When $\theta = \pi + \alpha$, $r = OB$ so $OB = \dfrac{4}{2+\sin(\pi+\alpha)} = \dfrac{4}{2-\sin\alpha}$

$\Rightarrow \dfrac{1}{OB} = \dfrac{2-\sin\alpha}{4}$

$\dfrac{1}{OA} + \dfrac{1}{OB} = \dfrac{2+\sin\alpha}{4} + \dfrac{2-\sin\alpha}{4} = \dfrac{4}{4} = 1$

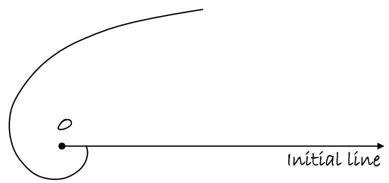
4. (a) $r = \dfrac{1}{\theta}$, $\dfrac{\pi}{4} \le \theta \le 2\pi$.

$r$ is never equal to 0 in the interval $\dfrac{\pi}{4} \le \theta \le 2\pi$.

As $\theta$ increases, $r$ decreases.

| $\theta$ | $\frac{\pi}{4}$ | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ |
|---|---|---|---|---|---|
| $r$ | $\frac{4}{\pi}$ | $\frac{2}{\pi}$ | $\frac{1}{\pi}$ | $\frac{2}{3\pi}$ | $\frac{1}{2\pi}$ |

The curve starts at the point $\left(\dfrac{4}{\pi}, \dfrac{\pi}{4}\right)$ and ends at the point $\left(\dfrac{1}{2\pi}, 2\pi\right)$.



(b) (i) $A = \frac{1}{2}\int_{\frac{\pi}{4}}^{\alpha} \frac{1}{\theta^2}\, d\theta$

$= \left[-\frac{1}{2\theta}\right]_{\frac{\pi}{4}}^{\alpha}$

$= -\frac{1}{2\alpha} + \frac{2}{\pi}$

Area bounded by $OA$, $OP$ and curve $C$ is $\frac{2}{\pi} - \frac{1}{2\alpha}$

(ii) Putting $\alpha = 2\pi$, area bounded by $OA$, $OB$ and curve $C$ is $\frac{2}{\pi} - \frac{1}{4\pi}$.

$\frac{2}{\pi} - \frac{1}{2\alpha} = \frac{1}{2}\left(\frac{2}{\pi} - \frac{1}{4\pi}\right)$

$\frac{1}{2\alpha} = \frac{2}{\pi} - \frac{1}{\pi} + \frac{1}{8\pi}$

$\frac{1}{2\alpha} = \frac{9}{8\pi}$

$\alpha = \frac{4\pi}{9}$

Since $OP$ bisects the area bounded by $OA$, $OB$ and curve $C$, the area bounded by $OA$, $OP$ and curve $C$ is half that bounded by $OA$, $OB$ and curve $C$.

5.(a) $A = \frac{1}{2}\int_{\theta_1}^{\theta_2} e^{2k\theta}\, d\theta$

$r^2 = e^{k\theta} \times e^{k\theta} = e^{k\theta}$

$= \frac{1}{4k}\left[e^{2k\theta}\right]_{\theta_1}^{\theta_2}$

$= \frac{1}{4k}\left(e^{2k\theta_2} - e^{2k\theta_1}\right)$

$A = \frac{1}{4k}\left(r_2^{\,2} - r_1^{\,2}\right)$

Points $(r_1, \theta_1)$ and $(r_2, \theta_2)$ lie on the curve $r = e^{k\theta}$ so $r_1^{\,2} = e^{2k\theta_1}$ and $r_2^{\,2} = e^{2k\theta_2}$

(b) (i) At $K$, $r = 2$ and $r = e^{\theta} \Rightarrow 2 = e^{\theta}$

$\Rightarrow \theta = \ln 2$

Coordinates of $K$ are $(2, \ln 2)$

(ii) Area of sector of circle bounded by initial line and $OK$ is $\frac{1}{2} \times 2^2 \times \ln 2$

$= 2\ln 2$

Area bounded by the curve $r = e^{\theta}$ from $(1, 0)$ to $(2, \ln 2)$, initial line and $OK$ is $\frac{1}{4}\left(2^2 - 1^2\right) = \frac{3}{4}$

Using answer to part (a) with $k = 1$, $r_1 = 1$ and $r_2 = 2$

Area of shaded region $= 2\ln 2 - \frac{3}{4}$

# Past papers

# January 2008

Answer **all** questions.

**1** The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = f(x, y)$$

where $$f(x, y) = x^2 - y^2$$

and $$y(2) = 1$$

(a) Use the Euler formula

$$y_{r+1} = y_r + hf(x_r, y_r)$$

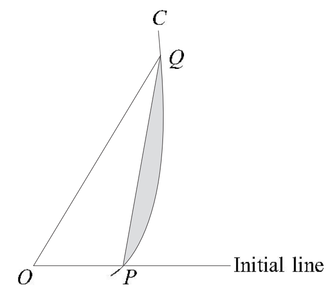
with $h = 0.1$, to obtain an approximation to $y(2.1)$. *(3 marks)*

(b) Use the formula

$$y_{r+1} = y_{r-1} + 2hf(x_r, y_r)$$

with your answer to part (a), to obtain an approximation to $y(2.2)$. *(3 marks)*

**2** The diagram shows a sketch of part of the curve $C$ whose polar equation is $r = 1 + \tan\theta$.
The point $O$ is the pole.




The points $P$ and $Q$ on the curve are given by $\theta = 0$ and $\theta = \frac{\pi}{3}$ respectively.

(a) Show that the area of the region bounded by the curve $C$ and the lines $OP$ and $OQ$ is

$$\frac{1}{2}\sqrt{3} + \ln 2$$ *(6 marks)*

(b) Hence find the area of the shaded region bounded by the line $PQ$ and the arc $PQ$ of $C$. *(3 marks)*

**3** (a) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 5$$ *(6 marks)*

(b) Hence express $y$ in terms of $x$, given that $y = 2$ and $\frac{dy}{dx} = 3$ when $x = 0$. *(4 marks)*

**4** (a) Explain why $\int_1^\infty xe^{-3x}\,dx$ is an improper integral. *(1 mark)*

(b) Find $\int xe^{-3x}\,dx$. *(3 marks)*

(c) Hence evaluate $\int_1^\infty xe^{-3x}\,dx$, showing the limiting process used. *(3 marks)*

**5** By using an integrating factor, find the solution of the differential equation

$$\frac{dy}{dx} + \frac{4x}{x^2+1}y = x$$

given that $y = 1$ when $x = 0$. Give your answer in the form $y = f(x)$. *(9 marks)*

**6** A curve $C$ has polar equation

$$r^2 \sin 2\theta = 8$$

(a) Find the cartesian equation of $C$ in the form $y = f(x)$. *(3 marks)*

(b) Sketch the curve $C$. *(1 mark)*

(c) The line with polar equation $r = 2\sec\theta$ intersects $C$ at the point $A$. Find the polar coordinates of $A$. *(4 marks)*

**7** (a) (i) Write down the expansion of $\ln(1+2x)$ in ascending powers of $x$ up to and including the term in $x^3$. *(2 marks)*

(ii) State the range of values of $x$ for which this expansion is valid. *(1 mark)*

(b) (i) Given that $y = \ln\cos x$, find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$. *(4 marks)*

(ii) Find the value of $\frac{d^4y}{dx^4}$ when $x = 0$. *(3 marks)*

(iii) Hence, by using Maclaurin's theorem, show that the first two non-zero terms in the expansion, in ascending powers of $x$, of $\ln\cos x$ are

$$-\frac{x^2}{2} - \frac{x^4}{12}$$

*(2 marks)*

(c) Find

$$\lim_{x \to 0}\left[\frac{x\ln(1+2x)}{x^2 - \ln\cos x}\right]$$

*(3 marks)*

**8** (a) Given that $x = e^t$ and that $y$ is a function of $x$, show that:

(i) $x\frac{dy}{dx} = \frac{dy}{dt}$; *(3 marks)*

(ii) $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$. *(3 marks)*

(b) Hence find the general solution of the differential equation

$$x^2\frac{d^2y}{dx^2} - 6x\frac{dy}{dx} + 6y = 0$$

*(5 marks)*

**END OF QUESTIONS**

# AQA – Further pure 3 – Jan 2008 – Answers

| Question 1: | Exam report |
|---|---|
| $\frac{dy}{dx} = f(x,y) = x^2 - y^2 \quad and \quad y(2) = 1$<br><br>$a)\ y(2.1) = 1 + 0.1\left(2^2 - 1^2\right) = 1.3$<br>$b)\ y(2.2) = y(2) + 2(0.1)(f(2.1, y(2.1)))$<br>$= 1 + 2\times 0.1\left(2.1^2 - 1.3^2\right) = 1.544$ | Numerical solutions of first order differential equations continues to be a good source of marks for all candidates. This was the best answered question on the paper. Although almost all candidates obtained the correct answer to part (a), some less able candidates showed a lack of understanding of the notation used in the given formula in part (b). |

| Question 2: | Exam report |
|---|---|
| $r = 1 + Tan\theta$<br>$P(1,0) \ \ and \ \ Q(\sqrt{3}, \frac{\pi}{3})$<br>$a)\ A = \frac{1}{2}\int_0^{\frac{\pi}{3}} (1+Tan\theta)^2 d\theta = \frac{1}{2}\int_0^{\frac{\pi}{3}} 1 + Tan^2\theta + 2Tan\theta d\theta$<br>$A = \left[\frac{1}{2}Tan\theta - \ln\lvert Cosx\rvert\right]_0^{\frac{\pi}{3}} = \left(\frac{\sqrt{3}}{2} - \ln\frac{1}{2}\right) - (0-0) = \frac{\sqrt{3}}{2} + \ln(2)$<br>$b)$ The area of the triangle OPQ is $\frac{1}{2}OP\times OQ\times SinPOQ$<br>$Area_{OPQ} = \frac{1}{2}\times 1\times\sqrt{3}Sin\frac{\pi}{3} = \frac{3}{4}$<br>The area shaded is therefore : $\frac{\sqrt{3}}{2} + \ln(2) - \frac{3}{4}$ | This question, which tested the areas of regions involving a curve whose equation was given in polar form, was relatively poorly answered. Full correct solutions were not often seen. In part (a), candidates generally wrote down the correct definite integral, then expanded $(1+ \tan\theta)^2$ and integrated 2 tanθ correctly, but could not find a correct method to integrate $1+ \tan^2\theta$ . Those who used the correct trigonometrical identity had no problem integrating the resulting $\sec^2\theta$ and completing the solution to reach the printed answer convincingly.<br>It was disappointing to find a significant minority of candidates not attempting part (b) having failed to obtain the printed answer in part (a). Most of the other candidates found the correct lengths for *OP* and *OQ* but some then wrote down an incorrect formula for the area of triangle *OPQ*. Some others lost the final mark because they did not give the area of the triangle in an exact form anywhere in their working despite the form of the printed answer in part (a). |

| Question 3: | Exam report |
|---|---|
| $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 5$<br>The complementary function:<br>Auxiliary equation: $\lambda^2 + 4\lambda + 5 = 0$<br>Discriminant: $4^2 - 4\times 1\times 5 = -4 = (2i)^2$<br>$\lambda_1 = \frac{-4+2i}{2} = -2+i \ \ and \ \ \lambda_2 = -2-i$<br>The complementary function is $y_c = e^{-2x}(ACos(x) + BSin(x))$<br><br>The particular integral $y = a$<br>It is obvious that $a = 1$ the solution<br>The general solution is : $y = 1 + e^{-2x}(ACosx + BSinx)$<br>$b)$ When $x = 0, y = 2 \ \ so \ \ 2 = 1 + A \qquad A = 1$<br>$y = 1 + e^{-2x}(Cosx + BSinx)$<br>$\frac{dy}{dx} = -2e^{-2x}(Cosx + BSinx) + e^{-2x}(-Sinx + B\cos x)$<br>When $x = 0, \frac{dy}{dx} = 3 \ \ so \ \ 3 = -2 + B \qquad B = 5$<br>$y = 1 + e^{-2x}(Cosx + 5Sinx)$ | This question, which required candidates to solve a second order differential equation, was generally a good source of marks. It was disappointing to see some candidates trying to solve the auxiliary equation, $m^2 + 4m + 5 = 0$ , by factorisation. They obtained real solutions and this error was penalised heavily. Better candidates were able to write down the correct complementary function and find the particular integral but some wasted valuable time by starting with $y_p = ax^2 + bx + c$ and showing that both *a* and *b* were zero. Candidates who were able to find the correct general solution in part (a) usually went on to apply the given boundary conditions correctly in their answers to part (b). |

| Question 4: | Exam report |
| --- | --- |
| a) $\int_1^{\infty} xe^{-3x}dx$ is an improper integral because the interval of integration is infinite<br><br>b) $\int xe^{-3x}dx = -\frac{1}{3}xe^{-3x} + \frac{1}{3}\int e^{-3x}dx$<br>$= -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + c$<br><br>c) $\int_1^N xe^{-3x}dx = \left[-\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x}\right]_1^N$<br>$= -\frac{1}{3}Ne^{-3N} - \frac{1}{9}e^{-3N} + \frac{1}{3}e^{-3} + \frac{1}{9}e^{-3}$<br>$\lim_{N\to\infty} Ne^{-3N} = 0 \ \ and \ \ \lim_{N\to\infty} e^{-3N} = 0$<br>$\int_1^{\infty} xe^{-3x}dx \ \ exists \ and \ \int_1^{\infty} xe^{-3x}dx = \frac{4}{9}e^{-3}$ | Part (a) was generally not well answered with a significant minority either not attempting it or making a statement which they then contradicted in part (c). The method of integration by parts was understood with the great majority obtaining the correct answer to part (b). Although there continues to be an improvement in candidates' solutions to the evaluation of an improper integral, there were still a significant minority who made no attempt to show the limiting process used. |

| Question 5: | Exam report |
| --- | --- |
| $\frac{dy}{dx} + \frac{4x}{x^2+1}y = x$<br>An integrating factor is<br>$I = e^{\int \frac{4x}{x^2+1}dx} = e^{2\int \frac{2x}{x^2+1}} = e^{2\ln(x^2+1)} = (x^2+1)^2$<br>The equation becomes:<br>$(x^2+1)^2\frac{dy}{dx} + 4x(x^2+1)^2 y = x(x^2+1)^2$<br>$\frac{d}{dx}\left((x^2+1)^2 y\right) = x(x^2+1)^2$<br>$(x^2+1)^2 y = \int x(x^2+1)^2dx = \frac{1}{2}\int 2x(x^2+1)^2dx$<br>$(x^2+1)^2 y = \frac{1}{2}\times\frac{1}{3}(x^2+1)^3 + c$<br>$y = \frac{1}{6}(x^2+1) + \frac{c}{(x^2+1)^2}$<br>When $x=0, y=1 \ \ so \ \ 1 = \frac{1}{6} + c \ \ so \ \ c = \frac{5}{6}$<br>$y = \frac{(x^2+1)+5}{6(x^2+1)^2}$ | Although many candidates were able to write down the integrating factor in terms of an integral, a significant minority could not then integrate $\frac{4x}{x^2+1}$ correctly. Those who found the correct simplified integrating factor generally used it appropriately and either solved the resulting integral by a suitable substitution or, more frequently, just multiplied out and integrated $x^5 + 2x^3 + x$. Although some candidates failed to insert the constant of integration and so lost the final two marks, this was not a common error. |

| Question 6: | Exam report |
|---|---|
| $r^2 Sin2\theta = 8$<br>$a)\ r^2 \times 2Sin\theta Cos\theta = 8$<br>$rSin\theta rCos\theta = 4$<br>$xy = 4 \qquad y = \frac{4}{x}$<br>$c)\ r = 2Sec\theta \ \text{ and } \ r^2 Sin2\theta = 8$<br>$\text{so } (2Sec\theta)^2 Sin2\theta = 8$<br>$\frac{4}{Cos^2\theta} \times 2Sin\theta Cos\theta = 8$<br>$Tan\theta = 1 \ \text{ so } \theta = \frac{\pi}{4}$<br>$For\ \theta = \frac{\pi}{4},\ r = 2Sec\frac{\pi}{4} = 2\sqrt{2} \qquad A(2\sqrt{2}, \frac{\pi}{4})$ | Those candidates who replaced sin 2θ by 2sinθ cosθ generally obtained the correct Cartesian equation in part (a). The sketch of the curve *C* (rectangular hyperbola) required in part (b) was not answered as well as expected with many sketches consisting of closed loops. Candidates presented a variety of acceptable methods for part (c). Those who eliminated *r* were required to obtain a trigonometrical equation in a single angle before any mark was awarded. Usually candidates who had found the correct equation went on to obtain the correct exact values for the polar coordinates of *A*. The second most popular method involved working with the cartesian form of the equation of *C*. Candidates who recognised the cartesian form of the equation of the line equivalent to the polar form given in part (c) of the question generally had no difficulty getting the cartesian coordinates for *A* as (2, 2), but then a significant minority could make no further progress. |

| Question 7: | Exam report |
|---|---|
| $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + ...$<br>$\ln(1+2x) = (2x) - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} + ...$<br>$\ln(1+2x) = 2x - 2x^2 + \frac{8}{3}x^3 + ...$<br>$ii)$ This is valid for $-1 < 2x \le 1$<br>$-\frac{1}{2} < x \le \frac{1}{2}$<br>$b)i)\ y = \ln(Cosx) \qquad \frac{dy}{dx} = -\frac{Sinx}{Cosx} = -Tanx$<br>$\frac{d^2y}{dx^2} = -Sec^2x = -\frac{1}{Cos^2x} \qquad \frac{d^3y}{dx^3} = -\frac{2Sinx}{Cos^3x}$<br>$\frac{d^4y}{dx^4} = \frac{-2Cosx \times Cos^3x + 2Sinx(-3SinxCos^2x)}{Cos^6x}$<br>$y(0) = \ln(1) = 0 \qquad y'(0) = -Tan(0) = 0$<br>$y''(0) = -1 \qquad y^{(3)}(0) = 0 \qquad y^{(4)}(0) = -2$<br>$iii)\ \ln(Cosx) = 0 + 0x - \frac{1}{2}x^2 + 0 - \frac{2}{4!}x^4 + ...$<br>$\ln(Cosx) = -\frac{1}{2}x^2 - \frac{1}{12}x^4 + ...$<br>$c)\ x\ln(1+2x) = 2x^2 - 2x^3 + ...$<br>$x^2 - \ln(Cosx) = \frac{3}{2}x^2 + \frac{1}{12}x^4 + ...$<br>$so\ \frac{x\ln(1+2x)}{x^2 - \ln(Cosx)} = \frac{2x^2 - 2x^3 + ...}{\frac{3}{2}x^2 + \frac{1}{12}x^4 + ..} = \frac{2 - 2x + ...}{\frac{3}{2} + \frac{1}{12}x^2 + ...}$<br>$\lim_{x \to 0} \frac{x\ln(1+2x)}{x^2 - \ln(Cosx)} = \frac{4}{3}$ | Although many candidates were able to answer part (a) correctly, it was clear, since factorials were present, that some had not realised that the series expansion for ln (1+ *x*) and its range of validity are both given in the formulae booklet. Methods of differentiation, chain rule and product/quotient rules, required in part (b) were either not known or not applied correctly by a significant number of candidates. Some candidates left expressions unsimplified before carrying out further differentiation, often leading to expressions which required multiple applications of the chain, product and quotient rules. Such an approach clearly used up valuable examination time. Those who did simplify their expressions generally scored the available marks for parts (b). Although careless sign errors were seen in solutions to find the limit for part (c), the general understanding of the principles involved was better displayed than in some previous series. |

**Question 8:**

$x = e^t \quad \dfrac{dx}{dt} = e^t = x \;\; and \;\; \dfrac{dt}{dx} = \dfrac{1}{e^t} = \dfrac{1}{x}$

$i)\dfrac{dy}{dt} = \dfrac{dy}{dx} \times \dfrac{dx}{dt} = x\dfrac{dy}{dx}$

$ii)\dfrac{d}{dx}\left(x\dfrac{dy}{dx}\right) = \dfrac{d}{dx}\left(\dfrac{dy}{dt}\right)$

$\dfrac{dy}{dx} + x\dfrac{d^2y}{dx^2} = \dfrac{dt}{dx} \times \dfrac{d}{dt}\left(\dfrac{dy}{dt}\right)$

$\dfrac{dy}{dx} + x\dfrac{d^2y}{dx^2} = \dfrac{1}{x}\dfrac{d^2y}{dt^2} \quad (\times x)$

$x\dfrac{dy}{dx} + x^2\dfrac{d^2y}{dx^2} = \dfrac{d^2y}{dt^2}$

$from\ i)\ we\ know\ that\ \ x\dfrac{dy}{dx} = \dfrac{dy}{dt}$

$so\ \ x^2\dfrac{d^2y}{dx^2} = \dfrac{d^2y}{dt^2} - \dfrac{dy}{dt}$

$b)\ x^2\dfrac{d^2y}{dx^2} - 6x\dfrac{dy}{dx} + 6y = 0 \ \ becomes$

$\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt} - 6\dfrac{dy}{dt} + 6y = 0$

$\dfrac{d^2y}{dt^2} - 7\dfrac{dy}{dt} + 6y = 0$

The auxiliary equation is

$\lambda^2 - 7\lambda + 6 = 0$

$(\lambda - 6)(\lambda - 1) = 0$

$\lambda = 6 \ or \ \lambda = 1$

$y = Ae^{6t} + Be^t = A(e^t)^6 + Be^t$

$y = Ax^6 + Bx$

**Exam report**

This question tested a relatively new part of the specification. Part (a) required candidates to produce 'proofs' of forms of some standard results for the topic, which could be classed as standard 'bookwork'. Part (a) was generally poorly answered, especially part (a)(ii). It was encouraging to see most candidates attempting to use the results given in part (a) to write the given differential equation in part (b) into a more useful form. Those who attempted this usually obtained the correct differential equation involving *y* and *t* and applied correct methods to solve it. Most of these candidates, however, gave their final answer as '$y = Ae^{6x} + Be^{x}$' instead of '$y = Ae^{6t} + Be^{t}$ so $y = Ax^6 + Bx$'. On this occasion, solutions with this error were generously marked and were not heavily penalised.

| Grade boundaries | | | | | | | |
|---|---|---|---|---|---|---|---|
| Grade | | | A | B | C | D | E |
| Mark | Max 75 | | 57 | 49 | 42 | 35 | 28 |

## Key to mark scheme and abbreviations used in marking

| | |
|---|---|
| M | mark is for method |
| m or dM | mark is dependent on one or more M marks and is for method |
| A | mark is dependent on M or m marks and is for accuracy |
| B | mark is independent of M or m marks and is for method and accuracy |
| E | mark is for explanation |

| | | | |
|---|---|---|---|
| √or ft or F | follow through from previous incorrect result | MC | mis-copy |
| CAO | correct answer only | MR | mis-read |
| CSO | correct solution only | RA | required accuracy |
| AWFW | anything which falls within | FW | further work |
| AWRT | anything which rounds to | ISW | ignore subsequent work |
| ACF | any correct form | FIW | from incorrect work |
| AG | answer given | BOD | given benefit of doubt |
| SC | special case | WR | work replaced by candidate |
| OE | or equivalent | FB | formulae book |
| A2,1 | 2 or 1 (or 0) accuracy marks | NOS | not on scheme |
| $-x$ EE | deduct $x$ marks for each error | G | graph |
| NMS | no method shown | c | candidate |
| PI | possibly implied | sf | significant figure(s) |
| SCA | substantially correct approach | dp | decimal place(s) |

### No Method Shown



Where the question specifically requires a particular method to be used, we must usually see evidence of use of this method for any marks to be awarded. However, there are situations in some units where part marks would be appropriate, particularly when similar techniques are involved. Your Principal Examiner will alert you to these and details will be provided on the mark scheme.

Where the answer can be reasonably obtained without showing working and it is very unlikely that the correct answer can be obtained by using an incorrect method, we must award **full marks**. However, the obvious penalty to candidates showing no working is that incorrect answers, however close, earn **no marks**.

Where a question asks the candidate to state or write down a result, no method need be shown for full marks.

Where the permitted calculator has functions which reasonably allow the solution of the question directly, the correct answer without working earns **full marks**, unless it is given to less than the degree of accuracy accepted in the mark scheme, when it gains **no marks**.

**Otherwise we require evidence of a correct method for any marks to be awarded.**



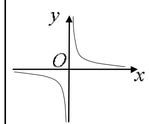
MFP3

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| 1(a) | $y(2.1) = y(2) + 0.1[2^2 - 1^2]$ | M1A1 | | |
| | $= 1+0.1\times3 = 1.3$ | A1 | 3 | |
| (b) | $y(2.2) = y(2) + 2(0.1)[\text{f}(2.1, y(2.1))]$ | M1 | | |
| | $\ldots = 1+2(0.1)[2.1^2 - 1.3^2]$ | A1√ | | Ft on cand's answer to (a) |
| | $\ldots = 1+0.2\times2.72 = 1.544$ | A1 | 3 | CAO |
| | **Total** | | 6 | |
| 2(a) | Area $= \frac{1}{2}\int(1+\tan\theta)^2\,d\theta$ | M1 | | Use of $\frac{1}{2}\int r^2\,d\theta$ |
| | $\ldots = \frac{1}{2}\int(1+2\tan\theta+\tan^2\theta)\,d\theta$ | B1 | | Correct expansion of $(1+\tan\theta)^2$ |
| | $=\frac{1}{2}\int(\sec^2\theta+2\tan\theta)\,d\theta$ | M1 | | $1+\tan^2\theta=\sec^2\theta$ used |
| | $=\frac{1}{2}[\tan\theta+2\ln(\sec\theta)]_0^{\frac{\pi}{3}}$ | A1√<br>B1√ | | Integrating $p\sec^2\theta$ correctly<br>Integrating $q\tan\theta$ correctly |
| | $=\frac{1}{2}[(\sqrt{3}+2\ln2)-0]=\frac{\sqrt{3}}{2}+\ln2$ | A1 | 6 | Completion. AG CSO be convinced |
| (b) | $OP = 1;\ OQ = 1+\tan\frac{\pi}{3}$ | B1 | | Both needed. Accept 2.73 for $OQ$ |
| | Shaded area = | | | |
| | 'answer (a)' $-\frac{1}{2}OP\times OQ\times\sin\left(\frac{\pi}{3}\right)$ | M1 | | |
| | $=\frac{\sqrt{3}}{2}+\ln2-\frac{\sqrt{3}}{4}(1+\sqrt{3})$ | A1 | 3 | ACF. Condone 0.376… if exact 'value' for area of triangle seen |
| | $=\frac{\sqrt{3}}{4}+\ln2-\frac{3}{4}$ | | | |
| | **Total** | | 9 | |

MFP3 (cont)

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| 3(a) | $(m+2)^2=-1$ | M1 | | Completing sq or formula |
| | $m=-2\pm i$ | A1 | | |
| | CF is $e^{-2x}(A\cos x+B\sin x)$ {or $e^{-x}A\cos(x+B)$ but not $Ae^{(-1+i)x}+Be^{(-1-i)x}$} | M1<br>A1√ | | If $m$ is real give M0<br>Ft on wrong $a$'s and $b$'s but roots must be complex |
| | PI try $y=p \Rightarrow 5p=5$ PI is $y=1$ | B1 | | |
| | GS $y=e^{-2x}(A\cos x+B\sin x)+1$ | B1√ | 6 | Their CF + their PI with two arbitrary constants. |
| (b) | $x=0,\ y=2 \Rightarrow A=1$ | B1√ | | Provided previous B1√ awarded |
| | $y'(x)=-2e^{-2x}(A\cos x+B\sin x)+e^{-2x}(-A\sin x+B\cos x)$ | M1<br>A1√ | | Product rule used |
| | $y'(0)=3 \Rightarrow 3=-2A+B \Rightarrow B=5$<br>$y=e^{-2x}(\cos x+5\sin x)+1$ | A1√ | 4 | Ft on one slip |
| | **Total** | | **10** | |
| 4(a) | The interval of integration is infinite | E1 | 1 | OE |
| (b) | $\int xe^{-3x}dx=-\frac{1}{3}xe^{-3x}-\int -\frac{1}{3}e^{-3x}dx$ | M1<br>A1 | | Reasonable attempt at parts |
| | $=-\frac{1}{3}xe^{-3x}-\frac{1}{9}e^{-3x}$ {+c} | A1√ | 3 | Condone absence of $+c$ |
| (c) | $I=\int_1^\infty xe^{-3x}\,dx=\lim_{a\to\infty}\int_1^a xe^{-3x}\,dx$ | | | |
| | $\lim_{a\to\infty}\{-\frac{1}{3}ae^{-3a}-\frac{1}{9}e^{-3a}\}-\left[-\frac{4}{9}e^{-3}\right]$ | M1 | | $F(a)-F(1)$ with an indication of limit '$a\to\infty$' |
| | $\lim_{a\to\infty} ae^{-3a}=0$ | M1 | | For statement with limit/limiting process shown |
| | $I=\frac{4}{9}e^{-3}$ | A1 | 3 | |
| | **Total** | | **7** | |

MFP3 (cont)

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| 5 | IF is $e^{\int \frac{4x}{x^2+1}dx}$ | M1 | | |
| | $=e^{2\ln(x^2+1)}$ | A1 | | |
| | $=e^{\ln(x^2+1)^2}=(x^2+1)^2$ | A1√ | | Ft on $e^{p\ln(x^2+1)}$ |
| | $\frac{d}{dx}\left(y(x^2+1)^2\right)=x(x^2+1)^2$ | M1<br>A1√ | | LHS as d/dx($y$×cand's IF) PI and also RHS of form $kx(x^2+1)^p$ |
| | $y(x^2+1)^2=\int x(x^2+1)^2\,dx$ | | | |
| | $y(x^2+1)^2=\frac{1}{6}(x^2+1)^3+c$ | M1<br>A1 | | Use of suitable substitution to find RHS or reaching $k(x^2+1)^3$ OE<br>Condone missing $c$ |
| | $y(0)=1 \Rightarrow c=\frac{5}{6}$ | m1 | | |
| | $y=\frac{1}{6}(x^2+1)+\frac{5}{6(x^2+1)^2}$ | A1 | 9 | Accept other forms of f($x$)<br>eg $y=\dfrac{\left(\frac{x^6}{6}+\frac{2x^4}{4}+\frac{x^2}{2}+1\right)}{(x^2+1)^2}$ |
| | **Total** | | **9** | |
| 6(a) | $r^2 2\sin\theta\cos\theta=8$ | M1 | | $\sin 2\theta=2\sin\theta\cos\theta$ used |
| | $x=r\cos\theta \quad y=r\sin\theta$ | M1 | | Either **one** stated or used |
| | $xy=4\ ,\ \ y=\frac{4}{x}$ | A1 | 3 | Either OE eg $y=\frac{8}{2x}$ |
| (b) |  | B1 | 1 | |
| (c) | $r=2\sec\theta$ is $x=2$ | B1 | | |
| | Sub $x=2$ in $xy=4 \Rightarrow 2y=4$<br>In cartesian, $A(2, 2)$ | M1 | | |
| | $\Rightarrow \tan\theta-\frac{y}{x}=1 \Rightarrow \theta=\frac{\pi}{4}$<br>$\Rightarrow r=\sqrt{x^2+y^2}=\sqrt{8}$ | M1 | | Used either $\tan\theta=\frac{y}{x}$ or $r=\sqrt{x^2+y^2}$ |
| | $\theta=\frac{\pi}{4}$ ; $r=\sqrt{8}$ | A1 | 4 | $r$ must be given in surd form |
| | Altn2: Eliminating $r$ to reach eqn. in $\cos\theta$ and $\sin\theta$ only (M1) $\quad \theta=\frac{\pi}{4}$ (A1)<br>Substitution $r=2\sec\left(\frac{\pi}{4}\right)$ (m1)<br>$r=\sqrt{8}$ (A1) OE surd | | | Altn3: $r\sin\theta=2$ (B1)<br>Solving $r\cos\theta=2$ and $r\sin\theta=2$ simultaneously (M1)<br>$\tan\theta=1$ or $r^2=2^2+2^2$ (M1)<br>$\theta=\frac{\pi}{4}$ ; $r-\sqrt{8}$ (A1) need both |
| | **Total** | | **8** | |

MFP3 (cont)

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| 7(a)(i) | $\ln(1+2x) = 2x - 2x^2 + \frac{8}{3}x^3 \ldots$ | M1<br>A1 | 2 | Use of expansion of $\ln(1+x)$<br>Simplified 'numerators'. |
| (ii) | $-\frac{1}{2} < x \le \frac{1}{2}$ | B1 | 1 | |
| (b)(i) | $y = \ln\cos x \Rightarrow y'(x) = \frac{1}{\cos x}(-\sin x)$ | M1 | | |
| | $y''(x) = -\sec^2 x$ | A1 | | ACF |
| | $y'''(x) = -2\sec x(\sec x \tan x)$ | M1 | | Chain rule OE |
| | $\{y'''(x) = -2\tan x(\sec^2 x)\}$ | A1√ | 4 | Ft a slip…accept unsimplified |
| (ii) | $y''''(x) = -2[\sec^2 x(\sec^2 x) + \tan x(2\sec x(\sec x \tan x))]$ | M1<br>A1 | | Product rule OE<br>ACF |
| | $y''''(0) = -2[(1)^2 + 0] = -2$ | A1√ | 3 | Ft a slip |
| (iii) | $\ln\cos x \approx 0 + 0 + \frac{x^2}{2}(-1) + 0 + \frac{x^4}{4!}(-2)$ | M1 | | |
| | $\approx -\frac{x^2}{2} - \frac{x^4}{12}$ | A1 | 2 | CSO throughout part (b). AG |
| (c) | Limit = $\lim_{x\to 0}\left[\frac{x\ln(1+2x)}{x^2 - \ln\cos x}\right]$ | | | |
| | $= \lim_{x\to 0}\left[\frac{x(2x - 2x^2 + ..)}{x^2 - \left(-\frac{x^2}{2} - \frac{x^4}{12}..\right)}\right]$ | M1 | | Using earlier expansions |
| | Limit = $\lim_{x\to 0}\frac{2x^2 - o(x^3)}{1.5x^2 + o(x^4)}$ | | | The notation $o(x^n)$ can be replaced by a term of the form $kx^n$ |
| | $= \lim_{x\to 0}\frac{2 - o(x)}{1.5 + o(x^2)} = \frac{4}{3}$ | M1<br>A1 | 3 | Need to see stage, division by $x^2$ |
| | **Total** | | **15** | |

MFP3 (cont)

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| 8(a)(i) | $\frac{dx}{dt} = e^t \ \{= x\}$ | B1 | | |
| | $x\frac{dy}{dx} = x\frac{dy}{dt}\frac{dt}{dx}$ | M1 | | Chain rule |
| | $= x\frac{dy}{dt}\frac{1}{x} = \frac{dy}{dt}$ | A1 | 3 | Completion. AG |
| (ii) | $\frac{d^2y}{dt^2} = \frac{d}{dt}\left(x\frac{dy}{dx}\right) =$ | | | |
| | $= \frac{dx}{dt}\frac{dy}{dx} + x\frac{d}{dt}\left(\frac{dy}{dx}\right)$ | M1 | | Product rule |
| | $\ldots = \frac{dy}{dt} + x\frac{dx}{dt}\frac{d}{dx}\left(\frac{dy}{dx}\right)$ | M1 | | |
| | $\ldots = \frac{dy}{dt} + x^2\left(\frac{d^2y}{dx^2}\right)$ | A1 | 3 | Condone leaving in this form |
| | $\Rightarrow x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$ | | | AG |
| (b) | $x^2\frac{d^2y}{dx^2} - 6x\frac{dy}{dx} + 6y = 0$ | | | |
| | $\Rightarrow \frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 6y = 0$ | M1 | | Using results in (a) to reach DE of this form |
| | Auxl eqn $m^2 - 7m + 6 = 0$ | | | |
| | $(m-6)(m-1) = 0$ | m1 | | PI |
| | $m = 1$ and $6$ | A1 | | PI |
| | $y = Ae^{6t} + Be^t$ | M1 | | Must be solving the 'correct' DE. (Give M1A0 for $y = Ae^{6x} + Be^x$) |
| | $y = Ax^6 + Bx$ | A1√ | 5 | Ft a minor slip only if previous A0 and all three method marks gained |
| | **Total** | | **11** | |
| | **TOTAL** | | **75** | |

# Notes about Jan 2008:

General Certificate of Education
June 2008
Advanced Level Examination

**MATHEMATICS** **MFP3**
**Unit Further Pure 3**

Monday 16 June 2008 1.30 pm to 3.00 pm

**For this paper you must have:**
- a 12-page answer book
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed: 1 hour 30 minutes

**Instructions**
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The *Examining Body* for this paper is AQA. The *Paper Reference* is MFP3.
- Answer **all** questions.
- Show all necessary working; otherwise marks for method may be lost.

**Information**
- The maximum mark for this paper is 75.
- The marks for questions are shown in brackets.

**Advice**
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

Answer **all** questions.

**1** The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = f(x, y)$$

where $f(x, y) = \ln(x + y)$

and $y(2) = 3$

Use the improved Euler formula

$$y_{r+1} = y_r + \tfrac{1}{2}(k_1 + k_2)$$

where $k_1 = hf(x_r, y_r)$ and $k_2 = hf(x_r + h, y_r + k_1)$ and $h = 0.1$, to obtain an approximation to $y(2.1)$, giving your answer to four decimal places. *(6 marks)*

**2** (a) Find the values of the constants $a$, $b$, $c$ and $d$ for which $a + bx + c\sin x + d\cos x$ is a particular integral of the differential equation

$$\frac{dy}{dx} - 3y = 10\sin x - 3x$$ *(4 marks)*

(b) Hence find the general solution of this differential equation. *(3 marks)*

**3** (a) Show that $x^2 = 1 - 2y$ can be written in the form $x^2 + y^2 = (1 - y)^2$. *(1 mark)*

(b) A curve has cartesian equation $x^2 = 1 - 2y$.

Find its polar equation in the form $r = f(\theta)$, given that $r > 0$. *(5 marks)*

**4** (a) A differential equation is given by

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} = 3x^2$$

Show that the substitution

$$u = \frac{dy}{dx}$$

transforms this differential equation into

$$\frac{du}{dx} - \frac{1}{x}u = 3x$$ *(2 marks)*

(b) By using an integrating factor, find the general solution of

$$\frac{du}{dx} - \frac{1}{x}u = 3x$$

giving your answer in the form $u = f(x)$. *(6 marks)*

(c) Hence find the general solution of the differential equation

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} = 3x^2$$

giving your answer in the form $y = g(x)$. *(2 marks)*

**5** (a) Find $\int x^3 \ln x \, dx$. *(3 marks)*

(b) Explain why $\int_0^e x^3 \ln x \, dx$ is an improper integral. *(1 mark)*

(c) Evaluate $\int_0^e x^3 \ln x \, dx$, showing the limiting process used. *(3 marks)*

**6** (a) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 10e^{-2x} - 9$$ *(10 marks)*

(b) Hence express $y$ in terms of $x$, given that $y = 7$ when $x = 0$ and that $\frac{dy}{dx} \to 0$ as $x \to \infty$. *(4 marks)*

**7** (a) Write down the expansion of $\sin 2x$ in ascending powers of $x$ up to and including the term in $x^3$. *(1 mark)*

(b) (i) Given that $y = \sqrt{3 + e^x}$, find the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when $x = 0$. *(5 marks)*

(ii) Using Maclaurin's theorem, show that, for small values of $x$,

$$\sqrt{3 + e^x} \approx 2 + \frac{1}{4}x + \frac{7}{64}x^2$$ *(2 marks)*

(c) Find

$$\lim_{x \to 0} \left[ \frac{\sqrt{3 + e^x} - 2}{\sin 2x} \right]$$ *(3 marks)*

**8** The polar equation of a curve $C$ is

$$r = 5 + 2\cos\theta, \qquad -\pi \leqslant \theta \leqslant \pi$$

(a) Verify that the points $A$ and $B$, with **polar coordinates** $(7, 0)$ and $(3, \pi)$ respectively, lie on the curve $C$. *(2 marks)*

(b) Sketch the curve $C$. *(2 marks)*

(c) Find the area of the region bounded by the curve $C$. *(6 marks)*

(d) The point $P$ is the point on the curve $C$ for which $\theta = \alpha$, where $0 < \alpha \leqslant \frac{\pi}{2}$. The point $Q$ lies on the curve such that $POQ$ is a straight line, where the point $O$ is the pole. Find, in terms of $\alpha$, the area of triangle $OQB$. *(4 marks)*

**END OF QUESTIONS**


AQA - FP3

# AQA – Further pure 3 – Jun 2008 – Answers

## Question 1:

$$\frac{dy}{dx}=f(x,y)=\ln(x+y) \qquad and \quad y(2)=3$$

$$k_1=0.1\times\ln(2+3)=0.16094$$
$$y_0+k_1=3+0.16094=3.16094$$
$$k_2=0.1\times\ln\left(2.1+3.16094\right)=0.16603$$
$$y(2.1)=3+\frac{1}{2}\left(0.16094+0.16603\right)=3.1635$$

**Exam report**

The majority of candidates were able to correctly use the given improved Euler formula to find the approximate value of $y(2.1)$ to four decimal places. Otherwise, the most common error was to use $y_r$ + h instead of 1 $y_r$ +$k_1$ in writing $k_2$ as 0.1ln (2.1 + 3.1). There was a minority of candidates who failed to gain marks because they gave the wrong answer for $y(2.1)$ and showed no method in their working, instead just giving a table of incorrect values.

## Question 2:

$a)\; y=a+bx+cSinx+dCosx$

$$\frac{dy}{dx}=b+cCosx-dSinx$$

$$\frac{dy}{dx}-3y=10Sinx-3x \qquad becomes$$

$$b+cCosx-dSinx-3(a+bx+cSinx+dCosx)=10Sinx-3x$$
$$(-d-3c)Sinx+(c-3d)Cosx-3bx+b-3a=10Sinx-3x$$

This gives: $\begin{cases}-d-3c=10\\ c-3d=0\end{cases}$ and $\begin{cases}-3b=-3\\ b-3a=0\end{cases}$

$$d=-1, c=-3\;,\; b=1\;,\; a=\frac{1}{3}$$

A particular integral is $y=\frac{1}{3}+x-3\;Sinx-Cosx$

$b)$ The complementary function:

The auxiliary equation : $\lambda-3=0 \qquad \lambda=3$

$y=Ae^{3x}$

The general solution $y=\frac{1}{3}+x-3Sinx-Cosx+Ae^{3x}$

**Exam report**

Most candidates differentiated the given expression correctly and substituted the result into the given differential equation. Subsequently, however, there were many cases of an incorrect expansion for the term $-3y$ which lead to the values for two of the four constants being incorrect.

In part (b), it was pleasing to find only a small number of candidates trying to use an integrating factor with the original differential equation. The successful candidates either used the auxiliary equation $m-3=0$ or used the reduced equation and then separated the variables. A common error, after finding $m=3$, was to assume that this represented repeated roots, leading to a complementary function of the form $e^{3x}(Ax+B)$, with two arbitrary constants for this first order differential equation. The error for those using the separation of variables method was to omit the arbitrary constant.

## Question 3:

$a)\; x^2=1-2y \quad by\ adding\ y^2\ on\ both\ sides$

$$x^2+y^2=y^2-2y+1=(y-1)^2$$

$b)\; x^2+y^2=(y-1)^2$

$$r^2=(rSin\theta-1)^2$$
$$r=rSin\theta-1 \;or\; r=-rSin\theta+1$$
$$r(1-Sin\theta)=-1 \;or\; r(1+Sin\theta)=1$$
$$r=\frac{1}{Sin\theta-1} \;or\; r=\frac{1}{1+Sin\theta}$$

We want $r>0$, so we keep the expression

$$r=\frac{1}{1+Sin\theta}$$

**Exam report**

Part (a) was generally answered correctly, with most candidates showing sufficient detail in reaching the printed result. Although most candidates started their solution for part (b) by successfully substituting $r\cos\theta$ for $x$ and $r\sin\theta$ for $y$ into either the given equation or the alternative form given in part (a), many failed to go on to score full marks because they did not consider and eliminate the negative square root (or the second solution of the quadratic equation).

| Question 4: | Exam report |
| --- | --- |
| a) $x\frac{d^2y}{dx^2}-\frac{dy}{dx}=3x^2 \qquad u=\frac{dy}{dx}$<br>$x\frac{du}{dx}-u=3x^2$<br>$\frac{du}{dx}-\frac{1}{x}u=3x$<br>b) An integrating factor is $I=e^{\int -\frac{1}{x}dx}=e^{\ln\left(\frac{1}{x}\right)}=\frac{1}{x}$<br>The equation becomes:<br>$\frac{1}{x}\frac{du}{dx}-\frac{1}{x^2}u=3$<br>$\frac{d}{dx}\left(\frac{u}{x}\right)=3 \qquad \frac{u}{x}=3x+A$<br>$u=3x^2+Ax$<br>c) $u=\frac{dy}{dx}=3x^2+Ax$ so $y=x^3+\frac{A}{2}x^2+B$ | Most candidates scored full marks for their solution to part (a). Again, part (b) was answered well by the majority of candidates, but it was not uncommon to find solutions which used the wrong integrating factor, $x$, or lacked an arbitrary constant. Those candidates who started part (c) by equating their answer for (b) to $\frac{dy}{dx}$ and integrating normally scored both marks, although some lost the accuracy mark because their general solution of this second order differential equation did not contain two arbitrary constants. |

| Question 5: | Exam report |
| --- | --- |
| a) $\int x^3\ln x\,dx=\frac{1}{4}x^4\ln x-\int \frac{1}{4}x^4\times\frac{1}{x}dx$<br>$=\frac{1}{4}x^4\ln x-\int\frac{1}{4}x^3dx$<br>$=\frac{1}{4}x^4\ln x-\frac{1}{16}x^4+c$<br>b) $\int_0^e x^3\ln x\,dx$ is an improper integral because<br>the function $x^3\ln x$ is not defined at $x=0$.<br>c) $\int_a^e x^3\ln x\,dx=\left[\frac{1}{4}x^4\ln x-\frac{1}{16}x^4\right]_a^e=\frac{e^4}{4}-\frac{e^4}{16}-\frac{1}{4}a^4\ln a+\frac{1}{16}a^4$<br>$\lim_{x\to 0}a^4\ln a=0$ and $\lim_{x\to 0}a^4=0$<br>$\int_0^e x^3\ln x\,dx$ exists and $\int_0^e x^3\ln x\,dx=\frac{3e^4}{16}$ | Most candidates applied integration by parts accurately to obtain the correct answer for the integral of $x^3\ln x$. Although many candidates gave a correct explanation in part (b) for why the integral was improper, there were others whose incorrect explanations centred on the limit e or the interval of integration being infinite. For full marks in part (c), candidates were expected to pay particular attention to the value of the limit of, for example, $a^4\ln a$ as $a$ tended to 0. |

| Question 6: | Exam report |
|---|---|
| $a)\ \frac{d^2y}{dx^2}-2\frac{dy}{dx}-3y=10e^{-2x}-9$<br>The auxiliary equation is $\lambda^2-2\lambda-3=0$<br>$(\lambda-3)(\lambda+1)=0$<br>$\lambda=3\ or\ \lambda=-1$<br>•The complementary function is $y_c=Ae^{3x}+Be^{-x}$<br>•The particular integral $y=ae^{-2x}+b$<br>$\frac{dy}{dx}=-2ae^{-2x}$ $\quad\frac{d^2y}{dx^2}=4ae^{-2x}$<br>$\frac{d^2y}{dx^2}-2\frac{dy}{dx}-3y=10e^{-2x}-9\quad becomes$<br>$4ae^{-2x}+4ae^{-2x}-3ae^{-2x}-3b=10e^{-2x}-9$<br>$5ae^{-2x}-3b=10e^{-2x}-9$<br>This gives $a=2\ \ and\ \ b=3$<br>The particular integral $y=2e^{-2x}+3$<br>The general solution is $y=3+2e^{-2x}+Ae^{3x}+Be^{-x}$<br>$b)$ When $x=0, y=7\ \ so\ \ 7=3+2+A+B\qquad A+B=2$<br>$\frac{dy}{dx}=-4e^{-2x}+3Ae^{3x}-Be^{-x}$<br>When $x\to\infty$, we want $\frac{dy}{dx}\to 0$, this mean $A=0$.<br>$(Because\ if\ \ A\neq 0, Ae^{3x}\xrightarrow[x\to\infty]{}\infty)\quad and\ \ B=2$<br>$y=3+2e^{-2x}+2e^{-x}$ | In part (a), many candidates scored high marks for finding the general solution of the second order differential equation, with the most common slip being a wrong expansion of brackets, which led to −9 instead of −3 for the constant part of the particular integral. A more serious but less common error was to look for a particular integral of the form $axe^{-2x}+b$ . Part (b) proved, as expected, to be more of a challenge to candidates, with many unable to deal with the boundary condition expressed as a limit. This was the first time that this had appeared on an examination paper for this unit and, in general, only the better candidates could handle it. |

| Question 7: | Exam report |
|---|---|
| $a)\ Sin(x)=x-\frac{x^3}{6}+...\ \ so\ \ Sin(2x)=2x-\frac{(2x)^3}{6}=2x-\frac{4}{3}x^3+...$<br>$b)i)\ y=\sqrt{3+e^x}=\left(3+e^x\right)^{\frac{1}{2}}$<br>$\frac{dy}{dx}=\frac{1}{2}\times e^x\times\left(3+e^x\right)^{-\frac{1}{2}}$<br>$\frac{d^2y}{dx^2}=\frac{1}{2}e^x\left(3+e^x\right)^{-\frac{1}{2}}+\frac{1}{2}e^x\times-\frac{1}{2}e^x\left(3+e^x\right)^{-\frac{3}{2}}$<br>$y(0)=\sqrt{4}=2$<br>$y'(0)=\frac{1}{2}(4)^{-\frac{1}{2}}=\frac{1}{4}$<br>$y''(0)=\frac{1}{4}-\frac{1}{4}(4)^{-\frac{3}{2}}=\frac{7}{32}$<br>$Conclusion: \sqrt{3+e^x}=2+\frac{1}{4}x+\frac{7}{64}x^2+...$<br>$c)\ \sqrt{3+e^x}-2=\frac{1}{4}x+\frac{7}{64}x^2+...$<br>$Sin(2x)=2x-\frac{4}{3}x^2+...$<br>$so\ \frac{\sqrt{3+e^x}-2}{Sin(2x)}=\frac{\frac{1}{4}x+\frac{7}{64}x^2+...}{2x-\frac{4}{3}x^2+...}=\frac{\frac{1}{4}+\frac{7}{64}x+...}{2-\frac{4}{3}x+...}$<br>$\lim_{x\to 0}\frac{\sqrt{3+e^x}-2}{Sin(2x)}=\frac{1}{8}$ | The examiners expected candidates to evaluate 3! in the expansion for sin 2*x*, although credit was given retrospectively if the evaluation was left until part (c). Most candidates gave the correct expansion in part (a), and it was pleasing to find a greater proportion of the candidates than last summer applying the chain rule and product rule correctly in part (b)(i). Although Maclaurin's theorem was well known, a significant number of candidates who had obtained a wrong value for the second derivative in part (b)(i) tried to convince the examiners that it led to the printed result in part (b)(ii). Such candidates would have been better advised to look for their error in part (b)(i). Although many candidates scored the three marks in part (c), there were others who did not show the division of the numerator and denominator by *x* before finding the limit as *x* tended to zero. |

**Question 8:**

$$r = 5 + 2Cos\theta \qquad -\pi \le \theta \le \pi$$

$a)$ For $\theta = 0,\ \ r = 5 + 2Cos0 = 7$

For $\theta = \pi,\ \ r = 5 + 2Cos\pi = 3$

$A(7,0)$ *and* $B(3,\pi)$ *belong to the curve C.*

$b)$

$$c)\ A = \frac{1}{2}\int_{-\pi}^{\pi}(5+2Cos\theta)^2\,d\theta = \frac{1}{2}\int_{-\pi}^{\pi} 25 + 4Cos^2\theta + 20Cos\theta d\theta$$

$$A = \frac{1}{2}\int_{-\pi}^{\pi} 25 + 2Cos2\theta + 2 + 20Cos\theta d\theta$$

$$A = \frac{1}{2}\int_{-\pi}^{\pi} 27 + 2Cos2\theta + 20Cos\theta d\theta$$

$$A = \left[\frac{27}{2}\theta + \frac{1}{2}Sin2\theta + 10Sin\theta\right]_{-\pi}^{\pi} = 27\pi$$

$$d)\ B(3,\pi) \quad P(5+2Cos\alpha,\alpha)\ \ and\ \ Q(5+2Cos(-\pi+\alpha), -\pi+\alpha)$$

$$Notice\ \cos(-\pi+\alpha) = -Cos(\alpha)$$

The length $OB = 3$, the length $OQ = 5 - 2Cos\alpha$

The angle BOQ is $\alpha$

The area of the triangle BOQ is $\frac{1}{2}OB \times OQ \times Sin(BOQ)$

$$Area = \frac{1}{2}\times 3 \times (5 - 2Cos\alpha)Sin\alpha = \frac{15}{2}Sin\alpha - \frac{3}{2}Sin2\alpha$$

$$Area = \frac{3}{2}(5Sin\alpha - Sin2\alpha)$$

**Exam report**

Parts (a) and (b) were generally answered well, but some candidates did not identify the critical value of 5 on their sketches. In part (c), although the usual errors were seen — for example the wrong expansion $(5 + 2\cos\theta)^2 = 25 + 10\cos\theta + 4\cos^2\theta$ or a sign error in the identity for $4\cos^2\theta$ in terms of $\cos2\theta$ — most candidates had a thorough understanding of the method required to find the area of the region bounded by the curve. Although only a minority of candidates scored full marks for the final part of this last question, it is pleasing to report that a significant number of other candidates were awarded partial credit for finding an expression for *OQ*, although many lost at least one mark because they used $\pi - \alpha$ instead of $\alpha - \pi$ for $\theta$. The most common wrong method seen involved the integral $\int_{-\pi+\alpha}^{\pi}\frac{1}{2}r^2 d\theta$, for which no credit was awarded.

**Grade boundaries**

| Grade | | | A | B | C | D | E |
|---|---|---|---|---|---|---|---|
| Mark | Max 75 | | 63 | 55 | 47 | 39 | 31 |

**Key to mark scheme and abbreviations used in marking**

| | |
|---|---|
| M | mark is for method |
| m or dM | mark is dependent on one or more M marks and is for method |
| A | mark is dependent on M or m marks and is for accuracy |
| B | mark is independent of M or m marks and is for method and accuracy |
| E | mark is for explanation |

| | | | |
|---|---|---|---|
| √or ft or F | follow through from previous incorrect result | MC | mis-copy |
| CAO | correct answer only | MR | mis-read |
| CSO | correct solution only | RA | required accuracy |
| AWFW | anything which falls within | FW | further work |
| AWRT | anything which rounds to | ISW | ignore subsequent work |
| ACF | any correct form | FIW | from incorrect work |
| AG | answer given | BOD | given benefit of doubt |
| SC | special case | WR | work replaced by candidate |
| OE | or equivalent | FB | formulae book |
| A2,1 | 2 or 1 (or 0) accuracy marks | NOS | not on scheme |
| –*x* EE | deduct *x* marks for each error | G | graph |
| NMS | no method shown | c | candidate |
| PI | possibly implied | sf | significant figure(s) |
| SCA | substantially correct approach | dp | decimal place(s) |

**No Method Shown**

Where the question specifically requires a particular method to be used, we must usually see evidence of use of this method for any marks to be awarded. However, there are situations in some units where part marks would be appropriate, particularly when similar techniques are involved. Your Principal Examiner will alert you to these and details will be provided on the mark scheme.

Where the answer can be reasonably obtained without showing working and it is very unlikely that the correct answer can be obtained by using an incorrect method, we must award **full marks**. However, the obvious penalty to candidates showing no working is that incorrect answers, however close, earn **no marks**.

Where a question asks the candidate to state or write down a result, no method need be shown for full marks.

Where the permitted calculator has functions which reasonably allow the solution of the question directly, the correct answer without working earns **full marks**, unless it is given to less than the degree of accuracy accepted in the mark scheme, when it gains **no marks**.

**Otherwise we require evidence of a correct method for any marks to be awarded.**



MFP3

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| 1 | $k_1 = 0.1\times\ln(2+3)$ | M1 | | |
| | $= 0.1609(4379\ldots)\quad (= *)$ | A1 | | PI |
| | $k_2 = 0.1\times f(2.1,\ 3+*\ldots)$<br>$\ldots = 0.1\times\ln(2.1+3.16094\ldots)]$ | M1 | | |
| | $\ldots. = 0.1660(31\ldots)$ | A1 | | PI |
| | $y(2.1) = y(2) + \frac{1}{2}[k_1+k_2]$<br>$= 3 + 0.5\times 0.3269748\ldots.$ | m1 | | Dep on previous two Ms and numerical values for $k$'s |
| | $= 3.163487\ldots = 3.1635$ to 4dp | A1 | 6 | Must be 3.1635 |
| | **Total** | | **6** | |
| **2(a)** | PI: $y_{PI} = a + bx + c\sin x + d\cos x$<br>$y'_{PI} = b + c\cos x - d\sin x$ | | | |
| | $b + c\cos x - d\sin x - 3a - 3bx - 3c\sin x$<br>$-3d\cos x = 10\sin x - 3x$ | M1 | | Substituting into DE |
| | $b-3a=0;\ -3b=-3;\ c-3d=0;\ -d-3c=10$ | M1 | | Equating coefficients (at least 2 eqns) |
| | $a=\frac{1}{3};\ b=1;\ c=-3;\ d=-1$ | A2,1 | 4 | A1 for any two correct |
| | $y_{PI} = \frac{1}{3} + x - 3\sin x - \cos x$ | | | |
| **(b)** | Aux. eqn. $m - 3 = 0$ | M1 | | Altn. $\int y^{-1}\,dy = \int 3\,dx$ OE (M1) |
| | $(y_{CF} =)Ae^{3x}$ | A1 | | $Ae^{3x}$ OE |
| | $(y_{GS} =)Ae^{3x} + \frac{1}{3} + x - 3\sin x - \cos x$ | B1F | 3 | (c's CF + c's PI ) with 1 arbitrary constant |
| | **Total** | | **7** | |
| **3(a)** | $x^2+y^2 = 1 - 2y + y^2 \Rightarrow x^2 + y^2 = (1-y)^2$ | B1 | 1 | AG |
| **(b)** | $x^2 + y^2 = r^2$ | M1 | | Or $x = r\cos\theta$ |
| | $y = r\sin\theta$ | M1 | | |
| | $x^2 = 1 - 2y$ so $x^2 + y^2 = (1-y)^2$<br>$\Rightarrow r^2 = (1 - r\sin\theta)^2$ | A1 | | OE eg $r^2\cos^2\theta = 1 - 2r\sin\theta$<br>PI by the next line |
| | $r = 1 - r\sin\theta$ or $r = -(1 - r\sin\theta)$<br>$r(1+\sin\theta) = 1$ or $r(1-\sin\theta) = -1$ | m1 | | Either |
| | $r > 0$ so $r = \frac{1}{1+\sin\theta}$ | A1 | 5 | CSO |
| | **Total** | | **6** | |

MFP3 (cont)

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| 4(a) | $u=\frac{\mathrm{d}y}{\mathrm{d}x}\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ | M1 | | |
| | $x\frac{\mathrm{d}u}{\mathrm{d}x}-u=3x^2\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}x}-\frac{1}{x}u=3x$ | A1 | 2 | AG Substitution into LHS of DE and completion |
| (b) | IF is $\exp\left(\int -\frac{1}{x}\,\mathrm{d}x\right)$ | M1 | | and with integration attempted |
| | $=\mathrm{e}^{-\ln x}$ | A1 | | |
| | $=x^{-1}$ or $\frac{1}{x}$ | A1 | | or multiple of $x^{-1}$ |
| | $\frac{\mathrm{d}}{\mathrm{d}x}\left[ux^{-1}\right]=3$ | M1 | | LHS as differential of $u\times$IF . PI |
| | $\Rightarrow ux^{-1}=3x+A$ | m1 | | Must have an arbitrary constant (Dep. on previous M1 only) |
| | $u=3x^2+Ax$ | A1 | 6 | |
| (c) | $\frac{\mathrm{d}y}{\mathrm{d}x}=3x^2+Ax$ | M1 | | Replaces $u$ by $\frac{\mathrm{d}y}{\mathrm{d}x}$ and attempts to integrate |
| | $y=x^3+\frac{Ax^2}{2}+B$ | A1F | 2 | ft on cand's $u$ but solution must have two arbitrary constants |
| | **Total** | | **10** | |
| 5(a) | $\int x^3\ln x\,\mathrm{d}x=\frac{x^4}{4}\ln x-\int\frac{x^4}{4}\left(\frac{1}{x}\right)\mathrm{d}x$ | M1<br>A1 | | $...=kx^4\ln x\pm\int f(x)$, with f($x$) not involving the 'original' $\ln x$ |
| | $......=\frac{x^4}{4}\ln x-\frac{x^4}{16}+c$ | A1 | 3 | Condone absence of '$+c$' |
| (b) | Integrand is not defined at $x=0$ | E1 | 1 | OE |
| (c) | $\int_0^{\mathrm{e}}x^3\ln x\,\mathrm{d}x=\left\{\lim_{a\to0}\int_a^{\mathrm{e}}x^3\ln x\,\mathrm{d}x\right\}$ | | | |
| | $=\frac{3\mathrm{e}^4}{16}-\lim_{a\to0}\left[\frac{a^4}{4}\ln a-\frac{a^4}{16}\right]$ | M1 | | F(e) – F($a$) |
| | But $\lim_{a\to0}a^4\ln a=0$ | B1 | | Accept a general form eg $\lim_{x\to0}x^k\ln x=0$ |
| | So $\int_0^{\mathrm{e}}x^3\ln x\,\mathrm{d}x$ exists and $=\frac{3\mathrm{e}^4}{16}$ | A1 | 3 | CSO |
| | **Total** | | **7** | |

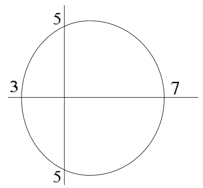
MFP3 (cont)

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| 6(a) | Aux eqn: $m^2-2m-3=0$ | M1 | | |
| | $m - -1,\ 3$ | A1 | | PI |
| | CF $(y_C=)A\mathrm{e}^{3x}+B\mathrm{e}^{-x}$ | M1 | | |
| | Try $(y_{PI}=)a\mathrm{e}^{-2x}\ (+b)$ | M1 | | |
| | $\frac{\mathrm{d}y}{\mathrm{d}x}=-2a\mathrm{e}^{-2x}$ | A1 | | |
| | $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=4a\mathrm{e}^{-2x}$ | A1 | | |
| | Substitute into DE gives $4a\mathrm{e}^{-2x}+4a\mathrm{e}^{-2x}-3a\mathrm{e}^{-2x}-3b=10\mathrm{e}^{-2x}-9$ | M1 | | |
| | $\Rightarrow a=2$ | A1 | | |
| | $b=3$ | B1 | | |
| | $(y_{GS}=)A\mathrm{e}^{3x}+B\mathrm{e}^{-x}+2\mathrm{e}^{-2x}+3$ | B1F | 10 | (c's CF+c's PI) with 2 arbitrary constants |
| (b) | $x=0,\ y=7\ \Rightarrow 7=A+B+2+3$ | B1F | | Only ft if exponentials in GS and two arbitrary constants remain |
| | $\frac{\mathrm{d}y}{\mathrm{d}x}=3A\mathrm{e}^{3x}-B\mathrm{e}^{-x}-4\mathrm{e}^{-2x}$ | | | |
| | As $x\to\infty$, $\mathrm{e}^{-kx}\to0$, $\frac{\mathrm{d}y}{\mathrm{d}x}\to0$ so $A=0$ | B1 | | |
| | When $A=0$, $5=0+B+3\ \Rightarrow B=2$ | B1F | | Must be using '$A$' = 0 |
| | $y=2\mathrm{e}^{-x}+2\mathrm{e}^{-2x}+3$ | A1 | 4 | CSO |
| | **Total** | | **14** | |

MFP3 (cont)

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| 7(a) | $\sin 2x \approx 2x - \frac{(2x)^3}{3!} + .. = 2x - \frac{4}{3}x^3 + ..$ | B1 | 1 | |
| (b)(i) | $\frac{dy}{dx} = \frac{1}{2}(3+e^x)^{-\frac{1}{2}}(e^x)$ | M1<br>A1 | | Chain rule |
| | $\frac{d^2y}{dx^2} = \frac{1}{2}e^x(3+e^x)^{-\frac{1}{2}} - \frac{1}{4}(3+e^x)^{-\frac{3}{2}}(e^{2x})$ | M1<br>A1 | | Product rule OE<br>OE |
| | $y'(0) = \frac{1}{4};\ y''(0) = \frac{1}{4} - \frac{1}{32} = \frac{7}{32}$ | A1 | 5 | CSO |
| (ii) | $y(0) = 2;\ y'(0) = \frac{1}{4};\ y''(0) = \frac{1}{4} - \frac{1}{32} = \frac{7}{32}$<br>McC. Thm: $y(0) + x\,y'(0) + \frac{x^2}{2}\,y''(0)$<br>$\sqrt{3+e^x} \approx 2 + \frac{1}{4}x + \frac{7}{64}x^2$ | M1<br>A1 | 2 | CSO; AG |
| (c) | $\left[\frac{\sqrt{3+e^x}-2}{\sin 2x}\right] - \left[\frac{2+\frac{1}{4}x+\frac{7}{64}x^2-2}{2x-\frac{4}{3}x^3}\right]$ | M1 | | |
| | $-\left[\frac{\frac{1}{4}+\frac{7}{64}x+...}{2-\frac{4}{3}x^2+..}\right]$ | m1 | | Dividing numerator and denominator by $x$ to get constant term in each |
| | $\lim_{x\to 0}\left[\frac{\sqrt{3+e^x}-2}{\sin 2x}\right] = \frac{\frac{1}{4}}{2} = \frac{1}{8}$ | A1F | 3 | Ft on cand's answer to (a) provided of the form $ax+bx^3$ |
| | Total | | 11 | |

MFP3 (cont)

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| 8(a) | $\theta = 0,\ r = 5 + 2\cos 0 = 7$ {$A$ lies on $C$} | B1 | | |
| | $\theta = \pi,\ r = 5 + 2\cos\pi = 3$ {$B$ lies on $C$} | B1 | 2 | |
| (b) |   | B1<br>B1 | <br>2 | Closed single loop curve, with (indication of) symmetry<br>Critical values, 3,5,7 indicated |
| (c) | Area $= \frac{1}{2}\int(5+2\cos\theta)^2 d\theta$ | M1 | | Use of $\frac{1}{2}\int r^2 d\theta$ |
| | $= \frac{1}{2}\int_{-\pi}^{\pi}(25+20\cos\theta+4\cos^2\theta)\,d\theta$ | B1<br>B1 | | OE for correct expansion of $(5+2\cos\theta)^2$<br>For correct limits |
| | $= \frac{1}{2}\int_{-\pi}^{\pi}(25+20\cos\theta+2(\cos 2\theta+1))\,d\theta$ | M1 | | Attempt to write $\cos^2\theta$ in terms of $\cos 2\theta$ |
| | $= \frac{1}{2}[27\theta+20\sin\theta+\sin 2\theta]_{-\pi}^{\pi}$ | A1F | | Correct integration ft wrong non-zero coefficients in $a + b\cos\theta + c\cos 2\theta$ |
| | $= 27\pi$ | A1 | 6 | CSO |
| (d) | Triangle $OBQ$ with<br>$OB = 3$ and angle $BOQ = \alpha$ | B1 | | PI |
| | $OQ = 5 + 2\cos(-\pi + \alpha)$ | M1 | | OE |
| | Area of triangle $OQB - \frac{1}{2}OB \times OQ \sin\alpha$ | m1 | | Dep. on correct method to find $OQ$ |
| | $= \frac{3}{2}(5 - 2\cos\alpha)\sin\alpha$ | A1 | 4 | CSO |
| | Total | | 14 | |
| | TOTAL | | 75 | |

# Notes about Jun 2008:

General Certificate of Education
January 2009
Advanced Level Examination

**MATHEMATICS** **MFP3**
**Unit Further Pure 3**

Wednesday 21 January 2009 1.30 pm to 3.00 pm

| For this paper you must have: |
| --- |
| • a 12-page answer book<br>• the blue AQA booklet of formulae and statistical tables.<br>You may use a graphics calculator. |

Time allowed: 1 hour 30 minutes

**Instructions**
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The *Examining Body* for this paper is AQA. The *Paper Reference* is MFP3.
- Answer **all** questions.
- Show all necessary working; otherwise marks for method may be lost.

**Information**
- The maximum mark for this paper is 75.
- The marks for questions are shown in brackets.

**Advice**
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

---

Answer **all** questions.

---

**1** The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = f(x, y)$$

where $$f(x, y) = \frac{x^2 + y^2}{x + y}$$

and $$y(1) = 3$$

(a) Use the Euler formula

$$y_{r+1} = y_r + hf(x_r, y_r)$$

with $h = 0.2$, to obtain an approximation to $y(1.2)$. *(3 marks)*

(b) Use the improved Euler formula

$$y_{r+1} = y_r + \tfrac{1}{2}(k_1 + k_2)$$

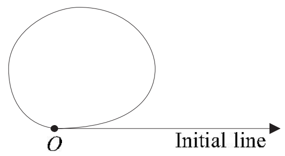
where $k_1 = hf(x_r, y_r)$ and $k_2 = hf(x_r + h, y_r + k_1)$ and $h = 0.2$, to obtain an approximation to $y(1.2)$, giving your answer to four decimal places. *(5 marks)*

**2** (a) Show that $\frac{1}{x^2}$ is an integrating factor for the first-order differential equation

$$\frac{dy}{dx} - \frac{2}{x}y = x$$ *(3 marks)*

(b) Hence find the general solution of this differential equation, giving your answer in the form $y = f(x)$. *(4 marks)*

**3** The diagram shows a sketch of a loop, the pole $O$ and the initial line.




The polar equation of the loop is

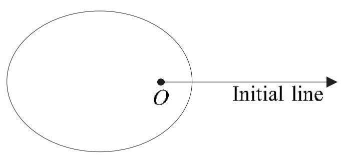
$$r = (2 + \cos\theta)\sqrt{\sin\theta}, \quad 0 \leqslant \theta \leqslant \pi$$

Find the area enclosed by the loop. *(6 marks)*

**4** (a) Use integration by parts to show that $\int \ln x \, dx = x \ln x - x + c$, where $c$ is an arbitrary constant. *(2 marks)*

(b) Hence evaluate $\int_0^1 \ln x \, dx$, showing the limiting process used. *(4 marks)*

**5** The diagram shows a sketch of a curve $C$, the pole $O$ and the initial line.




The curve $C$ has polar equation

$$r = \frac{2}{3 + 2\cos\theta}, \quad 0 \leqslant \theta \leqslant 2\pi$$

(a) Verify that the point $L$ with polar coordinates $(2, \pi)$ lies on $C$. *(1 mark)*

(b) The circle with polar equation $r = 1$ intersects $C$ at the points $M$ and $N$.

(i) Find the polar coordinates of $M$ and $N$. *(3 marks)*

(ii) Find the area of triangle $LMN$. *(4 marks)*

(c) Find a cartesian equation of $C$, giving your answer in the form $9y^2 = \mathrm{f}(x)$. *(5 marks)*

**6** The function f is defined by $\mathrm{f}(x) = \mathrm{e}^{2x}(1 + 3x)^{-\frac{2}{3}}$.

(a) (i) Use the series expansion for $\mathrm{e}^x$ to write down the first four terms in the series expansion of $\mathrm{e}^{2x}$. *(2 marks)*

(ii) Use the binomial series expansion of $(1 + 3x)^{-\frac{2}{3}}$ and your answer to part (a)(i) to show that the first three non-zero terms in the series expansion of $\mathrm{f}(x)$ are $1 + 3x^2 - 6x^3$. *(5 marks)*

(b) (i) Given that $y = \ln(1 + 2\sin x)$, find $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$. *(4 marks)*

(ii) By using Maclaurin's theorem, show that, for small values of $x$,

$$\ln(1 + 2\sin x) \approx 2x - 2x^2$$ *(2 marks)*

(c) Find

$$\lim_{x \to 0} \frac{1 - \mathrm{f}(x)}{x \ln(1 + 2\sin x)}$$ *(3 marks)*

**7** (a) Given that $x = \mathrm{e}^t$ and that $y$ is a function of $x$, show that

$$x^2 \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}^2y}{\mathrm{d}t^2} - \frac{\mathrm{d}y}{\mathrm{d}t}$$ *(7 marks)*

(b) Hence show that the substitution $x = \mathrm{e}^t$ transforms the differential equation

$$x^2 \frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4x\frac{\mathrm{d}y}{\mathrm{d}x} = 10$$

into

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 5\frac{\mathrm{d}y}{\mathrm{d}t} = 10$$ *(2 marks)*

(c) Find the general solution of the differential equation $\dfrac{\mathrm{d}^2y}{\mathrm{d}t^2} - 5\dfrac{\mathrm{d}y}{\mathrm{d}t} = 10$. *(5 marks)*

(d) Hence solve the differential equation $x^2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4x\dfrac{\mathrm{d}y}{\mathrm{d}x} = 10$, given that $y = 0$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 8$ when $x = 1$. *(5 marks)*

**END OF QUESTIONS**

# AQA – Further pure 3 – Jan 2009 – Answers

| Question 1: | Exam report |
|---|---|
| a) $\frac{dy}{dx}=\frac{x^2+y^2}{x+y}$ with $y(1)=3$<br>$y(1.2)=3+0.2\times\frac{1^2+3^2}{1+3}=3.5$<br>b) $k_1=0.2\times\frac{1^2+3^3}{1+3}=0.5$<br>$k_2=0.2\times\frac{1.2^2+3.5^2}{1.2+3.5}=0.5826$<br>$y(1.2)=3+\frac{1}{2}(0.5+0.5826)=3.5413$ | Numerical solutions of first order differential equations continue to be a good source of marks for all candidates. This was the best answered question on the paper and it was pleasing to see full working with clear substitutions into relevant formulae. The vast majority of candidates obtained the correct answers, and errors were generally limited to incorrect evaluations rather than any lack of understanding of the methods or notation involved. |

| Question 2: | Exam report |
|---|---|
| a) An integrating factor is<br>$I=e^{\int -\frac{2}{x}dx}=e^{-2\ln(x)}=e^{\ln\frac{1}{x^2}}=\frac{1}{x^2}$<br>b) Multiplying by the integrating factor:<br>$\frac{1}{x^2}\frac{dy}{dx}-\frac{2}{x^3}y=\frac{1}{x}$<br>$\frac{d}{dx}\left(\frac{y}{x^2}\right)=\frac{1}{x}$ so $\frac{y}{x^2}=\int \frac{1}{x}dx=\ln(x)+c$<br>$y=x^2\ln(x)+cx^2$ | Most candidates were able to show that $\frac{1}{x^2}$ was an integrating factor for the given first-order differential equation and the majority of them went on to correctly find its general solution. A small minority of candidates failed to insert the constant of integration and ended with a general solution which contained no arbitrary constant. Candidates are advised to check that their general solutions contain as many arbitrary constants as the order of the differential equation being solved. The error '$y=x^{-2}(\ln x+c)$' following a correct previous line '$\frac{y}{x^2}=\ln x+c$' was seen more often than expected. |

| Question 3: | Exam report |
|---|---|
| $A=\frac{1}{2}\int_0^{\pi}(2+Cos\theta)^2 Sin\theta d\theta$<br>$A=-\frac{1}{6}\int_0^{\pi}3\times(-Sin\theta)(2+Cos\theta)^2 d\theta$<br>This integral is of the form: $\int 3f'f^2=f^3+c$<br>$A=-\frac{1}{6}\left[(2+Cos\theta)^3\right]_0^{\pi}=-\frac{1}{6}(1)+\frac{1}{6}(3^3)=\frac{13}{3}$ | This question, which tested the area enclosed by a curve whose equation was given in polar form, was relatively poorly answered. Although almost all candidates obtained the correct integral for the area of the loop, a high proportion of these candidates could not then correctly integrate $(2+\cos\theta)^2\sin\theta$ with respect to $\theta$. Only a minority of candidates realised that the integrand was the same as the derivative of $-\frac{1}{3}(2+Cos\theta)^3$. Most other candidates attempted to solve the integral by first writing the integrand as $2\sin\theta + 2\sin 2\theta + \cos^2\theta\sin\theta$ but the majority of these candidates could not then integrate $\cos^2\theta\sin\theta$. |

| Question 4: | Exam report |
|---|---|
| a) $\int \ln(x)dx=\int 1\times\ln(x)dx$<br>$=x\ln(x)-\int x\times\frac{1}{x}dx=x\ln(x)-x+c$<br>b) $\int_a^1\ln(x)dx=\left[x\ln(x)-x\right]_0^a=1\ln(1)-1-a\ln(a)+a$<br>$=-1-a+a\ln(a)$<br>$\lim_{a\to 0}a\ln(a)=0$ so $\int_0^1\ln(x)dx$ exists and $\int_0^1\ln(x)dx=-1$ | Part (a) was, as expected, very well answered with candidates clearly showing the method of integration by parts. In part (b), a smaller proportion of candidates than in June 2008 showed the limiting process fully. For full marks candidates were also expected to pay particular attention to the value of the limit of ($a\ln a$) as $a$ tended to 0. |

| Question 5: | Exam report |
|---|---|
| $a)\ for\ \theta=\pi, r=\frac{2}{3+2Cos\pi}=\frac{2}{3-2}=2$<br>$L(2,\pi)$ belongs to the curve.<br>$b)\,i)\,r=\frac{2}{3+2Cos\theta}=1$<br>$3+2Cos\theta=2$<br>$Cos\theta=-\frac{1}{2}\ so\ \theta=\frac{2\pi}{3}\ or\ \theta=\frac{4\pi}{3}$<br>$This\ gives\ M(1,\frac{2\pi}{3})\ and\ N(1,\frac{4\pi}{3})$<br>$ii)$ Distance $MN^2=OM^2+ON^2-2\times OM\times ON\times Cos(MON)$<br>$MN^2=1+1-2Cos(\frac{4\pi}{3}-\frac{2\pi}{3})=3$<br>$MN=\sqrt{3}$<br>$Area\ of\ OMN=\frac{1}{2}OM\times ON\times Sin(MON)=\frac{\sqrt{3}}{4}$<br>$Area\ of\ OMLN=\frac{1}{2}MN\times OL=\sqrt{3}$<br>$Area\ of\ LMN=Area\ OMLN-Area\ OMN=\frac{3\sqrt{3}}{4}$<br>$c)\,r=\frac{2}{3+2Cos\theta}\ so\ 3r+2rCos\theta=2$<br>$3r=2-2rCos\theta$<br>$by\ squaring\ both\ sides\ 9r^2=(2-2rCos\theta)^2$<br>$9(x^2+y^2)=(2-2x)^2$<br>$9x^2+9y^2=4+4x^2-8x$<br>$9y^2=-5x^2-8x+4$ | Most candidates were able, in part (a), to verify that the point with polar coordinates (2,π) lay on the curve *C*. In part (b)(i), most candidates found the polar coordinates for the two points of intersection, although candidates should continue to be encouraged to give values of θ (in this case for which $Cos\theta=-\frac{1}{2}$) in the given range (in this case 0≤θ ≤2π ).<br>Although some excellent solutions were seen in part (b)(ii), in general candidates found this part to be a challenge. Many candidates gave a final answer which matched the area of triangle *OLM*. Without any clear identification of which area was being calculated, the examiner had to assume that a candidate's final expression was the candidate's answer for the area of triangle *LMN*.<br>Part (c), which required candidates to change a polar equation into a cartesian equation, caused more problems for candidates than anticipated. Those candidates who started by squaring both sides of the polar equation rarely scored more than one mark. In comparison, those candidates who rearranged to obtain $3r=2-2x$ before squaring, usually went on to find a correct cartesian equation of the curve. |

| Question 6: | Exam report |
| --- | --- |
| a) i) $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots$ so<br>$e^{2x} = 1 + 2x + \frac{(2x)^2}{2} + \frac{(2x)^3}{6} + \ldots$<br>$e^{2x} = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \ldots$<br>ii) $(1+3x)^{-\frac{2}{3}} = 1 - \frac{2}{3}\times 3x + \frac{-\frac{2}{3}\times -\frac{5}{3}}{2}\times(3x)^2 + \frac{-\frac{2}{3}\times -\frac{5}{3}\times -\frac{8}{3}}{6}\times(3x)^3 + \ldots$<br>$= 1 - 2x + 5x^2 - \frac{40}{3}x^3 + \ldots$<br>$f(x) = e^{2x}(1+3x)^{-\frac{2}{3}} = \left(1 + 2x + 2x^2 + \frac{4}{3}x^3 + \ldots\right)\left(1 - 2x + 5x^2 - \frac{40}{3}x^3 + \ldots\right)$<br>$f(x) = 1 - 2x + 5x^2 - \frac{40}{3}x^3 + 2x - 4x^2 + 10x^3 + 2x^2 - 4x^3 + \frac{4}{3}x^3 + \ldots.$<br>$f(x) = 1 + 3x^2 - 6x^3 + \ldots$<br>b) i) $y = \ln(1+2Sinx)$ $\quad \frac{dy}{dx} = \frac{2Cosx}{1+2Sinx}$<br>and $\frac{d^2y}{dx^2} = \frac{-2Sinx(1+2Sinx) - 2Cosx(2Cosx)}{(1+2Sinx)^2} = \frac{-2Sinx - 4}{(1+2Sinx)^2}$<br>ii) $\ln(1+2Sinx) = y(0) + y'(0)x + \frac{y''(0)}{2}x^2 + \ldots$<br>$= 0 + 2x - \frac{4}{2}x^2 + \ldots = 2x - 2x^2 + \ldots$<br>c) $1 - f(x) = 3x^2 - 6x^3 + \ldots$<br>$x\ln(1+2Sinx) = x(2x - 2x^2 + \ldots) = 2x^2 - 2x^3 + \ldots$<br>so $\frac{1-f(x)}{x\ln(1+2Sinx)} = \frac{3x^2 - 6x^3 + \ldots}{2x^2 - 2x^3 + \ldots} = \frac{3 - 6x + ..}{2 - 2x + \ldots}$<br>Therefore, $\lim_{x\to 0}\frac{1-f(x)}{x\ln(1+2Sinx)} = \frac{3}{2}$ | Most candidates were able to write down the first four terms in the series expansion of e2*x*, but in part (a)(ii) a significant minority of candidates ignored the request to use the binomial series expansion and instead resorted to differentiation of f(*x*) and the use of Maclaurin's theorem. In part (b)(i), the most common error was to fail to use the chain rule and thus to write the derivative of ln(1+ 2sin *x*) as $\frac{1}{1+2Sinx}$ Those candidates who correctly applied the chain rule usually went on to find the correct expression for $\frac{d^2y}{dx^2}$ by applying either the quotient rule or the product rule. Almost all candidates who obtained an expression for $\frac{d^2y}{dx^2}$ showed a correct understanding of Maclaurin's theorem in part (b)(ii). Although the majority of candidates could correctly find the limiting value of the given expression as *x* tended to 0 in part (c), there were some incorrect solutions seen which just involved substituting *x* = 0 into the given expression. |

**Question 7:**

a) $x = e^t \qquad \frac{dx}{dt} = e^t = x \;\; and \;\; \frac{dt}{dx} = \frac{1}{e^t} = \frac{1}{x}$

$$\frac{dy}{dx} = \frac{dt}{dx} \times \frac{dy}{dt} = e^{-t}\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dt}{dx} \times \frac{d}{dt}\left(e^{-t}\frac{dy}{dt}\right)$$

$$= e^{-t}\left(-e^{-t}\frac{dy}{dt} + e^{-t}\frac{d^2y}{dt^2}\right) = e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

$$e^{-2t} = \frac{1}{(e^t)^2} = \frac{1}{x^2} \;\; so$$

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$

b) The equation $x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} = 10$ *becomes*

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 4e^t \times e^{-t}\frac{dy}{dt} = 10$$

$$\frac{d^2y}{dt^2} - 5\frac{dy}{dt} = 10$$

c) The auxiliary equation is $\lambda^2 - 5\lambda = 0$

$$\lambda(\lambda - 5) = 0$$

$$\lambda = 0 \;\; or \;\; \lambda = 5$$

The complementary function $y_c = Ae^{0t} + Be^{5t} = A + Be^{5t}$

A particular integral $y = at \qquad \frac{dy}{dx} = a \;\; and \;\; \frac{d^2y}{dt^2} = 0$

$$0 - 5a = 10 \qquad a = -2$$

The general solution is $y = -2t + A + Be^{5t}$

$x = e^t \;\; and \;\; t = \ln x \qquad so \;\; y = -2\ln x + A + Bx^5$

d) When $x = 1, y = 0 \qquad 0 = 0 + A + B$

$$\frac{dy}{dx} = -\frac{2}{x} + 5Bx^4$$

When $x = 1, \frac{dy}{dx} = 8 \quad so \;\; 8 = -2 + 5B$

This gives $B = 2 \;\; and \;\; A = -2$

$$y = -2\ln x - 2 + 2x^5$$

**Exam report**

The bookwork, which was tested in part (a), was last tested in the January 2008 paper, although in a more structured manner. In general candidates still find this to be a demanding exercise and marks awarded in Question 7(a) were again not high. Those candidates who correctly answered part (a) normally gained the two marks for correctly transforming the differential equation but the remaining candidates generally could not deal with the term '$4x\frac{dy}{dx}$' in a convincing way.

Part (c) was not answered as well as expected with a significant number of candidates either starting with the wrong auxiliary equation '$m^2 - 5m - 10 = 0$' or making errors in solving the correct auxiliary equation '$m^2 - 5m = 0$'. Finding the particular integral was a major problem for a significant number of candidates. Many tried to find it by using $y = a$ instead of $y = at$ despite their form of the complementary function having a constant term. It was also very common to see the general solution of the differential equation in $y$ and $t$ being written in the form $y = f(x)$ instead of $y = f(t)$.
Although many candidates realised that the solution to the differential equation in part (d) was related to their answer for part (c), only the more able candidates could apply the correct conversion or apply the relevant boundary conditions to reached the simplified answer '$y = 2x^5 - 2 - 2\ln x$'.


| Grade boundaries | | | | | | | |
|---|---|---|---|---|---|---|---|
| Grade | | | A | B | C | D | E |
| Mark | Max 75 | | 59 | 51 | 44 | 37 | 30 |

MFP3

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| 1(a) | $y_1 = 3 + 0.2\times\left[\frac{1^2+3^2}{1+3}\right]$ | M1A1 | | |
| | $= 3.5$ | A1 | 3 | |
| (b) | $k_1 = 0.2\times 2.5 = 0.5$ | B1ft | | PI ft from (a) |
| | $k_2 = 0.2\times f(1.2,\ 3.5)$ | M1 | | ft on (a) |
| | $\ldots = 0.2\times\frac{1.2^2+3.5^2}{1.2+3.5} = 0.5825(53\ldots)$ | A1ft | | PI condone 3dp |
| | $y(1.2) = y(1) + \frac{1}{2}\left[0.5 + 0.5825(53\ldots)\right]$ | m1 | | |
| | $= 3.54127\ldots = 3.5413$ to 4dp | A1ft | 5 | ft one slip<br>If answer not to 4dp withhold this mark |
| | **Total** | | **8** | |
| 2(a) | IF is $e^{\int -\frac{2}{x}\,dx}$ | M1 | | $e^{\int \pm\frac{2}{x}\,dx}$ |
| | $= e^{-2\ln x}$ | A1 | | P1 |
| | $= e^{\ln x^{-2}} = x^{-2} = \frac{1}{x^2}$ | A1 | 3 | AG Be convinced |
| (b) | $\frac{d}{dx}\left(\frac{y}{x^2}\right) = \frac{1}{x^2}x$ | M1 | | LHS as $d/dx(y\times IF)$ |
| | | A1 | | PI |
| | $\frac{y}{x^2} = \int \frac{1}{x}\,dx = \ln x + c$ | A1 | | RHS Condone missing '+ c' here |
| | $y = x^2\ln x + cx^2$ | A1 | 4 | |
| | **Total** | | **7** | |
| 3 | Area $= \frac{1}{2}\int_0^{\pi}(2+\cos\theta)^2\sin\theta\,d\theta$ | M1 | | use of $\frac{1}{2}\int r^2\,d\theta$ |
| | | B1 | | Correct limits |
| | $-\frac{1}{2}\left[-\frac{1}{3}(2+\cos\theta)^3\right]_0^{\pi}$ | M2 | | Valid method to reach $k(2+\cos\theta)^3$ or $a\cos\theta + b\cos 2\theta + c\cos^3\theta$ OE<br>{SC: M1 if expands then integrates to get either $a\cos\theta + b\cos 2\theta$ OE or $c\cos^3\theta$ OE in a valid way} |
| | | A1 | | OE eg $-4\cos\theta - \cos 2\theta - \frac{1}{3}\cos^3\theta$ |
| | $= \frac{1}{2}\left\{-\frac{1}{3} + \frac{1}{3}\times 3^3\right\} = \frac{13}{3}$ | A1 | 6 | CSO |
| | **Total** | | **6** | |

MFP3 (cont)

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| 4(a) | $\int \ln x\,dx - x\ln x - \int x\left(\frac{1}{x}\right)dx$ | M1 | | Integration by parts |
| | $= x\ln x - x + c$ | A1 | 2 | CSO AG |
| (b) | $\int_0^1 \ln x\,dx = \lim_{a\to 0}\int_a^1 \ln x\,dx$ | M1 | | OE |
| | $= \lim_{a\to 0}\{0 - 1 - [a\ln a - a]\}$ | M1 | | $F(1) - F(a)$ OE |
| | But $\lim_{a\to 0} a\ln a = 0$ | E1 | | Accept a general form eg $\lim_{a\to 0} a^k\ln a = 0$ |
| | So $\int_0^1 \ln x\,dx = -1$ | A1 | 4 | |
| | **Total** | | **6** | |
| 5(a) | When $\theta = \pi$,<br>$r = \frac{2}{3+2\cos\pi} = \frac{2}{3+2(-1)} = 2$ | B1 | 1 | Correct verification |
| (b)(i) | $\frac{2}{3+2\cos\theta} = 1 \Rightarrow \cos\theta = -\frac{1}{2}$ | M1 | | Equates $r$'s and attempts to solve. |
| | Points of intersection $\left(1, \frac{2\pi}{3}\right), \left(1, \frac{4\pi}{3}\right)$ | A2,1 | 3 | Condone eg $-2\pi/3$ for $4\pi/3$<br>A1 if either one point correct or two correct solutions of $\cos\theta = -0.5$ |
| (ii) | Area $OMN = \frac{1}{2}\times 1\times 1\times\sin(\lvert\theta_M - \theta_N\rvert)$ | M1 | | **ALT** $MN = 2\times 1\times\sin\frac{\pi}{3}$ M1 |
| | $= \frac{1}{2}\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{4}$ | A1 | | Perp. from $L$ to $MN$<br>$= 2 - 1\cos\frac{\pi}{3} = \frac{3}{2}$ M1A1 |
| | Area $OMLN = 2\times\frac{1}{2}\times 1\times 2\times\sin\frac{\pi}{3}$ | M1 | | Area $LMN = \frac{1}{2}\times\sqrt{3}\times\frac{3}{2} = \frac{3\sqrt{3}}{4}$ A1 |
| | Area $LMN = \sqrt{3} - \frac{\sqrt{3}}{4} = \frac{3\sqrt{3}}{4}$ | A1 | 4 | |
| (c) | $3r + 2r\cos\theta = 2$ | M1 | | |
| | $3r + 2x = 2$ | B1 | | $r\cos\theta = x$ stated or used |
| | $3r = 2 - 2x$ | A1 | | $3r = \pm(2 - 2x)$ |
| | $9(x^2 + y^2) = (2 - 2x)^2$ | M1 | | $r^2 - x^2 + y^2$ used |
| | $9y^2 = (2 - 2x)^2 - 9x^2$ | A1 | 5 | CSO<br>ACF for f(x) eg $9y^2 = -5x^2 - 8x + 4$ |
| | **Total** | | **13** | |

MFP3 (cont)

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| 6(a)(i) | $e^{2x} = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \ldots$ | M1 | | Clear use of $x \to 2x$ in expansion of $e^x$ |
| | | A1 | 2 | ACF |
| (ii) | $\{f(x)\} = e^{2x}(1+3x)^{-\frac{2}{3}}$ | | | |
| | $(1+3x)^{-\frac{2}{3}} = 1 + \left(-\frac{2}{3}\right)(3x) + \frac{\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)(3x)^2}{2} - \frac{40}{3}x^3$ | M1 | | First three terms as $1 + \left(-\frac{2}{3}\right)(3x) + kx^2$ OE |
| | $= 1 - 2x + 5x^2 - \frac{40}{3}x^3$ | A1 | | |
| | $\{f(x) \approx\}$ $1 + 2x + 2x^2 + \frac{4x^3}{3} - 2x - 4x^2 - 4x^3 + 5x^2 + 10x^3 - \frac{40x^3}{3}$ | m1 A1ft | | Dep on both prev MS Condone one sign or numerical slip in mult. |
| | $= 1 + 3x^2 - 6x^3$ | A1 | 5 | CSO AG A0 if binominal series not used |
| (b)(i) | $y - \ln(1+2\sin x) \Rightarrow \frac{dy}{dx} = \frac{1}{1+2\sin x} \times 2\cos x$ | M1 A1 | | Chain rule |
| | $\frac{d^2y}{dx^2} = \frac{(1+2\sin x)(-2\sin x) - 2\cos x(2\cos x)}{(1+2\sin x)^2} = \frac{-2(\sin x + 2)}{(1+2\sin x)^2}$ | M1 | | Quotient rule OE with $u$ and $v$ non constant |
| | | A1 | 4 | ACF |
| (ii) | $y(0) = 0,\ y'(0) = 2,\ y''(0) = -4$ | M1 | | |
| | McL Thm.: $\{\ln(1+2\sin x)\} \approx 0 + 2x - 4\left(\frac{x^2}{2}\right) + .. \approx 2x - 2x^2$ | A1 | 2 | CSO AG |
| (c) | $\lim_{x \to 0} \frac{1 - f(x)}{x\ln(1+2\sin x)} = \lim_{x \to 0} \frac{-3x^2 + 6x^3}{2x^2 - 2x^3}$ | M1 | | Using expansions |
| | $= \lim_{x \to 0} \frac{-3+6x}{2-2x}$ | m1 | | Division by $x^2$ stage before taking limit. |
| | $= -\frac{3}{2}$ | A1 | 3 | CSO |
| | **Total** | | **16** | |

MFP3 (cont)

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| 7(a) | $\frac{dx}{dt} = e^t \ \{= x\}$ | B1 | | OE |
| | $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = e^{-t}\frac{dy}{dt}$ | M1 A1 | | Chain rule; OE eg $x\frac{dy}{dx} = \frac{dy}{dt}$ |
| | $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(e^{-t}\frac{dy}{dt}\right) = \frac{dt}{dx}\frac{d}{dt}\left(e^{-t}\frac{dy}{dt}\right)$ | M1 | | $\frac{d}{dx}(\ ) = \frac{dt}{dx}\frac{d}{dt}(\ )$ OE |
| | $= \frac{dt}{dx}\left(-e^{-t}\frac{dy}{dt} + e^{-t}\frac{d^2y}{dt^2}\right)$ | M1 | | Product rule OE |
| | $\ldots = e^{-t}\left(-e^{-t}\frac{dy}{dt} + e^{-t}\frac{d^2y}{dt^2}\right)$ | A1 | | OE |
| | $\ldots = x^{-2}\left(-\frac{dy}{dt} + \frac{d^2y}{dt^2}\right)$ | | | |
| | $\Rightarrow x^2\frac{d^2y}{dx^2} = \left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$ | A1 | 7 | CSO AG Completion. Be convinced |
| (b) | $x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} = 10$ | | | |
| | $\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) - 4\left(\frac{dy}{dt}\right) = 10$ | M1 | | |
| | $\frac{d^2y}{dt^2} - 5\frac{dy}{dt} = 10$ | A1 | 2 | CSO AG Completion. Be convinced |
| (c) | $\frac{d^2y}{dt^2} - 5\frac{dy}{dt} = 10$ (*) | | | |
| | Auxl eqn $m^2 - 5m = 0$ | M1 | | PI |
| | $m(m-5) = 0$ | | | |
| | $m = 0$ and $5$ | A1 | | |
| | CF: $(y_C =) A + Be^{5t}$ | M1 | | ft wrong values of $m$ provided 2 arb. constants in CF. condone $x$ for $t$ here |
| | PI: $(y_P =) -2t$ | B1 | | |
| | GS of (*) $\{y\} = A + B\,e^{5t} - 2t$ | B1ft | 5 | ft on c's CF + PI, provided PI is non-zero and CF has two arbitrary constants |
| (d) | $\Rightarrow y = A + Bx^5 - 2\ln x$ | M1 | | |
| | $y'(x) = 5Bx^4 - 2x^{-1}$ | A1ft | | Must involve differentiating $a \ln x$ ft slip |
| | Using boundary conditions to find $A$ & $B$ | M1 | | |
| | $B = 2;\ A = -2;$ $\{y = -2 + 2x^5 - 2\ln x\}$ | A1;A1ft | 5 | ft a slip. |
| | **Total** | | **19** | |
| | **TOTAL** | | **75** | |

# June 2009

Answer **all** questions.

**1** The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = f(x, y)$$

where $f(x, y) = \sqrt{x^2 + y + 1}$

and $y(3) = 2$

(a) Use the Euler formula

$$y_{r+1} = y_r + h\,f(x_r, y_r)$$

with $h = 0.1$, to obtain an approximation to $y(3.1)$, giving your answer to four decimal places. *(3 marks)*

(b) Use the formula

$$y_{r+1} = y_{r-1} + 2h\,f(x_r, y_r)$$

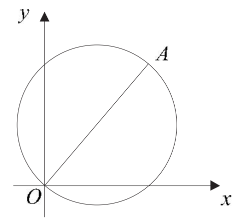
with your answer to part (a), to obtain an approximation to $y(3.2)$, giving your answer to three decimal places. *(3 marks)*

**2** By using an integrating factor, find the solution of the differential equation

$$\frac{dy}{dx} - y\tan x = 2\sin x$$

given that $y = 2$ when $x = 0$. *(9 marks)*

**3** The diagram shows a sketch of a circle which passes through the origin $O$.




The equation of the circle is $(x-3)^2 + (y-4)^2 = 25$ and $OA$ is a diameter.

(a) Find the cartesian coordinates of the point $A$. *(2 marks)*

(b) Using $O$ as the pole and the positive $x$-axis as the initial line, the polar coordinates of $A$ are $(k, \alpha)$.

(i) Find the value of $k$ and the value of $\tan\alpha$. *(2 marks)*

(ii) Find the polar equation of the circle $(x-3)^2 + (y-4)^2 = 25$, giving your answer in the form $r = p\cos\theta + q\sin\theta$. *(4 marks)*

**4** Evaluate the improper integral

$$\int_1^{\infty} \left(\frac{1}{x} - \frac{4}{4x+1}\right) dx$$

showing the limiting process used and giving your answer in the form $\ln k$, where $k$ is a constant to be found. *(5 marks)*

**5** It is given that $y$ satisfies the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 8\sin x + 4\cos x$$

(a) Find the value of the constant $k$ for which $y = k\sin x$ is a particular integral of the given differential equation. *(3 marks)*

(b) Solve the differential equation, expressing $y$ in terms of $x$, given that $y = 1$ and $\frac{dy}{dx} = 4$ when $x = 0$. *(8 marks)*

**6** The function f is defined by

$$f(x) = (9 + \tan x)^{\frac{1}{2}}$$

(a) (i) Find $f''(x)$. *(4 marks)*

(ii) By using Maclaurin's theorem, show that, for small values of $x$,

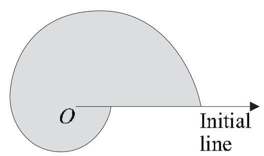
$$(9 + \tan x)^{\frac{1}{2}} \approx 3 + \frac{x}{6} - \frac{x^2}{216}$$ *(3 marks)*

(b) Find

$$\lim_{x \to 0} \left[ \frac{f(x) - 3}{\sin 3x} \right]$$ *(3 marks)*

**7** The diagram shows the curve $C_1$ with polar equation

$$r = 1 + 6e^{-\frac{\theta}{\pi}}, \quad 0 \leqslant \theta \leqslant 2\pi$$




(a) Find, in terms of π and e, the area of the shaded region bounded by $C_1$ and the initial line. *(5 marks)*

(b) The polar equation of a curve $C_2$ is

$$r = e^{\frac{\theta}{\pi}}, \quad 0 \leqslant \theta \leqslant 2\pi$$

Sketch the curve $C_2$ and state the polar coordinates of the end-points of this curve. *(4 marks)*

(c) The curves $C_1$ and $C_2$ intersect at the point $P$. Find the polar coordinates of $P$. *(5 marks)*

**8** (a) Given that $x = t^2$, where $t \geqslant 0$, and that $y$ is a function of $x$, show that:

(i) $2\sqrt{x}\frac{dy}{dx} = \frac{dy}{dt}$; *(3 marks)*

(ii) $4x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = \frac{d^2y}{dt^2}$. *(3 marks)*

(b) Hence show that the substitution $x = t^2$, where $t \geqslant 0$, transforms the differential equation

$$4x\frac{d^2y}{dx^2} + 2(1 + 2\sqrt{x})\frac{dy}{dx} - 3y = 0$$

into

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = 0$$ *(2 marks)*

(c) Hence find the general solution of the differential equation

$$4x\frac{d^2y}{dx^2} + 2(1 + 2\sqrt{x})\frac{dy}{dx} - 3y = 0$$

giving your answer in the form $y = g(x)$. *(4 marks)*

**END OF QUESTIONS**

# AQA – Further pure 3 – Jun 2009 – Answers

| Question 1: | Exam report |
| --- | --- |
| $a)\ \frac{dy}{dx}=\sqrt{x^2+y+1}\ \ and\ \ y(3)=2$<br>$y_1=2+0.1\sqrt{3^2+2+1}=2.3464\ to\ 4d.p.$<br><br>$b)\ y_2=y_0+2hf(x_1,y_1)$<br>$=2+2\times0.1\times\sqrt{3.1^2+2.3464+1}=2.720\ to\ 3d.p.$ | Numerical solutions of first order differential equations continue to be a good source of marks for all candidates. Although it was the best answered question on the paper, more candidates than usual mixed up the *x* and *y* values in applying the given formulae, or incorrectly used f(3,2) instead of f(3.1, *y*(3.1)) in the given formula in part (b). There were very few candidates who lost the final accuracy mark for failing to give their answer correct to three decimal places. |

| Question 2: | Exam report |
| --- | --- |
| $\frac{dy}{dx}-yTan(x)=2Sin(x)$<br>An integrating factor is $I=e^{\int -\tan(x)dx}=e^{\ln(\cos x)}=\cos(x)$<br>The equation becomes<br>$Cos(x)\frac{dy}{dx}-Sin(x)y=2Sin(x)Cos(x)$<br>$\frac{d}{dx}\left(Cos(x)y\right)=Sin(2x)$<br>$Cos(x)y=\int Sin(2x)dx=-\frac{1}{2}Cos(2x)+c$<br>$y=\frac{-Cos(2x)}{2Cos(x)}+\frac{c}{Cos(x)}$<br>When $x=0, y=2\ gives\ 2=-\frac{1}{2}+c\ \ so\ c=\frac{5}{2}$<br>$y=\frac{5-Cos(2x)}{2Cos(x)}$ | Most candidates were able to show that they knew how to find and use an integrating factor to solve a first order differential equation. However, a significant number failed to find the correct integrating factor because they missed the negative sign and used $e^{\int \tan(x)dx}$. Candidates should be aware that, unless told otherwise in the question, it is acceptable to leave the solution in a form other than *y* = f(*x*). Some candidates lost the final accuracy mark because either they had attempted to divide throughout by cos(x) but forgot to divide the + *c* as well before substituting in the boundary condition, or had made the arithmetical error "$2=-\frac{1}{2}+c\Rightarrow c=\frac{3}{2}$". |

| Question 3: | Exam report |
| --- | --- |
| $a)$ The centre of the circle (3,4) is the midpoint of O(0,0) and A<br>This gives A(6,8).<br>$b)\,i)\,k=\sqrt{6^2+8^2}=10$<br>$Tan\,\alpha=\frac{8}{6}=\frac{4}{3}$<br>$ii)\,(x-3)^2+(y-4)^2=25$<br>$x^2-6x+9+y^2-8y+16=25$<br>$x^2+y^2-6x-8y=0$<br>$r^2-6rCos\theta-8rSin\theta=0$<br>$r=6Cos\theta+8Sin\theta$ | This question, which tested the relationship between cartesian and polar coordinates, caused candidates more problems than anticipated. In part (a), it was not uncommon to see solutions which assumed incorrectly that the angle between *OA* and the *x*-axis was $45^o$, which led to the incorrect coordinates $\left(\sqrt{50},\sqrt{50}\right)$ for *A*. In part (b)(i), a common wrong value for *k* was 5 but the value of tanα was usually stated correctly. Many candidates gave the correct polar equation for the circle in part (b)(ii). |

| Question 4: | Exam report |
|---|---|
| $\int_1^N \left(\frac{1}{x}-\frac{4}{4x+1}\right)dx = \left[\ln x - \ln(4x+1)\right]_1^N$<br>$= \left[\ln\left(\frac{x}{4x+1}\right)\right]_1^N = \ln\left(\frac{N}{4N+1}\right)-\ln\left(\frac{1}{5}\right)$<br>$When\ N\to\infty, \frac{N}{4N+1} = \frac{1}{4+\frac{1}{N}} \to \frac{1}{4}$<br>$so\ \ln\left(\frac{N}{4N+1}\right)\to \ln\left(\frac{1}{4}\right)$<br>$\int_1^\infty \left(\frac{1}{x}-\frac{4}{4x+1}\right)dx\ exists$<br>$and\ \int_1^\infty \left(\frac{1}{x}-\frac{4}{4x+1}\right)dx = \ln\left(\frac{1}{4}\right)-\ln\left(\frac{1}{5}\right) = \ln\left(\frac{5}{4}\right)$ | This question on improper integrals and limiting processes, which lacked the structure given in many previous papers, was the worse answered question on the paper. Examiners expected to see the infinite upper limit replaced by, for example, *a*, the integration then carried out and then consideration of the limiting process as $a\to\infty$. Most, although not all, reached 'ln*x*−ln(4*x*+1)' for 1 mark, but many of the weaker candidates just stated that ln*x*−ln(4*x*+1)=0 when $x\to\infty$ and gave the wrong value ' ln 5 ' as their answer. Better candidates scored 3 marks for reaching 'as $a\to\infty, \ln\left(\frac{5a}{4a+1}\right)=\ln\left(\frac{5}{4}\right)$' but only those who inserted the extra step to get, for example, 'as $a\to\infty, \ln\left(\frac{5}{4+\frac{1}{a}}\right)=\ln\left(\frac{5}{4}\right)$' were in line for full marks. |

| Question 5: | Exam report |
|---|---|
| $a)\ y = kSinx$<br>$\frac{dy}{dx} = kCos(x)$<br>$\frac{d^2y}{dx^2} = -kSin(x)$<br>$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+5y = 8Sinx+4Cosx$<br>$-kSinx+2kCosx+5kSinx = 8Sinx+4Cosx$<br>$4kSinx+2kCosx = 8Sinx+4Cosx$<br>$so\ k=2$<br>A particular integral is $y = 2Sinx$<br>$b)$ The auxiliary equation is $\lambda^2+2\lambda+5=0$<br>The discriminant is $2^2-4\times1\times5=-16$<br>$\lambda_1 = \frac{-2+4i}{2} = -1+2i\ and\ \lambda_2 = -1-2i$<br>The complementary function is $y = e^{-x}(A\cos(2x)+BCos(2x))$<br>The general solution is $y = 2Sinx + e^{-x}(ACos(2x)+BSin(2x))$<br>$when\ x=0, y=1\ this\ gives\ \ 1=0+A\ \ so\ \ A=1$<br>$\frac{dy}{dx} = 2Cos(x)+e^{-x}(-Cos(2x)-BSin(2x)-2Sin(2x)+2BCos(2x))$<br>$\frac{dy}{dx} = 4 = 2+(-1+0-0+2B)\ so\ B=\frac{3}{2}$<br>The particular solution is $y = 2Sin(x)+e^{-x}(Cos(2x)+\frac{3}{2}Sin(2x))$ | In part (a), many candidates decided to ignore the given form of the particular integral and worked with *a*cos*x*+*b*sin*x*. Such an approach was not penalised by examiners provided the candidate showed that both *a* = 0 and *b* = 2. The vast majority of candidates showed that they knew the methods required to solve the second order differential equation, but arithmetical errors in solving the auxiliary equation $m^2+2m+5=0$ or in applying the boundary condition $\frac{dy}{dx}=4$ when *x* = 0 or the wrong differentiation of cos2*x* were sources of loss of marks as well as the more serious error of applying the boundary conditions to the complementary function before adding on the particular integral. |

## Question 6:

$$f(x)=(9+Tanx)^{\frac{1}{2}}$$

$$a)\ f'(x)=\frac{1}{2}\times\frac{1}{Cos^2x}\times(9+Tanx)^{-\frac{1}{2}}$$

$$f''(x)=\frac{1}{2}\times\frac{-2\times -Sinx}{Cos^3x}(9+Tanx)^{-\frac{1}{2}}+\frac{1}{2Cos^2x}\times-\frac{1}{2}\times\frac{1}{Cos^2x}\times(9+Tanx)^{-\frac{3}{2}}$$

$$f''(x)=\frac{Sinx}{Cos^3x\sqrt{9+Tanx}}-\frac{1}{4Cos^4x(9+Tanx)^{\frac{3}{2}}}$$

$$ii)\ f(0)=9^{\frac{1}{2}}=3$$

$$f'(0)=\frac{1}{2}\times1\times\frac{1}{\sqrt{9}}=\frac{1}{6}$$

$$f''(0)=0-\frac{1}{4\times9^{\frac{3}{2}}}=-\frac{1}{108}$$

The Maclaurin's series is $(9+Tanx)^{\frac{1}{2}}=3+\frac{1}{6}x-\frac{1}{108}\times\frac{x^2}{2}+...$

$$(9+Tanx)^{\frac{1}{2}}=3+\frac{x}{6}-\frac{x^2}{216}+...$$

$b)\ f(x)-3=\frac{x}{6}+...$ *and* $Sin(3x)=3x+...$

So $\frac{f(x)-3}{Sin(3x)}=\frac{\frac{x}{6}+...}{3x+..}=\frac{1}{18}+...$ so $\lim_{x\to0}\left(\frac{f(x)-3}{Sin3x}\right)=\frac{1}{18}$

### Exam report

Most candidates were able to find f′($x$) correctly although some weaker candidates failed to apply the chain rule and were heavily penalised. The vast majority of the other candidates used the product rule (or quotient rule) but errors in differentiating $\sec^2 x$ were common. The majority of candidates showed good knowledge of Maclaurin's theorem but only those who had made no errors in earlier differentiations could score all 3 marks for showing the printed result in part (a)(i). Many weaker candidates failed to realise that the series expansion for sin 3$x$ was required in part (b) and just stated the incorrect answer 0 for the limit. A significant minority of other candidates who used the expansion for sin 3$x$ did not explicitly reach the stage of a constant term in both the numerator and denominator before taking the limit as $x\to0$.

## Question 7:

$$a)\ A=\frac{1}{2}\int_0^{2\pi}\left(1+6e^{-\frac{\theta}{\pi}}\right)^2d\theta=\frac{1}{2}\int_0^{2\pi}1+36e^{-\frac{2\theta}{\pi}}+12e^{-\frac{\theta}{\pi}}d\theta$$

$$A=\left[\frac{1}{2}\theta-9\pi e^{-\frac{2\theta}{\pi}}-6\pi e^{-\frac{\theta}{\pi}}\right]_0^{2\pi}$$

$$A=\left(\pi-9\pi e^{-4}-6\pi e^{-2}\right)-\left(0-9\pi-6\pi\right)$$

$$A=\pi\left(16-6e^{-2}-9e^{-4}\right)$$

$b)\ C_2: r=e^{\frac{\theta}{\pi}}$

When $\theta=0,\ r=1$ *In polar coordinates*: $(1,0)$

When $\theta=2\pi, r=e^2$ $(e^2,2\pi)$

$c)\ r=1+6e^{-\frac{\theta}{\pi}}=e^{\frac{\theta}{\pi}}$ *Multiplying by* $e^{\frac{\theta}{\pi}}$ *gives*:

$$\left(e^{\frac{\theta}{\pi}}\right)^2-e^{\frac{\theta}{\pi}}-6=0$$

$$\left(e^{\frac{\theta}{\pi}}-3\right)\left(e^{\frac{\theta}{\pi}}+2\right)=0$$

*so* $e^{\frac{\theta}{\pi}}=3$ *or* $e^{\frac{\theta}{\pi}}=-2$ (*no solutions*)

$\frac{\theta}{\pi}=\ln(3)$ $\theta=\pi\ln(3)$

*then* $r=e^{\frac{\theta}{\pi}}=e^{\ln(3)}=3$

The point of intersection is $(3,\pi\ln(3))$

### Exam report

Some candidates integrated $r$ instead of $r^2$ to find the area of the shaded region. If they had first stated the formula $A=\int_\alpha^\beta\frac{1}{2}r^2d\theta$, which is given in the formulae booklet, before substituting for $r$ further credit could have been awarded. The two most common errors were incorrectly squaring $6e^{-\frac{\theta}{\pi}}$ to get $6e^{-\frac{2\theta}{\pi}}$ and integrating $12e^{-\frac{\theta}{\pi}}$ incorrectly to get $-\frac{12}{\pi}e^{-\frac{\theta}{\pi}}$. The quality of sketches varied significantly with some even starting at the pole. A significant minority of candidates either just gave the coordinates of one end point, missing (1, 0) or gave a decimal approximation for $e^2$ or gave the incorrect answer ( $e^2$ , 0). It was surprising to find some candidates giving the coordinates in reverse order. In part (c), most candidates formed a correct equation by equating the $r$ terms but in general only the better candidates were then able to form and solve the resulting quadratic equation in $e^{\frac{\theta}{\pi}}$. Only a small number of candidates failed to reject the negative value for $e^{\frac{\theta}{\pi}}$ before going on to get the correct coordinates for the point $P$. Again candidates should use exact forms and not give a decimal approximation in place of π ln 3 . A common mistake seen in solving '$e^{\frac{\theta}{\pi}}=1+6e^{-\frac{\theta}{\pi}}$' is illustrated '$\frac{\theta}{\pi}=\ln1+6\left(-\frac{\theta}{\pi}\right)$'

| Question 8: | Exam report |
|---|---|

**Question 8:**

$$a)\ x = t^2 \quad for\ t \geq 0 \ and \quad t = \sqrt{x}$$

$$\frac{dx}{dt} = 2t = 2\sqrt{x} \quad and \quad \frac{dt}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2t}$$

$$i)\ \frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} = \frac{dy}{dx} \times 2\sqrt{x}$$

$$ii)\ \frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dx}\left(2\sqrt{x}\frac{dy}{dx}\right) \times \frac{dx}{dt}$$

$$= \left(\frac{1}{\sqrt{x}}\frac{dy}{dx} + 2\sqrt{x}\frac{d^2y}{dx^2}\right) \times 2\sqrt{x}$$

$$= 2\frac{dy}{dx} + 4x\frac{d^2y}{dx^2}$$

$$b)\ \underbrace{4x\frac{d^2y}{dx^2} + 2\frac{dy}{dx}}_{\frac{d^2y}{dt^2}} + \underbrace{4\sqrt{x}\frac{dy}{dx}}_{2\frac{dy}{dt}} - 3y = 0 \quad becomes$$

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = 0$$

c) The auxiliary equation is $\lambda^2 + 2\lambda - 3 = 0$

$$(\lambda + 3)(\lambda - 1) = 0$$

$$\lambda = -3 \ or \ \lambda = 1$$

The general solution is $y = Ae^{-3t} + Be^{t}$

but $t = \sqrt{x}$ so $y = Ae^{-3\sqrt{x}} + Be^{\sqrt{x}}$

**Exam report**

The bookwork in part (a) was similar to that tested in recent papers. Candidates generally answered part (a)(i) correctly but showing the printed result involving the second derivatives in part (a)(ii) proved to be more difficult. However, there were some excellent concise solutions to part (a)(ii) seen. As is the case in all questions which ask for results to be shown, sufficient detail in solutions must be provided that is fully correct. It was not uncommon to see incorrect working followed by the printed result. Candidates should realise that credit will not be given and in some cases marks can be lost for this. Candidates were generally able to use the printed results in part (a) to correctly transform the differential equation into the required form in part (b). Those candidates who attempted part (c) normally scored at least two marks. The most common error was to give the answer as $y=Ae^{-3x}+Be^{x}$ instead of $y = Ae^{-3\sqrt{x}} + Be^{\sqrt{x}}$ which followed from $y=Ae^{-3t}+Be^{t}$.

**Grade boundaries**

| Grade | | | A | B | C | D | E |
|---|---|---|---|---|---|---|---|
| Mark | Max 75 | | 62 | 54 | 46 | 38 | 30 |

MFP3

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| **1(a)** | $y(3.1) = y(3) + 0.1\sqrt{3^2+2+1}$ | M1A1 | | |
| | $= 2 + 0.1\times\sqrt{12} = 2.3464(10..)$ $= 2.3464$ | A1 | 3 | Condone > 4dp if correct |
| **(b)** | $y(3.2) = y(3) + 2(0.1)[f(3.1, y(3.1))]$ | M1 | | |
| | $\ldots = 2 + 2(0.1)[\sqrt{(3.1^2+2.3464+1)}]$ | A1F | | ft on candidate's answer to **(a)** |
| | $\ldots = 2 + 0.2\times 3.599499.. = 2.719(89..)$ $= 2.720$ | A1 | 3 | CAO Must be 2.720 |
| | **Total** | | 6 | |
| **2** | IF is $e^{\int -\tan x \, dx}$ | M1 | | Award even if negative sign missing |
| | $- e^{\ln(\cos x)\ (+c)}$ | A1 | | OE Condone missing $c$ |
| | $= (k)\cos x$ | A1F | | ft earlier sign error |
| | $\cos x\frac{dy}{dx} - y\tan x\cos x = 2\sin x\cos x$ | | | |
| | $\frac{d}{dx}(y\cos x) = 2\sin x\cos x$ | M1 | | LHS as $\frac{d}{dx}(y\times \text{IF})$ PI |
| | $y\cos x = \int 2\sin x\cos x\,dx\,dx$ | A1F | | ft on c's IF provided no exp or logs |
| | $y\cos x = \int \sin 2x\ dx$ | m1 | | Double angle or substitution OE for integrating $2\sin x\cos x$ |
| | $y\cos x = -\frac{1}{2}\cos 2x(+c)$ | A1 | | ACF |
| | $2 = -\frac{1}{2} + c$ | m1 | | Boundary condition used to find $c$ |
| | $c = \frac{5}{2}$ | | | |
| | $y\cos x = -\frac{1}{2}\cos 2x + \frac{5}{2}$ | A1 | 9 | ACF eg $y\cos x - 2 + \sin^2 x$ Apply ISW after ACF |
| | **Total** | | 9 | |

MFP3 (cont)

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| **3(a)** | Centre of circle is $M(3, 4)$ | B1 | | PI |
| | $A(6, 8)$ | B1 | 2 | |
| **(b)(i)** | $k = OA = 10$ | B1 | | |
| | $\tan\alpha = \frac{y_A}{x_A} = \frac{4}{3}$ | B1 | 2 | SC "$r = 10$ and $\tan\theta = \frac{8}{6}$" = B1 only |
| **(b)(ii)** | $x^2 + y^2 - 6x - 8y + 25 = 25$ | B1 | | If polar form before expansion award the B1 for correct expansions of both $(r\cos\theta - m)^2$ and $(r\sin\theta - n)^2$ where $(m,n) = (3,4)$ or $(m,n) = (4,3)$ |
| | $r^2 - 6r\cos\theta - 8r\sin\theta = 0$ | M1M1 | | 1st M1 for use of any one of $x^2+y^2-r^2$, $x - r\cos\theta$, $y - r\sin\theta$<br>2nd M1 for use of these to convert the form $x^2+y^2+ax+by=0$ correctly to the form $r^2 + ar\cos\theta + br\sin\theta = 0$ |
| | $\{r = 0$, origin$\}$ Circle: $r = 6\cos\theta + 8\sin\theta$ | A1 | 4 | NMS Mark as 4 or 0 |
| | **ALTn** | | | |
| | Circle has eqn $r = OA\cos(\alpha - \theta)$ | (M2) | | |
| | $r - OA\cos\alpha\cos\theta + OA\sin\alpha\sin\theta$ | (m1) | | OE |
| | Circle: $r = 6\cos\theta + 8\sin\theta$ | (A1) | | |
| | **Total** | | 8 | |

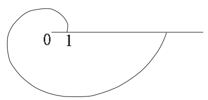
MFP3 (cont)

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| 4 | $\int\left(\frac{1}{x}-\frac{4}{4x+1}\right)\mathrm{d}x=\ln x-\ln(4x+1)\{+c\}$ | B1 | | OE |
| | $I=\lim_{a\to\infty}\int_1^a\left(\frac{1}{x}-\frac{4}{4x+1}\right)\mathrm{d}x$ | M1 | | $\infty$ replaced by $a$ (OE) and $\lim_{a\to\infty}$ |
| | $=\lim_{a\to\infty}\left[\ln x-\ln(4x+1)\right]_1^a$ | | | |
| | $=\lim_{a\to\infty}\left[\ln\left(\frac{a}{4a+1}\right)-\ln\frac{1}{5}\right]$ | m1 | | $\ln a-\ln(4a+1)=\ln\left(\frac{a}{4a+1}\right)$ **and** previous M1 scored |
| | $=\lim_{a\to\infty}\left[\ln\left(\frac{1}{4+\frac{1}{a}}\right)-\ln\frac{1}{5}\right]$ | m1 | | $\ln\left(\frac{a}{4a+1}\right)=\ln\left(\frac{1}{4+\frac{1}{a}}\right)$ **and** previous M1m1 scored |
| | $=\ln\frac{1}{4}-\ln\frac{1}{5}=\ln\frac{5}{4}$ | A1 | 5 | CSO |
| | **Total** | | **5** | |
| **5(a)** | $-k\sin x+2k\cos x+5k\sin x=8\sin x+4\cos x$ | M1<br>A1 | | Differentiation and subst. into DE |
| | $k=2$ | A1 | 3 | |
| **(b)** | Auxl eqn $m^2+2m+5=0$ | | | |
| | $m=\frac{-2\pm\sqrt{4-20}}{2}$ | M1 | | Formula or completing sq. PI |
| | $m=-1\pm 2\mathrm{i}$ | A1 | | |
| | CF: $\{y_C\}=\mathrm{e}^{-x}(A\sin 2x+B\cos 2x)$ | A1F | | ft provided $m$ is not real |
| | GS $\{y\}=\mathrm{e}^{-x}(A\sin 2x+B\cos 2x)+k\sin x$ | B1F | | ft on CF + PI; must have 2 arb consts |
| | When $x=0, y=1\Rightarrow B=1$ | B1F | | |
| | $\frac{\mathrm{d}y}{\mathrm{d}x}=-\mathrm{e}^{-x}(A\sin 2x+B\cos 2x)$ $+\mathrm{e}^{-x}(2A\cos 2x-2B\sin 2x)+k\cos x$ | M1 | | Product rule |
| | When $x=0$, $\frac{\mathrm{d}y}{\mathrm{d}x}=4\Rightarrow 4=-B+2A+k$ | A1 | | PI |
| | $\Rightarrow A=\frac{3}{2}$ | | | |
| | $y=\mathrm{e}^{-x}\left(\frac{3}{2}\sin 2x+\cos 2x\right)+2\sin x$ | A1 | 8 | CSO |
| | **Total** | | **11** | |

MFP3 (cont)

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| **6(a)(i)** | $f(x)=(9+\tan x)^{\frac{1}{2}}$ | | | |
| | so $f'(x)=\frac{1}{2}(9+\tan x)^{-\frac{1}{2}}\sec^2 x$ | M1<br>A1 | | Chain rule |
| | $f''(x)=-\frac{1}{4}(9+\tan x)^{-\frac{3}{2}}\sec^4 x$ | M1 | | Product rule, OE |
| | $+\frac{1}{2}(9+\tan x)^{-\frac{1}{2}}(2\sec^2 x\tan x)$ | A1 | 4 | ACF |
| **(a)(ii)** | $f(0)=3$ | B1 | | |
| | $f'(0)=\frac{1}{2}(9)^{-\frac{1}{2}}=\frac{1}{6}$; $f''(0)=-\frac{1}{4}(9)^{-\frac{3}{2}}=-\frac{1}{108}$ | M1 | | Both attempted and at least one correct ft on c's $f'(x)$ and $f''(x)$ |
| | $f(x)\approx f(0)+xf'(0)+\frac{1}{2}x^2f''(0)$ | | | |
| | $(9+\tan x)^{\frac{1}{2}}\approx 3+\frac{x}{6}-\frac{x^2}{216}$ | A1 | 3 | CSO AG |
| **(b)** | $\frac{f(x)-3}{\sin 3x}\approx\frac{\frac{x}{6}-\frac{x^2}{216}\ldots}{3x-\frac{(3x)^3}{3!}\ldots}$ | M1 | | Using series expns. |
| | $\approx\frac{\frac{1}{6}-\frac{x}{216}\ldots}{3-\ldots}$ | m1 | | Dividing numerator and denominator by $x$ to get constant term in each |
| | $\lim_{x\to 0}\left[\frac{f(x)-3}{\sin 3x}\right]=\frac{1}{18}$ | A1 | 3 | |
| | **Total** | | **10** | |

MFP3 (cont)

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| 7(a) | Area $= \frac{1}{2}\int \left(1+6e^{-\frac{\theta}{\pi}}\right)^2 \, d\theta$ | M1 | | Use of $\frac{1}{2}\int r^2 \, d\theta$ |
| | $= \frac{1}{2}\int_0^{2\pi}\left(1+12e^{-\frac{\theta}{\pi}}+36e^{-\frac{2\theta}{\pi}}\right) d\theta$ | B1 | | Correct expansion of $\left(1+6e^{-\frac{\theta}{\pi}}\right)^2$ |
| | | B1 | | Correct limits |
| | $= \frac{1}{2}\left[\theta-12\pi e^{-\frac{\theta}{\pi}}-18\pi e^{-\frac{2\theta}{\pi}}\right]_0^{2\pi}$ | m1 | | Correct integration of at least two of the three terms 1, $p e^{-\frac{\theta}{\pi}}$, $q e^{-\frac{2\theta}{\pi}}$ |
| | $= \pi(16 - 6e^{-2} - 9e^{-4})$ | A1 | 5 | ACF |
| (b) |  | B1 | | Going the correct way round the pole |
| | | B1 | | Increasing in distance from the pole |
| | End-points (1, 0) and $(e^2, 2\pi)$ | B2,1,0 | 4 | Correct end-points<br>B1 for each pair or for 1 and $e^2$ shown on graph in correct positions |
| (c) | $e^{\frac{\theta}{\pi}} = 1+6e^{-\frac{\theta}{\pi}}$ | M1 | | Elimination of $r$ or $\theta$ [$r = 1 + \frac{6}{r}$] |
| | $\left(e^{\frac{\theta}{\pi}}\right)^2 - e^{\frac{\theta}{\pi}} - 6 = 0$ | m1 | | Forming quadratic in $e^{\frac{\theta}{\pi}}$ or in $e^{-\frac{\theta}{\pi}}$ or in $r$. [$r^2 - r - 6 = 0$] |
| | $\left(e^{\frac{\theta}{\pi}}-3\right)\left(e^{\frac{\theta}{\pi}}+2\right)=0$ | m1 | | OE |
| | $e^{\frac{\theta}{\pi}} > 0$ so $e^{\frac{\theta}{\pi}} = 3$ | E1 | | Rejection of negative 'solution' PI [$r = 3$] |
| | Polar coordinates of $P$ are $(3, \pi \ln 3)$ | A1 | 5 | |
| | **Total** | | **14** | |

MFP3 (cont)

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| 8(a)(i) | $\frac{dx}{dt} = 2t$ | B1 | | PI or for $\frac{dt}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$ |
| | $\frac{dx}{dt}\frac{dy}{dx} = \frac{dy}{dt}$ | M1 | | OE Chain rule $\frac{dy}{dx} = \ldots$ or $\frac{dy}{dt} = \ldots$ |
| | $2t\frac{dy}{dx} = \frac{dy}{dt}$ so $2\sqrt{x}\frac{dy}{dx} = \frac{dy}{dt}$ | A1 | 3 | AG |
| (a)(ii) | $\frac{d}{dx}\left(2\sqrt{x}\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dt}\right) - \frac{dt}{dx}\frac{d}{dt}\left(\frac{dy}{dt}\right)$ | M1 | | $\frac{d}{dx}(f(t)) = \frac{dt}{dx}\frac{d}{dt}(f(t))$ OE<br>eg $\frac{d}{dt}(g(x)) = \frac{dx}{dt}\frac{d}{dx}(g(x))$ |
| | $2\sqrt{x}\frac{d^2y}{dx^2} + x^{-\frac{1}{2}}\frac{dy}{dx} = \frac{1}{2t}\frac{d^2y}{dt^2}$ | M1 | | Product rule OE |
| | $4t\sqrt{x}\frac{d^2y}{dx^2} + 2tx^{-\frac{1}{2}}\frac{dy}{dx} = \frac{d^2y}{dt^2}$ | | | |
| | $\Rightarrow 4x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = \frac{d^2y}{dt^2}$ | A1 | 3 | AG Completion |
| (b) | $4x\frac{d^2y}{dx^2} + 2(1+2\sqrt{x})\frac{dy}{dx} - 3y = 0$ | | | |
| | $(4x\frac{d^2y}{dx^2} + 2\frac{dy}{dx}) + 2(2\sqrt{x}\frac{dy}{dx}) - 3y = 0$ | | | |
| | $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = 0$ | M1<br>A1 | 2 | Use of either (a)(i) or (a)(ii)<br>AG Completion |
| (c) | $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = 0$ (*) | | | |
| | Auxl. Eqn. $m^2 + 2m - 3 = 0$ | | | |
| | $(m+3)(m-1) = 0$ | M1 | | PI |
| | $m = -3$ and 1 | A1 | | PI |
| | GS of (*) $\{y\} = Ae^{-3t} + Be^t$ | M1 | | $Ae^{-3x} + Be^x$ scores M0 here |
| | $\Rightarrow y = Ae^{-3\sqrt{x}} + Be^{\sqrt{x}}$ | A1 | 4 | |
| | **Total** | | **12** | |
| | **TOTAL** | | **75** | |

# Notes about June 2009:

# January 2010

Answer **all** questions.

**1** The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = f(x, y)$$

where $f(x, y) = x\ln(2x + y)$

and $y(3) = 2$

(a) Use the Euler formula

$$y_{r+1} = y_r + hf(x_r, y_r)$$

with $h = 0.1$, to obtain an approximation to $y(3.1)$, giving your answer to four decimal places. *(3 marks)*

(b) Use the improved Euler formula

$$y_{r+1} = y_r + \tfrac{1}{2}(k_1 + k_2)$$

where $k_1 = hf(x_r, y_r)$ and $k_2 = hf(x_r + h, y_r + k_1)$ and $h = 0.1$, to obtain an approximation to $y(3.1)$, giving your answer to four decimal places. *(5 marks)*

**2** (a) Given that $y = \ln(4 + 3x)$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$. *(3 marks)*

(b) Hence, by using Maclaurin's theorem, find the first three terms in the expansion, in ascending powers of $x$, of $\ln(4 + 3x)$. *(2 marks)*

(c) Write down the first three terms in the expansion, in ascending powers of $x$, of $\ln(4 - 3x)$. *(1 mark)*

(d) Show that, for small values of $x$,

$$\ln\left(\frac{4 + 3x}{4 - 3x}\right) \approx \frac{3}{2}x$$ *(2 marks)*

**3** (a) A differential equation is given by

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 3x$$

Show that the substitution

$$u = \frac{dy}{dx}$$

transforms this differential equation into

$$\frac{du}{dx} + \frac{2}{x}u = 3$$ *(2 marks)*

(b) Find the general solution of

$$\frac{du}{dx} + \frac{2}{x}u = 3$$

giving your answer in the form $u = f(x)$. *(5 marks)*

(c) Hence find the general solution of the differential equation

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 3x$$

giving your answer in the form $y = g(x)$. *(2 marks)*

**4** (a) Write down the expansion of $\sin 3x$ in ascending powers of $x$ up to and including the term in $x^3$. *(1 mark)*

(b) Find

$$\lim_{x \to 0}\left[\frac{3x\cos 2x - \sin 3x}{5x^3}\right]$$ *(4 marks)*

**5** It is given that $y$ satisfies the differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 2e^{-2x}$$

(a) Find the value of the constant $p$ for which $y = pxe^{-2x}$ is a particular integral of the given differential equation. *(4 marks)*

(b) Solve the differential equation, expressing $y$ in terms of $x$, given that $y = 2$ and $\frac{dy}{dx} = 0$ when $x = 0$. *(8 marks)*

**6** (a) Explain why $\int_1^{\infty} \frac{\ln x^2}{x^3}\,dx$ is an improper integral. *(1 mark)*

(b) (i) Show that the substitution $y = \frac{1}{x}$ transforms $\int \frac{\ln x^2}{x^3}\,dx$ into $\int 2y \ln y\,dy$. *(2 marks)*

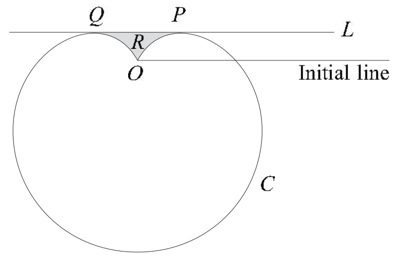
(ii) Evaluate $\int_0^1 2y \ln y\,dy$, showing the limiting process used. *(5 marks)*

(iii) Hence write down the value of $\int_1^{\infty} \frac{\ln x^2}{x^3}\,dx$. *(1 mark)*

**7** Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 4y = 8x^2 + 9\sin x$$ *(8 marks)*

**8** The diagram shows a sketch of a curve $C$ and a line $L$, which is parallel to the initial line and touches the curve at the points $P$ and $Q$.




The polar equation of the curve $C$ is

$$r = 4(1 - \sin\theta), \qquad 0 \leqslant \theta < 2\pi$$

and the polar equation of the line $L$ is

$$r\sin\theta = 1$$

(a) Show that the polar coordinates of $P$ are $\left(2, \frac{\pi}{6}\right)$ and find the polar coordinates of $Q$. *(5 marks)*

(b) Find the area of the shaded region $R$ bounded by the line $L$ and the curve $C$. Give your answer in the form $m\sqrt{3} + n\pi$, where $m$ and $n$ are integers. *(11 marks)*

**END OF QUESTIONS**

# AQA – Further pure 3 – Jan 2010 – Answers

| Question 1: | Exam report |
|---|---|
| $\frac{dy}{dx}=x\ln(2x+y) \qquad x_0=3 \text{ and } y_0=2$<br>$a)\ x_1=3.1 \text{ and } y_1=y(3.1)=y_0+hf(x_r,y_r)$<br>$=2+0.1(3\ln(2\times3+2))$<br>$y(3.1)=2.6238 \text{ to } 4d.p$<br>$b)\ k_1=hf(x_r,y_r)=0.1(3\ln(2\times3+2))=0.6238$<br>$y_0+k_1=2.6238$<br>$k_2=0.1(3.1\ln(2\times3.1+2.6238))=0.6750$<br>$y(3.1)=2+\frac{1}{2}(0.6238+0.6750)$<br>$y(3.1)=2.6494 \text{ to } 4d.p.$ | Numerical solutions of first order differential equations continue to be a good source of marks for all candidates and this was the best answered question on the paper. Very few candidates mixed up the *x* and *y* values in applying the given formulae. Almost all candidates gave their final answers to the required degree of accuracy. |

| Question 2: | Exam report |
|---|---|
| $a)\ y=\ln(4+3x)$<br>$\frac{dy}{dx}=\frac{3}{4+3x} \text{ and } \frac{d^2y}{dx^2}=-\frac{9}{(4+3x)^2}$<br>$b)\ y(0)=\ln4$<br>$y'(0)=\frac{3}{4} \text{ and } y''(0)=-\frac{9}{16}$<br>$Conclusion:\ \ln(4+3x)=\ln(4)+\frac{3}{4}x-\frac{9}{32}x^2+...$<br>$c)\ \ln(4-3x)=\ln(4)-\frac{3}{4}x-\frac{9}{32}x^2+...\ (substituting\ x\ with\ (-x))$<br>$d)\ \ln\left(\frac{4+3x}{4-3x}\right)=\ln(4+3x)-\ln(4-3x)$<br>$=\left(\ln(4)+\frac{3}{4}x-\frac{9}{32}x^2+...\right)-\left(\ln(4)-\frac{3}{4}x-\frac{9}{32}x^2+...\right)$<br>$\ln\left(\frac{4+3x}{4-3x}\right)=\frac{3}{4}x+\frac{3}{4}x+...\approx\frac{3}{2}x$ | Most candidates were able to find $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ correctly, although some less able candidates failed to apply the chain rule. Although a few candidates in part (b) attempted to use the printed expansion of ln(1 + *x*) from the formulae booklet instead of applying the prescribed method, the majority of candidates answered the question as instructed and showed good knowledge of Maclaurin's theorem. Many candidates failed to appreciate that in part (c) they had to replace *x* with −*x*, and instead multiplied both their *x* and $x^2$ terms by −1. A significant minority of candidates in part (d) did not realise that they had to write the expression as the difference of their two expansions. |

| Question 3: | Exam report |
|---|---|
| $a)\ x\frac{d^2y}{dx^2}+2\frac{dy}{dx}=3x \quad with \quad u=\frac{dy}{dx}$<br>$becomes\ \ x\frac{du}{dx}+2u=3x \qquad \frac{du}{dx}+\frac{2}{x}u=3$<br>$b)$ An integrating factor is $I=e^{\int\frac{2}{x}dx}=e^{2\ln x}=x^2$<br>The equation becomes $x^2\frac{du}{dx}+2xu=3x^2$<br>$\frac{d}{dx}(x^2u)=3x^2$ integrating $x^2u=x^3+A$<br>$u=x+\frac{A}{x^2}$<br>$c)\ u=\frac{dy}{dx}=x+\frac{A}{x^2} \quad so\ by\ integrating\ both\ sides:$<br>$y=\frac{1}{2}x^2-\frac{A}{x}+B \qquad A,B\in\mathbb{R}$ | This question was answered well with the majority of candidates able to use the given substitution correctly. Most candidates were able to show that they knew how to find and use an integrating factor to solve a first order differential equation. However, a significant number of candidates failed to write their general solutions with the required number of arbitrary constants. |

| Question 4: | Exam report |
|---|---|
| $a)\sin(x) = x - \frac{x^3}{6} + \ldots \ so$<br>$\sin(3x) = (3x) - \frac{(3x)^3}{6} + \ldots$<br>$\sin(3x) = 3x - \frac{9}{2}x^3 + \ldots$<br>$b)$ In the same manner<br>$Cos(2x) = 1 - 2x^2 + \ldots$<br>$Hence, \frac{3xCos(2x) - Sin(3x)}{5x^3}$<br>$= \frac{1}{5x^3}\left(3x(1-2x^2) - (3x - \frac{9}{2}x^3) + \ldots\right)$<br>$= \frac{1}{5x^3}\left(-6x^3 + \frac{9}{2}x^3 + \ldots\right) = -\frac{3}{10} + \ldots$<br>$\lim_{x\to 0}\left(\frac{3xCos(2x) - Sin(3x)}{5x^3}\right) = -\frac{3}{10}$ | This question on series expansions and the limiting process was generally answered very well, but a significant minority of candidates in part (b) did not explicitly reach the stage of a constant term in both the numerator and denominator before taking the limit as $x \to 0$. |

| Question 5: | Exam report |
|---|---|
| $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 2e^{-2x}$<br>$a)\ y = pxe^{-2x}$<br>$\frac{dy}{dx} = pe^{-2x} - 2pxe^{-2x} = (p - 2px)e^{-2x}$<br>$\frac{d^2y}{dx^2} = -2pe^{-2x} - 2(p - 2px)e^{-2x} = (-4p + 4px)e^{-2x}$<br>Substituting in the equation:<br>$(-4p + 4px)e^{-2x} + 3(p - 2px)e^{-2x} + 2pxe^{-2x} = 2e^{-2x}$<br>$-pe^{-2x} = 2e^{-2x} \quad so \quad p = -2$<br>$b)$ The auxiliary equation associated to the equation is :<br>$\lambda^2 + 3\lambda + 2 = 0$<br>$(\lambda + 1)(\lambda + 2) = 0$<br>$\lambda = -1 \ or \ \lambda = -2$<br>The complementary function is $y = Ae^{-x} + Be^{-2x}$<br>The general solution is $y = -2xe^{-2x} + Ae^{-x} + Be^{-2x}$<br>$when \ x = 0, y = 2 \quad this \ gives \quad 2 = 0 + A + B$<br>$when \ x = 0, \frac{dy}{dx} = 0 \quad this \ gives \quad 0 = -2 - A - 2B$<br>Solving these two equations simultaneously:<br>$B = -4 \ and \ A = 6$<br>The solution wanted is $y = -2xe^{-2x} + 6e^{-x} - 4e^{-2x}$ | In part (a), it was pleasing to see a higher proportion of candidates than in previous papers using the given form of the particular integral rather than introducing the extra term $qe^{-2x}$. The majority of candidates used the product rule correctly to find the value of the constant *p*. Most candidates showed that they knew the methods required to solve the second order differential equation, although a minority of candidates made the error of applying the boundary conditions to the complementary function before adding on the particular integral. |

| Question 6: | Exam report |
|---|---|
| a) $\int_1^{\infty} \frac{\ln x^2}{x^3} dx$ is an improper integral<br>because the interval of integration is infinite.<br>b)i) $y=\frac{1}{x}$ , $x=\frac{1}{y}$ and $\frac{dx}{dy}=-\frac{1}{y^2}$<br>$\int \frac{\ln x^2}{x^3} dx = \int \frac{1}{x^3}\times 2\times \ln(x)dx$<br>becomes $\int y^3 \times 2\times \ln\left(\frac{1}{y}\right)\times -\frac{1}{y^2}dy$<br>$= \int 2y\ln(y)dy$<br>ii) $\int_a^1 2y\ln(y)dy = \left[y^2\ln(y)\right]_a^1 - \int_a^1 y^2\times\frac{1}{y}dy$<br>$= a^2\ln(a) - 0 - \left[\frac{1}{2}y^2\right]_a^1$<br>$= a^2\ln(a) - \frac{1}{2} + \frac{1}{2}a^2$<br>$\lim_{a\to 0} a^2\ln(a) = 0$ and $\lim_{a\to 0}\frac{1}{2}a^2 = 0$<br>Conclusion: $\int_0^1 2y\ln(y)dy$ exists and $\int_0^1 2y\ln(y)dy = -\frac{1}{2}$<br>iii) When $x$ tends to $\infty$, $y=\frac{1}{x}$ tends to 0,<br>When $x=1, y=1$<br>Hence, $\int_1^{\infty}\frac{\ln x^2}{x^3}dx = \int_1^0 2y\ln(y)dy = -\int_0^1 2y\ln(y)dy = \frac{1}{2}$ | In part (a), a significant number of candidates failed to give the correct reason for why the integral was improper, with most giving the reason that the integrand was undefined for the upper limit rather than that the interval of integration was infinite. In part (b)(i), a large number of candidates failed to use the substitution correctly. Part (b)(ii) was generally answered well with most candidates integrating by parts correctly to score 3 of the 5 marks available, but some candidates did not gain the final two marks as examiners did not see the lower limit replaced by, for example, *a* and the consideration of the limiting process as $a\to 0$. Part (b)(iii) was answered incorrectly by the majority of candidates, who usually either stated the same value or the reciprocal value of their answer to part (b)(ii). |

| Question 7: | Exam report |
|---|---|
| $\frac{d^2y}{dx^2}+4y=8x^2+9Sin(x)$<br>Complementary function:<br>The auxiliary equation is $\lambda^2+4=0$<br>$\lambda=2i$ or $\lambda=-2i$<br>$y_{cf}=ACos(2x)+BSin(2x)$<br>Particular integral:<br>$y=ax^2+bx+c+d\sin(x)+eCos(x)$<br>$\frac{dy}{dx}=2ax+b+dCos(x)-eSin(x)$<br>$\frac{d^2y}{dx^2}=2a-dSin(x)-eCos(x)$<br>The equation $\frac{d^2y}{dx^2}+4y=8x^2+9Sin(x)$<br>becomes:<br>$2a-dSin(x)-eCos(x)+4\left(ax^2+bx+c+dSin(x)+eCos(x)\right)=8x^2+9Sin(x)$<br>$(4ax^2+4bx+4c+2a)+(-d+4d)Sin(x)+(-e+4e)Cos(x)=8x^2+9Sin(x)$<br>This gives: $a=2, b=0, c=-1, d=3$ and $e=0$<br>The general solution is $y=2x^2-1+3Sin(x)+ACos(2x)+BSin(2x)$ | This unstructured question was answered well by many of the candidates. However, a significant minority of candidates did not even write down the correct form of the auxiliary equation, or they solved it incorrectly to give real rather than imaginary values. When finding the particular integral, a large number of candidates considered more terms than they needed to, and they did not always go on to show that the relevant coefficients of the extra terms were zero. The majority of candidates knew that they had to find a complementary function and a particular integral and almost all scored the final mark for combining the two to give the general solution. |

**Question 8:**

$a)$ Solving the equations simultaneously

$$r = 4(1 - Sin\theta) \ and \ rSin\theta = 1$$

$$gives: \ 4(1 - Sin\theta)Sin\theta = 1$$

$$-4Sin^2\theta + 4Sin\theta - 1 = 0$$

$$4Sin^2\theta - 4Sin\theta + 1 = 0$$

$$(2Sin\theta - 1)^2 = 0$$

$$Sin\theta = \frac{1}{2} \ \ so \ \ \theta = \frac{\pi}{6} \ or \ \theta = \frac{5\pi}{6}$$

For $\theta = \frac{\pi}{6}, r = 4(1 - Sin\frac{\pi}{6}) = 2 \qquad P(2, \frac{\pi}{6})$

For $\theta = \frac{5\pi}{6}, r = 4(1 - Sin\frac{5\pi}{6}) = 2 \qquad Q(2, \frac{5\pi}{6})$

$b)$ The area shaded = Area of the triangle POQ $- 2 \times Area \ bounded \ by \ line \ OP$

$$Area \ OPQ = \frac{1}{2} \times 2 \times 2 \times Sin\left(\frac{5\pi}{6} - \frac{\pi}{6}\right) = 2Sin\left(\frac{2\pi}{3}\right) = \sqrt{3}$$

$$Area \ bounded \ by \ line \ OP \ is \ \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{2} \times 16(1 - Sin\theta)^2 \, d\theta = 8\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 1 + Sin^2\theta - 2Sin\theta d\theta$$

$$= 8\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 1 + \frac{1}{2} - \frac{1}{2}Cos2\theta - 2Sin\theta d\theta = 8\left[\frac{3}{2}\theta - \frac{1}{4}Sin2\theta + 2Cos\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= 8\left(\frac{3}{4}\pi - \left(\frac{3\pi}{12} - \frac{1}{4}Sin\frac{\pi}{3} + 2Cos\frac{\pi}{6}\right)\right) = 4\pi - 7\sqrt{3}$$

The area shaded is $\sqrt{3} - 2(4\pi - 7\sqrt{3}) = 15\sqrt{3} - 8\pi$

**Exam report**

Part (a) was generally answered well with most candidates forming an equation in either sinθ or *r* and solving it correctly to give the coordinates of *P* and *Q*. Candidates who tried to verify the coordinates of *P* generally failed to even verify them in both polar equations. Part (b) was the most demanding question on the paper. The majority of candidates scored at least 4 of the 11 marks by applying the formula $A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta$ to find an area of a region partly bounded by the curve *C* and in doing so they correctly expanded $(1-\sin\theta)^2$, wrote $\sin^2\theta$ in terms of cos2θ and integrated correctly. The errors occurred because many candidates did not use the correct limits. A significant number of candidates also failed to find, or even consider the need for, the area of triangle *OPQ* and so correct answers for the area of the shaded region *R* were generally only presented by the most able candidates.

**Grade boundaries**

| Grade | | | A | B | C | D | E |
|---|---|---|---|---|---|---|---|
| Mark | Max 75 | | 63 | 55 | 47 | 39 | 32 |

**Key to mark scheme and abbreviations used in marking**

| | | | |
|---|---|---|---|
| M | mark is for method | | |
| m or dM | mark is dependent on one or more M marks and is for method | | |
| A | mark is dependent on M or m marks and is for accuracy | | |
| B | mark is independent of M or m marks and is for method and accuracy | | |
| E | mark is for explanation | | |
| | | | |
| √or ft or F | follow through from previous incorrect result | MC | mis-copy |
| CAO | correct answer only | MR | mis-read |
| CSO | correct solution only | RA | required accuracy |
| AWFW | anything which falls within | FW | further work |
| AWRT | anything which rounds to | ISW | ignore subsequent work |
| ACF | any correct form | FIW | from incorrect work |
| AG | answer given | BOD | given benefit of doubt |
| SC | special case | WR | work replaced by candidate |
| OE | or equivalent | FB | formulae book |
| A2,1 | 2 or 1 (or 0) accuracy marks | NOS | not on scheme |
| $-x$ EE | deduct $x$ marks for each error | G | graph |
| NMS | no method shown | c | candidate |
| PI | possibly implied | sf | significant figure(s) |
| SCA | substantially correct approach | dp | decimal place(s) |

**No Method Shown**

Where the question specifically requires a particular method to be used, we must usually see evidence of use of this method for any marks to be awarded. However, there are situations in some units where part marks would be appropriate, particularly when similar techniques are involved. Your Principal Examiner will alert you to these and details will be provided on the mark scheme.

Where the answer can be reasonably obtained without showing working and it is very unlikely that the correct answer can be obtained by using an incorrect method, we must award **full marks**. However, the obvious penalty to candidates showing no working is that incorrect answers, however close, earn **no marks.**

Where a question asks the candidate to state or write down a result, no method need be shown for full marks.

Where the permitted calculator has functions which reasonably allow the solution of the question directly, the correct answer without working earns **full marks**, unless it is given to less than the degree of accuracy accepted in the mark scheme, when it gains **no marks.**

**Otherwise we require evidence of a correct method for any marks to be awarded.**

MFP3

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| 1(a) | $y_1 = 2 + 0.1\times[3\ln(2\times3+2)] = 2 + 0.3\ln 8$ $= 2.6238(3...)$ | M1A1 | | |
| | $y(3.1) = 2.6238$ (to 4dp) | A1 | 3 | Condone greater accuracy |
| (b) | $k_1 = 0.1\times3\ln 8 = 0.6238(32\ldots)$ | B1F | | PI ft from (a), 4dp or better |
| | $k_2 = 0.1\times f(3.1,\ 2.6238(32...))$ | M1 | | |
| | $\ldots =\ 0.1\times3.1\times\ln 8.8238(32..)$ | A1F | | PI; ft on $0.1\times3.1\times\ln[6.2 + \text{answer(a)}]$ |
| | $[= 0.6750(1\ldots)$ | | | |
| | $y(3.1) = 2 + \frac{1}{2}[0.6238(3..) + 0.6750(1..)]$ | m1 | | |
| | $= 2.6494(2\ldots) = 2.6494$ to 4dp | A1 | 5 | CAO Must be 2.6494 |
| | **Total** | | **8** | |
| 2(a) | $\frac{dy}{dx} = \frac{1}{4+3x}\times3$ | M1 | | Chain rule |
| | $\frac{d^2y}{dx^2} = -3(4+3x)^{-2}\times3 = -9(4+3x)^{-2}$ | M1A1 | 3 | M1 for quotient (PI) or chain rule used |
| (b) | $\ln(4+3x) = \ln 4 + y'(0)\,x + y''(0)\frac{1}{2}x^2 + ..$ | M1 | | Clear attempt to use Maclaurin's theorem with numerical values for $y'(0)$ and $y''(0)$ |
| | First three terms: $\ln 4 + \frac{3}{4}x - \frac{9}{32}x^2$ | A1F | 2 | ft on c's answers to (a) provided $y'(0)$ and $y''(0)$ are $\neq 0$. Accept 1.38(6..) for ln4 |
| (c) | $\ln(4-3x) = \ln 4 - \frac{3}{4}x - \frac{9}{32}x^2$ | B1F | 1 | ft $x\to -x$ in c's answer to (b) |
| (d) | $\ln\left(\frac{4+3x}{4-3x}\right) = \ln(4+3x) - \ln(4-3x)$ | M1 | | |
| | $\approx \ln 4 + \frac{3}{4}x - \frac{9}{32}x^2 - \ln 4 + \frac{3}{4}x + \frac{9}{32}x^2$ | | | |
| | $\approx \frac{3}{2}x$ | A1 | 2 | CSO AG |
| | **Total** | | **8** | |

MFP3 (cont)

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| 3(a) | $u=\frac{dy}{dx}\Rightarrow\frac{du}{dx}=\frac{d^2y}{dx^2}$ | M1 | | |
| | $x\frac{du}{dx}+2u=3x\Rightarrow\frac{du}{dx}+\frac{2}{x}u=3$ | A1 | 2 | CSO AG Substitution into LHS of DE and completion |
| 3(b) | IF is $\exp\left(\int\frac{2}{x}dx\right)$ | M1 | | $\exp\left(\int\frac{k}{x}dx\right)$, for $k=\pm2,\pm1$ and integration attempted |
| | $=e^{2\ln x}$; $=x^2$ | A1;A1 | | |
| | $\frac{d}{dx}(ux^2)=3x^2$ | M1 | | LHS as differential of $u\times$IF |
| | $ux^2=x^3+A\Rightarrow u=x+Ax^{-2}$ | A1 | 5 | Must have an arbitrary constant |
| (c) | $\frac{dy}{dx}=x+Ax^{-2}$ | M1 | | and with integration attempted |
| | $\frac{dy}{dx}=x+Ax^{-2}\Rightarrow y=\frac{1}{2}x^2-\frac{A}{x}+B$ | A1F | 2 | ft only if IF is M1A0A0 |
| | **Total** | | **9** | |
| 4(a) | $\sin 3x=3x-\frac{1}{3!}(3x)^3+\ldots=3x-4.5x^3+\ldots$ | B1 | 1 | |
| (b) | $\cos 2x=1-\frac{1}{2!}(2x)^2+\ldots$ | B1 | | |
| | $\lim_{x\to0}\left[\frac{3x\cos 2x-\sin 3x}{5x^3}\right]=$ $\lim_{x\to0}\frac{3x-6x^3-3x+4.5x^3+\ldots}{5x^3}$ | M1 | | Using expansions |
| | $=\lim_{x\to0}\frac{-1.5+(o(x^2))\ldots}{5}$ | m1 | | Division by $x^3$ stage to reach relevant form of quotient before taking limit. |
| | $=-\frac{3}{10}$ | A1 | 4 | CSO OE |
| | **Total** | | **5** | |

MFP3 (cont)

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| 5(a) | $y_{PI}=pxe^{-2x}\Rightarrow\frac{dy}{dx}=pe^{-2x}-2pxe^{-2x}$ | M1 | | Product Rule used |
| | $\Rightarrow\frac{d^2y}{dx^2}=-2pe^{-2x}-2pe^{-2x}+4pxe^{-2x}$ | A1 | | |
| | $-4pe^{-2x}+4pxe^{-2x}+3pe^{-2x}-6pxe^{-2x}+2pxe^{-2x}=2e^{-2x}$. | M1 | | Sub. into DE |
| | $-pe^{-2x}=2e^{-2x}\Rightarrow p=-2$ | A1F | 4 | ft one slip in differentiation |
| 5(b) | Aux. eqn. $m^2+3m+2=0$ $\Rightarrow m=-1,-2$ | B1 | | |
| | CF is $Ae^{-x}+Be^{-2x}$ | M1 | | ft on real values of $m$ only |
| | GS $y=Ae^{-x}+Be^{-2x}-2xe^{-2x}$. | B1F | | Their CF + their PI must have 2 arb consts |
| | When $x=0, y=2\Rightarrow A+B=2$ | B1F | | Must be using GS; ft on wrong non-zero values for $p$ and $m$ |
| | $\frac{dy}{dx}=-Ae^{-x}-2Be^{-2x}-2e^{-2x}+4xe^{-2x}$ | B1F | | Must be using GS; ft on wrong non-zero values for $p$ and $m$ |
| | When $x=0$, $\frac{dy}{dx}=0\Rightarrow-A-2B-2=0$ | B1F | | Must be using GS; ft on wrong non-zero values for $p$ and $m$ and slips in finding $y'(x)$ |
| | Solving simultaneously, 2 eqns each in two arbitrary constants | m1 | | |
| | $A=6, B=-4$; $y=6e^{-x}-4e^{-2x}-2xe^{-2x}$. | A1 | 8 | CSO |
| | **Total** | | **12** | |

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| 6(a) | The interval of integration is infinite | E1 | 1 | OE |
| (b)(i) | $x=\frac{1}{y} \Rightarrow$ '$dx = -y^{-2}\,dy$' | | | |
| | $\int \frac{\ln x^2}{x^3}\,dx \Rightarrow \int (y^3 \ln y^{-2})(-y^{-2})\,dy$ | M1 | | |
| | $-\int -y\ln y^{-2}\,dy = \int 2y\ln y\,dy$ | A1 | 2 | CSO AG |
| (ii) | $\int 2y\ln y\,dy = y^2\ln y - \int y^2\left(\frac{1}{y}\right)dy$ | M1 | | $\ldots = ky^2\ln y \pm \int f(y)\,dy$ with $f(y)$ not involving the 'original' $\ln y$ |
| | | A1 | | |
| | $\ldots\ldots = y^2\ln y - \frac{1}{2}y^2 + c$ | A1 | | Condone absence of '$+c$' |
| | $\int_0^1 2y\ln y\,dy = \lim_{a\to 0}\int_a^1 2y\ln y\,dy$ | | | |
| | $=\left(0-\frac{1}{2}\right) - \lim_{a\to 0}\left[a^2\ln a - \frac{a^2}{2}\right]$ | M1 | | |
| | $= -\frac{1}{2}$ since $\lim_{a\to 0} a^2\ln a = 0$ | A1 | 5 | CSO Must see clear indication that cand has correctly considered $\lim_{a\to 0} a^k\ln a = 0$ |
| (iii) | So $\int_1^\infty \frac{\ln x^2}{x^3}\,dx = \frac{1}{2}$ | B1F | 1 | ft on minus c's value as answer to (b)(ii) |
| | **Total** | | **9** | |
| 7 | Aux. eqn. $m^2+4=0 \Rightarrow m=\pm 2i$ | B1 | | |
| | CF is $A\cos 2x + B\sin 2x$ | M1<br>A1F | | OE. If $m$ is real give M0<br>ft on incorrect complex value for $m$ |
| | PI: Try $ax^2+b$ $+ c\sin x$ | M1<br>M1 | | Award even if extra terms, provided the relevant coefficients are shown to be zero. |
| | $2a - c\sin x + 4ax^2 + 4b + 4c\sin x = 8x^2 + 9\sin x$ | | | |
| | $a=2,\ b=-1,$ | A1 | | Dep on relevant M mark |
| | $c=3$ | A1 | | Dep on relevant M mark |
| | $(y=)\,A\cos 2x + B\sin 2x + 2x^2 - 1 + 3\sin x$ | B1F | 8 | Their CF + their PI. Must be exactly two arbitrary constants |
| | **Total** | | **8** | |

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| 8(a) | $4\sin\theta(1-\sin\theta)=1$ | M1 | | Elimination of $r$ or $\theta$ $\{r=4[1-(1/r)]\}$ |
| | $4\sin^2\theta - 4\sin\theta + 1 = 0$ | A1 | | $\{r^2-4r+4=0\}$ |
| | $(2\sin\theta-1)^2=0 \Rightarrow \sin\theta = 0.5$ | m1 | | Valid method to solve quadratic eqn. PI $\{(r-2)^2=0 \Rightarrow r=2\}$ |
| | $\theta=\frac{\pi}{6},\ \theta=\frac{5\pi}{6},\ r=2$ | A2,1 | | A1 for any two of the three. |
| | $[P\left(2,\ \frac{\pi}{6}\right)\quad Q\left(2,\ \frac{5\pi}{6}\right)]$ | | 5 | SC: Verification of $P\left(2,\ \frac{\pi}{6}\right)$ scores max of B1 & a further B1 if $Q\left(2,\ \frac{5\pi}{6}\right)$ stated |
| 8(b) | Area triangle OPQ $=\frac{1}{2}\times 2\times r_Q\times\sin POQ$ | M1 | | Any valid method to correct (ft eg on $r_Q$) expression with just one remaining unknown |
| | Angle $POQ=\frac{5\pi}{6}-\frac{\pi}{6}\quad\left(=\frac{2\pi}{3}\right)$ | m1 | | Valid method to find remaining unknown either relevant angle or relevant side |
| | Area triangle $OPQ - 2\sin\frac{2\pi}{3} - \sqrt{3}$ | A1 | | |
| | Unshaded area bounded by line $OP$ and arc $OP = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}[4(1-\sin\theta)]^2\,d\theta$ | M1 | | Use of $\frac{1}{2}\int r^2\,d\theta$ for relevant area(s) (condone missing/wrong limits) |
| | $=8\int(1-2\sin\theta+\sin^2\theta)\,d\theta$ | B1 | | Correct expn of $(1-\sin\theta)^2$ |
| | $=8\int\left(1-2\sin\theta+\frac{1-\cos 2\theta}{2}\right)d\theta$ | M1 | | Attempt to write $\sin^2\theta$ in terms of $\cos 2\theta$ |
| | $=8\left[\theta+2\cos\theta+\frac{\theta}{2}-\frac{\sin 2\theta}{4}\right]\ (+c)$ | A1F | | Correct integration ft wrong coeffs |
| | $8\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}(1-\sin\theta)^2\,d\theta -$ | | | |
| | $8\times\left[\frac{3\theta}{2}+2\cos\theta-\frac{\sin 2\theta}{4}\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$ | | | |
| | $=8\times\{\frac{3\pi}{4}-\left(\frac{3\pi}{12}+2\cos\frac{\pi}{6}-\frac{1}{4}\sin\frac{2\pi}{6}\right)\}$ | m1 | | $F\left(\frac{\pi}{2}\right)-F\left(\frac{\pi}{6}\right)$ OE for relevant area(s) |
| | $-8\times\left(\frac{\pi}{2}-\sqrt{3}+\frac{\sqrt{3}}{8}\right)\quad\{=4\pi-7\sqrt{3}\}$ | A1F | | ft one slip; accept terms in $\pi$, and $\sqrt{3}$ left unsimplified |
| | Shaded area = Area of triangle $OPQ$ − $2\times\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}[4(1-\sin\theta)]^2\,d\theta$ | M1 | | OE |
| | Shaded area = $\sqrt{3}-16\left(\frac{\pi}{2}-\sqrt{3}+\frac{\sqrt{3}}{8}\right)=15\sqrt{3}-8\pi$ | A1 | 11 | CSO Accept $m=15,\ n=-8$ |
| | **Total** | | **16** | |
| | **TOTAL** | | **75** | |

# Notes about Jan 2010:

AQA General Certificate of Education
Advanced Level Examination
June 2010

# Mathematics MFP3

**Unit Further Pure 3**

**Friday 11 June 2010 9.00 am to 10.30 am**

**For this paper you must have:**
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

**Time allowed**
- 1 hour 30 minutes

**Instructions**
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- Write the question part reference (eg (a), (b)(i) etc) in the left-hand margin.
- You must answer the questions in the spaces provided. Do not write
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work that you do not want to be marked.

**Information**
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 75.

**Advice**
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

**1** The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = f(x, y)$$

where $f(x, y) = x + 3 + \sin y$

and $y(1) = 1$

**(a)** Use the Euler formula

$$y_{r+1} = y_r + hf(x_r, y_r)$$

with $h = 0.1$, to obtain an approximation to $y(1.1)$, giving your answer to four decimal places. *(3 marks)*

**(b)** Use the formula

$$y_{r+1} = y_{r-1} + 2hf(x_r, y_r)$$

with your answer to part **(a)**, to obtain an approximation to $y(1.2)$, giving your answer to three decimal places. *(3 marks)*

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**2 (a)** Find the value of the constant $k$ for which $k \sin 2x$ is a particular integral of the differential equation

$$\frac{d^2y}{dx^2} + y = \sin 2x$$ *(3 marks)*

**(b)** Hence find the general solution of this differential equation. *(4 marks)*

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**3 (a)** Explain why $\int_1^{\infty} 4xe^{-4x}\,dx$ is an improper integral. *(1 mark)*

**(b)** Find $\int 4xe^{-4x}\,dx$. *(3 marks)*

**(c)** Hence evaluate $\int_1^{\infty} 4xe^{-4x}\,dx$, showing the limiting process used. *(3 marks)*

**4** By using an integrating factor, find the solution of the differential equation

$$\frac{dy}{dx} + \frac{3}{x}y = (x^4 + 3)^{\frac{3}{2}}$$

given that $y = \frac{1}{5}$ when $x = 1$. *(9 marks)*

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**5 (a)** Write down the expansion of $\cos 4x$ in ascending powers of $x$ up to and including the term in $x^4$. Give your answer in its simplest form. *(2 marks)*

**(b) (i)** Given that $y = \ln(2 - e^x)$, find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$.

(You may leave your expression for $\frac{d^3y}{dx^3}$ unsimplified.) *(6 marks)*

**(ii)** Hence, by using Maclaurin's theorem, show that the first three non-zero terms in the expansion, in ascending powers of $x$, of $\ln(2 - e^x)$ are

$$-x - x^2 - x^3$$

*(2 marks)*

**(c)** Find

$$\lim_{x \to 0}\left[\frac{x\ln(2 - e^x)}{1 - \cos 4x}\right]$$

*(3 marks)*

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**6** The polar equation of a curve $C_1$ is

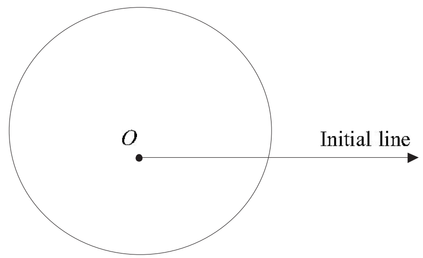
$$r = 2(\cos\theta - \sin\theta), \quad 0 \leqslant \theta \leqslant 2\pi$$

**(a) (i)** Find the cartesian equation of $C_1$. *(4 marks)*

**(ii)** Deduce that $C_1$ is a circle and find its radius and the cartesian coordinates of its centre. *(3 marks)*

**(b)** The diagram shows the curve $C_2$ with polar equation

$$r = 4 + \sin\theta, \quad 0 \leqslant \theta \leqslant 2\pi$$




**(i)** Find the area of the region that is bounded by $C_2$. *(6 marks)*

**(ii)** Prove that the curves $C_1$ and $C_2$ do not intersect. *(4 marks)*

**(iii)** Find the area of the region that is outside $C_1$ but inside $C_2$. *(2 marks)*

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**7 (a)** Given that $x = t^{\frac{1}{2}}$, $x > 0$, $t > 0$ and $y$ is a function of $x$, show that:

**(i)** $\frac{dy}{dx} = 2t^{\frac{1}{2}}\frac{dy}{dt}$; *(2 marks)*

**(ii)** $\frac{d^2y}{dx^2} = 4t\frac{d^2y}{dt^2} + 2\frac{dy}{dt}$. *(3 marks)*

**(b)** Hence show that the substitution $x = t^{\frac{1}{2}}$ transforms the differential equation

$$x\frac{d^2y}{dx^2} - (8x^2 + 1)\frac{dy}{dx} + 12x^3y = 12x^5$$

into

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 3y = 3t$$

*(2 marks)*

**(c)** Hence find the general solution of the differential equation

$$x\frac{d^2y}{dx^2} - (8x^2 + 1)\frac{dy}{dx} + 12x^3y = 12x^5$$

giving your answer in the form $y = f(x)$. *(7 marks)*

**END OF QUESTIONS**

# AQA – Further pure 3 – Jun 2010 – Answers

| Question 1: | Exam report |
|---|---|
| 1) $\frac{dy}{dx} = f(x,y) = x+3+Sin(y) \quad with\ y(1)=1$<br>a) $y_{r+1} = y_r + hf(x_r, y_r)$<br>$x_0 = 1\ and\ y_1 = y(1) = 1$<br>$x_1 = 1.1\ and\ y_2 = y(1.1) = 1 + 0.1(1+3+\sin(1))$<br>$y_2 = 1.4841$ to 4 decimal places<br>b) $y_{r+1} = y_{r-1} + 2hf(x_r, y_r)$<br>$y(1.2) = y(1) + 2\times 0.1 \times (1.1+3+\sin(1.4841))$<br>$y(1.2) = 2.019$ to $3d.p.$ | This was the best answered question on the paper. Numerical solutions of first order differential equations continue to be a good source of marks for all candidates. The most common loss of marks was due to calculators being set in degree mode. It is worth recording that slightly more candidates than in recent series have slipped back into not showing the necessary working. Without such working, wrong answers cannot be awarded credit. |

| Question 2: | Exam report |
|---|---|
| a) $y = kSin(2x) \quad \frac{dy}{dx} = 2kCos(2x) \quad \frac{d^2y}{dx^2} = -4kSin(2x)$<br>The equation $\frac{d^2y}{dx^2} + y = Sin(2x)$ becomes<br>$-4kSin(2x) + kSin(2x) = Sin(2x)$<br>$-3Sin(2x) = Sin(2x) \quad$ so $k = -\frac{1}{3}$<br>b) The auxiliary equation associated with the differential equation is<br>$\lambda^2 + 1 = 0 \quad \lambda = i\ or\ \lambda = -i$<br>The complementary function is $y = ACos(x) + BSin(x) \quad A, B \in \mathbb{R}$<br>The general solution is $y = ACos(x) + BSin(x) - \frac{1}{3}Sin(2x) \quad A, B \in \mathbb{R}$ | This was another very good source of marks for candidates, with many fully correct solutions presented. In part (a) some candidates decided to ignore the given form of the particular integral and worked with $p\cos 2x + q\sin 2x$. Such an approach was not penalised by examiners provided the candidate showed that both $p = 0$ and $q = -\frac{1}{3}$. The vast majority of candidates showed that they knew the methods to solve the second order differential equation but errors in forming and solving the auxiliary equation were sources of loss of marks. Real solutions from the auxiliary equation were more heavily penalised than other errors. |

| Question 3: | Exam report |
|---|---|
| a) The integral is improper because<br>the interval of integration is infinite.<br>b) $\int 4xe^{-4x}dx = \left[-xe^{-4x}\right] - \int -e^{-4x}dx = -xe^{-4x} - \frac{1}{4}e^{-4x} + c$<br>c) $\int_1^N 4xe^{-4x}dx = \left[-xe^{-4x} - \frac{1}{4}e^{-4x}\right]_1^N = -Ne^{-4N} - \frac{1}{4}e^{-4N} + e^{-4} + \frac{1}{4}e^{-4}$<br>$\lim_{N\to\infty} e^{-4N} = 0$ and $\lim_{N\to\infty} Ne^{-4N} = 0$ so $\int_1^\infty 4xe^{-4x}dx$ exists and<br>$\int_1^\infty 4xe^{-4x}dx = \frac{5}{4}e^{-4}$ | Improper integrals and limiting processes continue to cause problems for a significant minority of candidates. Part (a) was generally not answered well with too many candidates making a statement which they then contradicted in part (c). As highlighted in the MFP3 examiners' reports for January 2008 and January 2010, an integral, with $\infty$ as a limit, is improper because the interval of integration is infinite.<br>Most candidates correctly applied integration by parts to score the three marks available in part (b).<br>In part (c) examiners expected to see the infinite upper limit replaced by, for example, *a*, the integration carried out and then consideration of the limiting process as $a \to \infty$.<br>It was not uncommon to see in solutions the statement "as<br>"$a \to \infty, e^{-4a} \to 0$ so $e^{-4a}(-a - \frac{1}{4}) \to 0$" without seemingly any particular analysis of $\lim_{a\to\infty} ae^{-4a}$. This lack of analysis was penalised. |

| Question 4: | Exam report |
|---|---|
| $\frac{dy}{dx}+\frac{3}{x}y=\left(x^4+3\right)^{\frac{3}{2}}$ and $y(1)=\frac{1}{5}$<br>An integrating factor $I=e^{\int \frac{3}{x}dx}=e^{\ln(x^3)}=x^3$<br>The equation becomes:<br>$x^3\frac{dy}{dx}+3x^2=x^3\left(x^4+3\right)^{\frac{3}{2}}$<br>$\frac{d}{dx}\left(x^3y\right)=x^3\left(x^4+3\right)^{\frac{3}{2}}$<br>$x^3y=\int x^3\left(x^4+3\right)^{\frac{3}{2}}dx=\frac{1}{4}\int 4x^3\left(x^4+3\right)^{\frac{3}{2}}dx$<br>$x^3y=\frac{1}{4}\times\frac{2}{5}\left(x^4+3\right)^{\frac{5}{2}}+c$<br>$y=\frac{1}{10x^3}\left(x^4+3\right)^{\frac{5}{2}}+\frac{c}{x^3}$<br>For $x=1, y=\frac{1}{5}$ this gives<br>$\frac{1}{5}=\frac{1}{10}\times 32+c$ so $c=-3$<br>$y=\frac{1}{10x^3}\left(x^4+3\right)^{\frac{5}{2}}-\frac{3}{x^3}$ | Approximately half the candidates scored full marks for this question. Almost all candidates were able to show that they knew how to find and use an integrating factor to solve a first order differential equation. However, a significant minority did not recognise the form of the integral of<br>$\int x^3\left(x^4+3\right)^{\frac{3}{2}}dx$ as $k\left(x^4+3\right)^{\frac{5}{2}}$ or did not apply a suitable substitution, so made very little further progress. |

| Question 5: | Exam report |
|---|---|
| a) From the formulae book<br>$Cos(x)=1-\frac{x^2}{2}+\frac{x^4}{24}+....$<br>so $Cos(4x)=1-\frac{(4x)^2}{2}+\frac{(4x)^4}{24}+...$<br>$Cos(4x)=1-8x^2+\frac{32}{3}x^4+...$<br>b) i) $y=\ln(2-e^x)$ $\quad\frac{dy}{dx}=\frac{-e^x}{2-e^x}$<br>$\frac{d^2y}{dx^2}=\frac{-e^x(2-e^x)-e^{2x}}{(2-e^x)^2}=\frac{-2e^x}{(2-e^x)^2}$<br>$\frac{d^3y}{dx^3}=\frac{-2e^x(2-e^x)^2-(-2e^x)\times 2(-e^x)(2-e^x)}{(2-e^x)^4}$<br>$=\frac{-2e^x(2-e^x)-4e^{2x}}{(2-e^x)^3}=\frac{-4e^x-2e^{2x}}{(2-e^x)^3}$<br>ii) $y(0)=\ln\left(2-e^0\right)=0$<br>$y'(0)=\frac{-1}{2-1}=-1$ $\quad y''(0)=\frac{-2}{(2-1)^2}=-2$<br>$y^{(3)}(0)=\frac{-4-2}{(2-1)^3}=-6$<br>Conclusion: $\ln(2-e^x)=0-1x+\frac{-2}{2!}x^2+\frac{-6}{3!}x^3+...$<br>$\ln(2-e^x)=-x-x^2-x^3+...$<br>c) $x\ln(2-e^x)=x(-x-x^2-x^3+...)=-x^2-x^3-x^4+...$<br>$1-Cos(4x)=1-1+8x^2+...=8x^2+...$<br>Therefore, $\frac{x\ln(2-e^x)}{1-\cos(4x)}=\frac{-x^2+...}{8x^2+...}=-\frac{1}{8}+...$<br>Conclusion: $\lim_{x\to 0}\frac{x\ln(2-e^x)}{1-\cos(4x)}=-\frac{1}{8}$ | Candidates' differentiation skills have improved significantly over recent series. Almost all candidates could correctly quote the series expansion for cos4*x* although some failed to give their answer in its simplest form. The methods required to obtain the three derivatives were well understood and, although some algebraic errors were seen, there were many correct expressions presented.<br><br>Most candidates displayed good knowledge of Maclaurin's theorem but only those who had made no errors in earlier differentiations could score both marks for showing the printed result in part (b)(ii). There was a pleasing improvement in explicitly reaching the stage of a constant term in both the numerator and denominator before taking the limit as $x\to 0$ in the final part of the question. |

| Question 6: | Exam report |
|---|---|
| $C_1: r = 2(Cos\theta - Sin\theta) \qquad 0 \le \theta \le 2\pi$<br>$a)i)\ Multiplying\ by\ r:$<br>$r^2 = 2rCos\theta - 2rSin\theta$<br>$x^2 + y^2 = 2x - 2y \qquad x^2 + y^2 = 2(x-y)$<br>$ii)\ x^2 + y^2 - 2x + 2y = 0$<br>$(x-1)^2 - 1 + (y+1)^2 - 1 = 0$<br>$(x-1)^2 + (y+1)^2 = 2$<br>This is the circle centre $(1, -1)$, radius $\sqrt{2}$.<br>$b)\ C_2: r = 4 + Sin\theta \qquad 0 \le \theta \le 2\pi$<br>$i)\ Area = \frac{1}{2}\int_0^{2\pi} (4 + Sin\theta)^2 d\theta = \frac{1}{2}\int_0^{2\pi} (16 + 8Sin\theta + Sin^2\theta) d\theta$<br>$Area = \frac{1}{2}\int_0^{2\pi} (16 + 8Sin\theta + \frac{1}{2} - \frac{1}{2}Cos(2\theta)) d\theta$<br>$= \left[\frac{33}{4}\theta - 4Cos(\theta) - \frac{1}{8}Sin(2\theta)\right]_0^{2\pi} = \frac{33}{2}\pi$<br>$ii)\ r = 4 + Sin\theta\ \ and\ \ r = 2(Cos\theta - Sin\theta)$<br>$so\ \ 4 + Sin\theta = 2Cos\theta - 2Sin\theta$<br>$3Sin\theta - 2Cos\theta = -4$<br>$R = \sqrt{3^2 + 2^2} = \sqrt{13}$<br>$\frac{3}{\sqrt{13}}Sin\theta - \frac{2}{\sqrt{13}}Cos\theta = -\frac{4}{\sqrt{13}}$<br>$Sin(\alpha - \theta) = -\frac{4}{\sqrt{13}} < -1$<br>$No\ solution\ for\ \theta.$<br>$iii)\ Area = \frac{33}{2}\pi - \pi\left(\sqrt{2}\right)^2 = \frac{29}{2}\pi$ | This question on polar coordinates proved to be the most demanding question on the paper. In part (a), the better candidates had no problems in finding the cartesian equation for $C_1$ and rearranging it by completing the squares so as to be able to deduce that the curve was a circle.<br><br>Many other candidates failed to eliminate θ even after a page or more of working and abandoned part (a) of the question.<br><br>Many candidates scored heavily on the more familiar part (b)(i), finding the area bounded by the second curve $C_2$.<br>Proving that the two curves did not intersect in part (b)(ii) was a challenge, even for the better candidates, and it was rare to award all four marks for this part of the question. However, some excellent solutions were seen which used a variety of methods including forming and solving quadratic equations in either sin *x*, cos *x*, tan *x*, sec *x* , use of calculus and use of the *R*, α form.<br>In the final part of the question, those who used part (a)(ii) to find the area enclosed by $C_1$ and used it with their answer to part (b)(i) had no problems scoring both marks. Those who tried to use integration to find the area enclosed by $C_1$ did not analyse the problem fully and failed to score. A common misinterpretation of the word 'intersect' is indicated by the not uncommon answer 'Since $C_1$ and $C_2$ do not intersect the required area is just 16.5π , the area bounded by $C_2$.' |

**Question 7:**

$$a)\ x = t^{\frac{1}{2}} = \sqrt{t} \qquad \frac{dx}{dt} = \frac{1}{2\sqrt{t}} \ \ and \ \ \frac{dt}{dx} = 2\sqrt{t}$$

$$i)\ \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 2\sqrt{t}\,\frac{dy}{dt}$$

$$ii)\ \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(2\sqrt{t}\,\frac{dy}{dt}\right) \times \frac{dt}{dx}$$

$$= \left(\frac{1}{\sqrt{t}}\frac{dy}{dt} + 2\sqrt{t}\,\frac{dy}{dt}\right) \times 2\sqrt{t} = 2\frac{dy}{dt} + 4t\frac{d^2y}{dt^2}$$

$$b)\ x\frac{d^2y}{dx^2} - (8x^2+1)\frac{dy}{dx} + 12x^3y = 12x^5 \ \ becomes$$

$$\sqrt{t}\left(2\frac{dy}{dt} + 4t\frac{d^2y}{dt^2}\right) - (8t+1)\left(2\sqrt{t}\,\frac{dy}{dt}\right) + 12t\sqrt{t}\,y = 12t^2\sqrt{t} \qquad (\div\sqrt{t})$$

$$2\frac{dy}{dt} + 4t\frac{d^2y}{dt^2} - 16t\frac{dy}{dt} - 2\frac{dy}{dt} + 12t \times y = 12t^2 \qquad (\div 4t)$$

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 3y = 3t \qquad (E)$$

$c)$ The auxiliary equation associated with the diff.eq. is

$$\lambda^2 - 4\lambda + 3 = 0\ (\lambda-3)(\lambda-1) = 0$$

$$\lambda = 3 \ or \ \lambda = 1$$

The complementary function is $y = Ae^{3t} + Be^{t} \quad A, B \in \mathbb{R}$

The particular integral is of the form

$$y = at + b \qquad \frac{dy}{dt} = a \qquad \frac{d^2y}{dt^2} = 0$$

$(E)$ *becomes* $0 - 4a + 3at + 3b = 3t$

This gives $a = 1$ *and* $b = \frac{4}{3}$

$$y = t + \frac{4}{3} + Ae^{3t} + Be^{t} \ \ and \ with \qquad t = x^2$$

$$so \qquad y = x^2 + \frac{4}{3} + Ae^{3x^2} + Be^{x^2}$$

**Exam report**

Candidates generally answered part (a)(i) correctly but showing the printed result involving the second derivatives in part (a)(ii) proved to be more difficult, although candidates' attempts displayed a significant improvement over candidates' performances on previous papers.
Although most candidates realised what was required to answer part (b), careless work resulted in many less than convincing solutions.
In the final part of the question, the majority of candidates started out correctly to solve the differential equation for *y* in terms of *t* with many obtaining the correct form of the complementary function (although some used *x* instead of *t*), but finding the correct particular integral proved to be more problematic. A very common careless error resulted in candidates solving the equation "$-4p + 3pt + q = 3t$" instead of the correct equation "$-4p + 3pt + 3q = 3t$". However, correct answers were quite frequently seen and it was particularly pleasing to see a higher proportion of candidates correctly converting back from *t* to *x*.

**Grade boundaries**

| Grade | | A* | A | B | C | D | E |
|---|---|---|---|---|---|---|---|
| Mark | Max 75 | 69 | 64 | 56 | 49 | 42 | 35 |

**Key to mark scheme and abbreviations used in marking**

| | | | |
|---|---|---|---|
| M | mark is for method | | |
| m or dM | mark is dependent on one or more M marks and is for method | | |
| A | mark is dependent on M or m marks and is for accuracy | | |
| B | mark is independent of M or m marks and is for method and accuracy | | |
| E | mark is for explanation | | |
| √or ft or F | follow through from previous incorrect result | MC | mis-copy |
| CAO | correct answer only | MR | mis-read |
| CSO | correct solution only | RA | required accuracy |
| AWFW | anything which falls within | FW | further work |
| AWRT | anything which rounds to | ISW | ignore subsequent work |
| ACF | any correct form | FIW | from incorrect work |
| AG | answer given | BOD | given benefit of doubt |
| SC | special case | WR | work replaced by candidate |
| OE | or equivalent | FB | formulae book |
| A2,1 | 2 or 1 (or 0) accuracy marks | NOS | not on scheme |
| –x EE | deduct x marks for each error | G | graph |
| NMS | no method shown | c | candidate |
| PI | possibly implied | sf | significant figure(s) |
| SCA | substantially correct approach | dp | decimal place(s) |

**No Method Shown**

Where the question specifically requires a particular method to be used, we must usually see evidence of use of this method for any marks to be awarded. However, there are situations in some units where part marks would be appropriate, particularly when similar techniques are involved. Your Principal Examiner will alert you to these and details will be provided on the mark scheme.

Where the answer can be reasonably obtained without showing working and it is very unlikely that the correct answer can be obtained by using an incorrect method, we must award **full marks**. However, the obvious penalty to candidates showing no working is that incorrect answers, however close, earn **no marks**.

Where a question asks the candidate to state or write down a result, no method need be shown for full marks.

Where the permitted calculator has functions which reasonably allow the solution of the question directly, the correct answer without working earns **full marks**, unless it is given to less than the degree of accuracy accepted in the mark scheme, when it gains **no marks**.

**Otherwise we require evidence of a correct method for any marks to be awarded.**

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| 1(a) | $y(1.1) = y(1) + 0.1[1+3+\sin 1]$ | M1A1 | | |
| | $= 1 + 0.1\times 4.84147.= 1.4841(47..)$ $= 1.4841$ to 4dp | A1 | 3 | Condone > 4dp |
| (b) | $y(1.2) = y(1) + 2(0.1)\{f[1.1, y(1.1)]\}$ | M1 | | |
| | $.... = 1 + 2(0.1)\{1.1+3+\sin[1.4841(47..)]\}$ $= 2.019$ to 3dp | A1F<br>A1 | 3 | Ft on cand's answer to **(a)**<br>CAO Must be 2.019<br>**Note:** If using degrees max mark is 4/6 ie M1A1A0;M1A1FA0 |
| | **Total** | | 6 | |
| 2(a) | $-4k\sin 2x + k\sin 2x = \sin 2x$ | M1<br>A1 | | Substituting into the differential equation |
| | $k = -\frac{1}{3}$ | A1 | 3 | Accept correct PI |
| (b) | (Aux. eqn $m^2 + 1 = 0$) $m = \pm i$<br>CF: $A\cos x + B\sin x$ | B1<br>M1<br>A1F | | PI<br>M0 if $m$ is real<br>OE Ft on incorrect complex values for $m$<br>For the A1F do not accept if left in the form $Ae^{ix} + Be^{-ix}$ |
| | (GS: $y$ =) $A\cos x + B\sin x - \frac{1}{3}\sin 2x$ | B1F | 4 | c's CF +c's PI but must have 2 constants |
| | **Total** | | 7 | |
| 3(a) | The interval of integration is infinite | E1 | 1 | OE |
| (b) | $\int 4xe^{-4x}dx = -xe^{-4x} - \int -e^{-4x}dx$ | M1<br>A1 | | $kxe^{-4x} - \int ke^{-4x}dx$ for non-zero $k$ |
| | $= -xe^{-4x} - \frac{1}{4}e^{-4x}$ {+c} | A1F | 3 | Condone absence of $+c$ |
| (c) | $I = \int_1^\infty 4xe^{-4x}\,dx = \lim_{a\to\infty} \int_1^a 4xe^{-4x}\,dx$<br>$\lim_{a\to\infty} \{-ae^{-4a} - \frac{1}{4}e^{-4a}\} - \left[-\frac{5}{4}e^{-4}\right]$ | M1 | | F($a$) – F(1) with an indication of limit '$a\to\infty$' |
| | $\lim_{a\to\infty} ae^{-4a} = 0$ | M1 | | For statement with limit/ limiting process shown |
| | $I = \frac{5}{4}e^{-4}$ | A1 | 3 | CSO |
| | **Total** | | 7 | |

MFP3 (cont)

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| 4 | IF is $\exp\left(\int \frac{3}{x}\,dx\right)$ | M1 | | and with integration attempted |
| | $= e^{3\ln x}$ | A1 | | PI |
| | $= x^3$ | A1 | | |
| | $\frac{d}{dx}\left[ yx^3 \right] = x^3(x^4+3)^{\frac{3}{2}}$ | M1 A1 | | LHS. Use of c's IF. PI |
| | $\Rightarrow yx^3 - \frac{1}{10}\left( x^4+3 \right)^{\frac{5}{2}} + A$ | m1 A1 | | $k\left( x^4+3 \right)^{\frac{5}{2}}$ Condone missing '$A$' |
| | $\Rightarrow \frac{1}{5} = \frac{1}{10}(4)^{\frac{5}{2}} + A$ | m1 | | Use of boundary conditions in attempt to find constant after intgr. Dep on two M marks, not dep on m |
| | $\Rightarrow A = -3;$ (*) | | | |
| | $\Rightarrow yx^3 = \frac{1}{10}\left(x^4+3\right)^{\frac{5}{2}} - 3$ | A1 | 9 | ACF. The A1 can be awarded at line (*) provided a correct earlier eqn in $y$, $x$ and '$A$' is seen immediately before boundary conditions are substituted. |
| | **Total** | | **9** | |

MFP3 (cont)

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| 5(a) | $\cos 4x \approx 1 - \frac{(4x)^2}{2} + \frac{(4x)^4}{4!} \ldots$ | M1 | | Clear attempt to replace $x$ by $4x$ in expansion of $\cos x$…condone missing brackets for the M mark |
| | $\approx 1 - 8x^2 + \frac{32}{3}x^4 \ldots$ | A1 | 2 | |
| (b)(i) | $\frac{dy}{dx} = \frac{1}{2-e^x} \times (-e^x)$ | M1 A1 | | Chain rule |
| | $\frac{d^2y}{dx^2} = \frac{(2-e^x)(-e^x) - (-e^x)(-e^x)}{(2-e^x)^2}$ | M1 A1 | | Quotient rule OE ACF |
| | $= \frac{-2e^x}{(2-e^x)^2}$ | | | |
| | $\frac{d^3y}{dx^3} = \frac{(2-e^x)^2(-2e^x) - (-2e^x)2(2-e^x)(-e^x)}{(2-e^x)^4}$ | m1 | | All necessary rules attempted (dep on previous 2 M marks) |
| | | A1 | 6 | ACF |
| (ii) | $y(0) = 0;\ y'(0) = -1;\ y''(0) = -2;\ y'''(0) = -6$ | M1 | | At least three attempted |
| | $\text{Ln}(2-e^x) \approx y(0) + xy'(0) + \frac{x^2}{2}y''(0) + \frac{x^3}{6}y'''(0) \ldots$ | | | |
| | $\ldots \approx -x - x^2 - x^3 \ldots$ | A1 | 2 | CSO AG (The previous 7 marks must have been awarded and no double errors seen) |
| (c) | $\left[\frac{x\ln(2-e^x)}{1-\cos 4x}\right] \approx \frac{-x^2 - x^3 - x^4 \ldots}{8x^2 - \frac{32}{3}x^4}$ | M1 | | Using the expansions |
| | $\text{Limit} = \lim_{x \to 0} \frac{-x^2 - o(x^3)}{8x^2 - o(x^4)}$ | | | The notation $o(x^n)$ can be replaced by a term of the form $kx^n$ |
| | $\ldots = \lim_{x \to 0} \frac{-1 - o(x)}{8 - o(x^2)}$ | m1 | | Division by $x^2$ stage before taking the limit |
| | $\ldots = -\frac{1}{8}$ | A1 | 3 | CSO |
| | **Total** | | **13** | |

MFP3 (cont)

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| 6(a)(i) | $x^2+y^2=r^2,\ x=r\cos\theta,\ y=r\sin\theta$ | B2,1,0 | | B1 for one stated or used |
| | $r^2=2r(\cos\theta-\sin\theta)$ | M1 | | |
| | $x^2+y^2=2(x-y)$ | A1 | 4 | ACF |
| (ii) | $(x-1)^2+(y+1)^2=2$ | M1<br>A1F | | |
| | Centre (1, −1); radius √2 | A1F | 3 | |
| (b)(i) | Area = $\frac{1}{2}\int(4+\sin\theta)^2\,d\theta$ | M1 | | Use of $\frac{1}{2}\int r^2\,d\theta$. |
| | $-\frac{1}{2}\int_0^{2\pi}(16+8\sin\theta+\sin^2\theta)d\theta$ | B1<br>B1 | | Correct expn of $[4+\sin\theta]^2$<br>Correct limits |
| | $-\int_0^{2\pi}(8+4\sin\theta+0.25(1-\cos 2\theta))\,d\theta$ | M1 | | Attempt to write $\sin^2\theta$ in terms of $\cos 2\theta$ |
| | $=\left[8\theta-4\cos\theta+\frac{1}{4}\theta-\frac{1}{8}\sin 2\theta\right]_0^{2\pi}$ | A1F | | Correct integration ft wrong coefficients |
| | $=16.5\pi$ | A1 | 6 | CSO |
| (ii) | For the curves to intersect, the eqn $2(\cos\theta-\sin\theta)=4+\sin\theta$ must have a solution. | M1 | | Equating rs and simplifying to a suitable form |
| | $2\cos\theta-3\sin\theta=4$<br>$R\cos(\theta+\alpha)=4$, | M1 | | OE. Forming a relevant eqn from which valid explanation can be stated directly |
| | where $R=\sqrt{2^2+3^2}$ and $\cos\alpha=\frac{2}{R}$ | A1 | | OE. Correct relevant equation |
| | $\cos(\theta+\alpha)=\frac{4}{\sqrt{13}}>1$. Since must have $-1\le\cos X\le 1$ there are no solutions of the equation $2(\cos\theta-\sin\theta)=4+\sin\theta$ so the two curves do not intersect. | E1 | 4 | Accept other valid explanations. |
| (iii) | Required area = answer (b)(i) $-\pi$(radius of $C_1$)$^2$ | M1 | | |
| | $=16.5\pi-2\pi=14.5\pi$ | A1F | 2 | Ft on (a)(ii) and (b)(i) |
| | **Total** | | **19** | |

MFP3 (cont)

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| 7(a)(i) | $\frac{dx}{dt}\frac{dy}{dx}=\frac{dy}{dt}$ | M1 | | OE Chain rule |
| | $\frac{1}{2}t^{-\frac{1}{2}}\frac{dy}{dx}=\frac{dy}{dt}$ so $\frac{dy}{dx}=2t^{\frac{1}{2}}\frac{dy}{dt}$ | A1 | 2 | CSO A.G. |
| (a)(ii) | $\frac{d^2y}{dx^2}=\frac{d}{dx}\left(2t^{\frac{1}{2}}\frac{dy}{dt}\right)=\frac{dt}{dx}\frac{d}{dt}\left(2t^{\frac{1}{2}}\frac{dy}{dt}\right)$ | M1 | | $\frac{d}{dx}(\,f(t)\,)=\frac{dt}{dx}\frac{d}{dt}(f(t))$<br>O.E. eg<br>$\frac{d}{dt}(\,g(x)\,)=\frac{dx}{dt}\frac{d}{dx}(g(x))$ |
| | $\frac{d^2y}{dx^2}-2t^{\frac{1}{2}}\left[2t^{\frac{1}{2}}\frac{d^2y}{dt^2}+t^{-\frac{1}{2}}\frac{dy}{dt}\right]$ | m1 | | Product rule O.E. used dep on previous M1 being awarded at some stage |
| | $\frac{d^2y}{dx^2}=4t\frac{d^2y}{dt^2}+2\frac{dy}{dt}$ | A1 | 3 | CSO A.G. |
| (b) | $t^{\frac{1}{2}}\left[4t\frac{d^2y}{dt^2}+2\frac{dy}{dt}\right]-(8t+1)2t^{\frac{1}{2}}\frac{dy}{dt}$<br>$+12t^{\frac{3}{2}}y=12t^{\frac{5}{2}}$ | M1 | | Subst. using (a)(i), (a)(ii) into given DE to eliminate all $x$ |
| | $4t^{\frac{3}{2}}\frac{d^2y}{dt^2}-16t^{\frac{3}{2}}\frac{dy}{dt}+12t^{\frac{3}{2}}y=12t^{\frac{5}{2}}$<br>Divide by $4t^{\frac{3}{2}}$ gives | | | |
| | $\frac{d^2y}{dt^2}-4\frac{dy}{dt}+3y=3t$ | A1 | 2 | CSO A.G. |
| (c) | Solving $\frac{d^2y}{dt^2}-4\frac{dy}{dt}+3y=3t$ (*) | | | |
| | Auxl. Eqn. $m^2-4m+3=0$<br>$(m-1)(m-3)=0$ | M1 | | PI |
| | $m=1$ and 3 | A1 | | |
| | CF $Ae^t+Be^{3t}$ | M1 | | Condone $x$ for $t$ here; ft c's 2 real values for '$m$' |
| | For PI try $y=pt+q$ | M1 | | OE |
| | $-4p+3pt+3q=3t\ \Rightarrow\ p=1,\ q=\frac{4}{3}$ | A1 | | |
| | GS of (*) is $y-Ae^t+Be^{3t}+t+\frac{4}{3}$ | B1F | | CF + PI with 2 arb. constants and both CF and PI functions of $t$ only |
| | GS of<br>$x\frac{d^2y}{dx^2}-(8x^2+1)\frac{dy}{dx}+12x^3y=12x^5$<br>is $y=Ae^{x^2}+Be^{3x^2}+x^2+\frac{4}{3}$ | A1 | 7 | |
| | **Total** | | **14** | |
| | **TOTAL** | | **75** | |

# Notes about June 2010:

AQA

General Certificate of Education
Advanced Level Examination
January 2011

# Mathematics MFP3

**Unit Further Pure 3**

**Monday 24 January 2011 9.00 am to 10.30 am**

**For this paper you must have:**
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

**Time allowed**
- 1 hour 30 minutes

**Instructions**
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- Write the question part reference (eg (a), (b)(i) etc) in the left-hand margin.
- You must answer the questions in the spaces provided. Do not write outside the box around each page.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work that you do not want to be marked.

**Information**
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 75.

**Advice**
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

**1** The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = f(x, y)$$

where $f(x, y) = x + \sqrt{y}$

and $y(3) = 4$

Use the improved Euler formula

$$y_{r+1} = y_r + \frac{1}{2}(k_1 + k_2)$$

where $k_1 = hf(x_r, y_r)$ and $k_2 = hf(x_r + h, y_r + k_1)$ and $h = 0.1$, to obtain an approximation to $y(3.1)$, giving your answer to three decimal places. *(5 marks)*

---

**2 (a)** Find the values of the constants $p$ and $q$ for which $p\sin x + q\cos x$ is a particular integral of the differential equation

$$\frac{dy}{dx} + 5y = 13\cos x$$ *(3 marks)*

**(b)** Hence find the general solution of this differential equation. *(3 marks)*

---

**3** A curve $C$ has polar equation $r(1 + \cos\theta) = 2$.

**(a)** Find the cartesian equation of $C$, giving your answer in the form $y^2 = f(x)$. *(5 marks)*

**(b)** The straight line with polar equation $4r = 3\sec\theta$ intersects the curve $C$ at the points $P$ and $Q$. Find the length of $PQ$. *(4 marks)*

---

**4** By using an integrating factor, find the solution of the differential equation

$$\frac{dy}{dx} - \frac{2}{x}y = 2x^3e^{2x}$$

given that $y = e^4$ when $x = 2$. Give your answer in the form $y = f(x)$. *(9 marks)*

---

**5 (a)** Write $\frac{4}{4x+1} - \frac{3}{3x+2}$ in the form $\frac{C}{(4x+1)(3x+2)}$, where $C$ is a constant. *(1 mark)*

**(b)** Evaluate the improper integral

$$\int_1^{\infty} \frac{10}{(4x+1)(3x+2)} \, dx$$

showing the limiting process used and giving your answer in the form $\ln k$, where $k$ is a constant. *(6 marks)*

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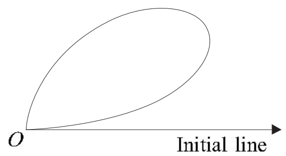
**6** The diagram shows a sketch of a curve $C$.




The polar equation of the curve is

$$r = 2\sin 2\theta\sqrt{\cos\theta}, \quad 0 \leqslant \theta \leqslant \frac{\pi}{2}$$

Show that the area of the region bounded by $C$ is $\frac{16}{15}$. *(7 marks)*

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**7 (a)** Write down the expansions in ascending powers of $x$ up to and including the term in $x^3$ of:

**(i)** $\cos x + \sin x$; *(1 mark)*

**(ii)** $\ln(1+3x)$. *(1 mark)*

**(b)** It is given that $y = e^{\tan x}$.

**(i)** Find $\frac{dy}{dx}$ and show that $\frac{d^2y}{dx^2} = (1+\tan x)^2 \frac{dy}{dx}$. *(5 marks)*

**(ii)** Find the value of $\frac{d^3y}{dx^3}$ when $x = 0$. *(2 marks)*

**(iii)** Hence, by using Maclaurin's theorem, show that the first four terms in the expansion, in ascending powers of $x$, of $e^{\tan x}$ are

$$1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{2}x^3$$ *(2 marks)*

**(c)** Find

$$\lim_{x \to 0} \left[ \frac{e^{\tan x} - (\cos x + \sin x)}{x \ln(1+3x)} \right]$$ *(3 marks)*

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**8 (a)** Given that $x = e^t$ and that $y$ is a function of $x$, show that

$$x\frac{dy}{dx} = \frac{dy}{dt}$$ *(2 marks)*

**(b)** Hence show that the substitution $x = e^t$ transforms the differential equation

$$x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 2\ln x$$

into

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 2t$$ *(5 marks)*

**(c)** Find the general solution of the differential equation

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 2t$$ *(6 marks)*

**(d)** Hence solve the differential equation $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 2\ln x$, given that $y = \frac{3}{2}$ and $\frac{dy}{dx} = \frac{1}{2}$ when $x = 1$. *(5 marks)*

# AQA – Further pure 3 – Jan 2011 – Answers

| Question 1: | Exam report |
|---|---|
| $\frac{dy}{dx} = f(x,y) = x + \sqrt{y} \quad and \quad y(3) = 4$<br>$k_1 = hf(x_0, y_0) = 0.1 \times (3 + \sqrt{4}) = 0.5$<br>$y_0 + k_1 = 4.5$<br>$k_2 = hf\left(x_1 + y_0 + k_1\right) = 0.1 \times \left(3.1 + \sqrt{4.5}\right) = 0.52213$<br>$y_1 = y(3.1) = 4 + \frac{1}{2}(0.5 + 0.52213) = 4.511 \text{ to } 3d.p.$ | Numerical solutions of first order differential equations continue to be a good source of marks for all candidates, and it was the best answered topic on the paper. However, a few candidates who had an incorrect value for $k_1$ and just gave a table of values without showing any methods gained no credit. Almost all candidates gave their final answer to the required degree of accuracy. |

| Question 2: | Exam report |
|---|---|
| a) $y = pSinx + qCosx$<br>$\frac{dy}{dx} = pCosx - qSinx$<br>$\frac{dy}{dx} + 5y = pCosx - qSinx + 5pSinx + 5qCosx = 13Cosx$<br>$(p+5q)Cosx + (5p-q)Sinx = 13Cosx$<br>so $\begin{cases} p + 5q = 13 \\ 5p - q = 0 \end{cases}$ $\begin{cases} p = \frac{1}{2} \\ q = \frac{5}{2} \end{cases}$<br>A particular integral is $y = \frac{1}{2}Sinx + \frac{5}{2}Cosx$<br>• The auxiliary equation is $\lambda + 5 = 0$<br>$\lambda = -5$<br>The complementary function is $y = Ae^{-5x}$<br>•The general solution is $y = \frac{1}{2}Sinx + \frac{5}{2}Cosx + Ae^{-5x}$ | Most candidates differentiated the given expression, substituted into the first order differential equation and equated coefficients to score the method marks. The most common error in solving the resulting equations was to write the solution of 26*p* = 13 as *p* = 2.<br>In part (b), those candidates who wrote down the correct auxiliary equation, '*m* + 5 = 0', generally had no problems scoring all three marks. Those candidates who solved $\frac{dy}{dx} + 5y = 0$ sometimes forgot to include the constant of integration and so ended up with a general solution which had no arbitrary constant.<br>There was a minority of candidates who tried to use an integrating factor and appeared not to know the method of solving a first order differential equation by using complementary function and particular integral. |

| Question 3: | Exam report |
|---|---|
| $r(1 + Cos\theta) = 2$<br>a) $r + rCos\theta = 2$<br>$r = 2 - rCos\theta$ (squaring both sides)<br>$r^2 = 4 + (rCos\theta)^2 - 4rCos\theta$<br>$x^2 + y^2 = 4 + x^2 - 4x$<br>$y^2 = 4 - 4x$<br>b) $4r = 3Sec\theta$<br>$rCos\theta = \frac{3}{4}$ $\quad x = \frac{3}{4}$<br>Solving simultaneously *gives*<br>$y^2 = 4 - 4x = 1 \;\; so \;\; y = 1 \; or \; y = -1$<br>The cartesian coordinates of *P and Q*:<br>$P(\frac{3}{4}, 1) \; and \; Q(\frac{3}{4}, -1)$<br>The distance $PQ = 2$ | This question on polar coordinates was generally answered well. In part (a), most candidates reached the stage $r + x = 2$, but some less able candidates squared incorrectly to reach $y^2 = 4 - 2x^2$. Those candidates who rearranged the equation to $r = 2 - x$ before squaring usually went on to gain all five marks.<br>In part (b), those candidates who wrote the given equation as 4*r*cosθ = 3, converted it to the cartesian equation $x = \frac{3}{4}$ and solved with their answer to part (a) usually had no difficulty in showing that the length of *PQ* is 2.<br>The other common approach was to solve the two polar equations simultaneously, but a significant proportion of candidates who used this approach stopped after reaching $Cos\theta = \frac{3}{5}$ not even finding the value for *r*. More able candidates, having found the values for cosθ and *r*, went on to use basic trigonometry to find the correct length for *PQ*. |

| **Question 4:** | **Exam report** |
| --- | --- |
| $\frac{dy}{dx}-\frac{2}{x}y=2x^3e^{2x}$<br>An Integrating factor is $I=e^{\int -\frac{2}{x}dx}=e^{-2\ln x}=\frac{1}{x^2}$<br>The equation becomes:<br>$\frac{1}{x^2}\frac{dy}{dx}-\frac{2}{x^3}y=2xe^{2x}$<br>$\frac{d}{dx}\left(\frac{y}{x^2}\right)=2xe^{2x}$<br>$\frac{y}{x^2}=\int 2xe^{2x}dx=xe^{2x}-\int e^{2x}dx$<br>$\frac{y}{x^2}=xe^{2x}-\frac{1}{2}e^{2x}+c$<br>$y=x^3e^{2x}-\frac{x^2}{2}e^{2x}+cx^2$<br>When $x=2, y=e^4$ so $e^4=8e^4-2e^4+4c$<br>$4c=-5e^4$<br>$c=-\frac{5}{4}e^4$<br>$y=x^3e^{2x}-\frac{x^2}{2}e^{2x}-\frac{5}{4}x^2$ | Most candidates were able to show that they knew how to find and use an integrating factor to solve a first order differential equation, and many gained either full marks for the question or just lost the final accuracy mark. The only other error of note was losing the negative sign in setting up the integrating factor. This led to a more complicated integral which few solved 'correctly'. |

| **Question 5:** | **Exam report** |
| --- | --- |
| a) $\frac{4}{4x+1}-\frac{3}{3x+2}=\frac{4(3x+2)-3(4x+1)}{(4x+1)(3x+2)}=\frac{5}{(4x+1)(3x+2)}$<br>b) $\int_1^N \frac{10}{(4x+1)(3x+2)}dx=2\int_1^N \frac{5}{(4x+1)(3x+2)}dx$<br>$=2\int_1^N \frac{4}{4x+1}-\frac{3}{3x+2}dx=2\left[\ln\lvert 4x+1\rvert-\ln\lvert 3x+2\rvert\right]_1^N$<br>$=2\left[\ln\left(\frac{4x+1}{3x+2}\right)\right]_1^N=2\ln\left(\frac{4N+1}{3N+2}\right)-2\ln 1=2\ln\left(\frac{4N+1}{3N+2}\right)$<br>$\lim_{N\to\infty}\frac{4N+1}{3N+2}=\lim_{N\to\infty}\frac{4+\frac{1}{N}}{3+\frac{2}{N}}=\frac{4}{3}$ so $\lim_{N\to\infty}\ln\left(\frac{4N+1}{3N+2}\right)=\ln\frac{4}{3}$<br>$\int_1^\infty \frac{10}{(4x+1)(3x+2)}dx$ exists and $\int_1^\infty \frac{10}{(4x+1)(3x+2)}dx=2\ln\frac{4}{3}=\ln\frac{16}{9}$ | Most candidates found the correct value for *C* in part (a). A large majority of candidates realised that part (a) had some relevance to part (b) and duly wrote the integrand in terms of partial fractions and generally integrated correctly. Those who missed this step gained little or no credit for their later work. Although showing the limiting process is an area of the specification that candidates still find difficult, overall improvement continues to be noted. |

## Question 6:

$$r = 2Sin2\theta\sqrt{\cos\theta} \quad with \quad 0\le\theta\le\frac{\pi}{2}$$

$$A=\frac{1}{2}\int_0^{\frac{\pi}{2}} 4Sin^2 2\theta Cos\theta d\theta = \int_0^{\frac{\pi}{2}} 2Cos\theta Sin^2 2\theta d\theta$$

$$A=\int_0^{\frac{\pi}{2}} 2Cos\theta\times 4Sin^2\theta Cos^2\theta d\theta = 8\int_0^{\frac{\pi}{2}} Cos\theta Sin^2\theta(1-Sin^2\theta)d\theta$$

$$A=8\int_0^{\frac{\pi}{2}} Cos\theta Sin^2\theta - Cos\theta Sin^4\theta \, d\theta$$

$$type: \int f'\times f^n = \frac{1}{n+1}f^{n+1}$$

$$A=8\left[\frac{1}{3}Sin^3\theta-\frac{1}{5}Sin^5\theta\right]_0^{\frac{\pi}{2}} = \frac{8}{3}-\frac{8}{5}-0=\frac{16}{15}$$

### Exam report

Although most candidates scored marks for substituting a correct expression for $r^2$ into $A=\int \frac{1}{2}r^2 d\theta$ and inserting the correct limits, less able candidates made little further progress. A significant proportion of other candidates went further by either using the identity sin 2θ = 2sinθ cosθ or writing $\sin^2 2\theta$ in terms of cos 4θ . A surprisingly common error amongst those solutions which used the latter approach is illustrated by '(1−cos4θ)cosθ = cosθ −$\cos^2 4\theta$ '. Some excellent solutions were seen from the more able candidates, which involved a variety of approaches including integration of $8\sin^2\theta\cos^3\theta$ by use of the substitution *s* = sinθ , direct integration by recalling that the integral of $\sin^n\theta\cos\theta$ with respect to θ is $\frac{Sin^{n+1}\theta}{n+1}$ or use of the identity 2cos4θ cosθ = cos5θ + cos3θ .Those candidates who used integration by parts were generally less successful. Only a few obtained the correct answer by applying integration by parts twice.

## Question 7:

$$i)\, Cosx+Sinx = 1-\frac{x^2}{2}+...+x-\frac{x^3}{6}+...$$

$$Cosx+Sinx = 1+x-\frac{x^2}{2}-\frac{x^3}{6}+...$$

$$ii)\, \ln(1+3x) = (3x)-\frac{(3x)^2}{2}+\frac{(3x)^3}{3}+...$$

$$=3x-\frac{9}{2}x^2+9x^3+...$$

$$b)\, y=e^{\tan x} \quad i)\,\frac{dy}{dx}=(1+Tan^2x)e^{\tan x}$$

$$\frac{d^2y}{dx^2}=2(1+Tan^2x)Tan(x)e^{\tan x}+(1+Tan^2x)^2e^{\tan x}$$

$$=2Tanx\frac{dy}{dx}+(1+Tan^2x)\frac{dy}{dx}=(1+2Tanx+Tan^2x)\frac{dy}{dx}$$

$$\frac{d^2y}{dx^2}=(1+Tanx)^2\frac{dy}{dx}$$

$$ii)\,\frac{d^3y}{dx^3}=2(1+Tan^2x)(1+Tanx)\frac{dy}{dx}+(1+Tanx)^2\frac{d^2y}{dx^2}$$

$$y(0)=1 \qquad y'(0)=1 \qquad y''(0)=y'(0)=1$$

$$y^{(3)}(0)=2+1=3$$

$$e^{\tan x}=1+x+\frac{1}{2}x^2+\frac{3}{6}x^3+... \qquad e^{\tan x}=1+x+\frac{1}{2}x^2+\frac{1}{2}x^3+...$$

$$c)\, e^{\tan x}-(Cosx+Sinx)=1+x+\frac{1}{2}x^2+\frac{1}{2}x^3-1-x+\frac{1}{2}x^2+\frac{1}{6}x^3+...$$

$$=x^2+\frac{2}{3}x^3+...$$

$$x\ln(1+3x)=3x^2-\frac{9}{2}x^3+...$$

$$so \quad \frac{e^{\tan x}-(Cosx+Sinx)}{x\ln(1+3x)}=\frac{x^2+\frac{2}{3}x^3+...}{3x^2-\frac{9}{2}x^3+...}=\frac{1+\frac{2}{3}x+...}{3-\frac{9}{2}x+...}$$

$$therefore \lim_{x\to 0}\frac{e^{\tan x}-(Cosx+Sinx)}{x\ln(1+3x)}=\frac{1}{3}$$

### Exam report

Most candidates were able to write down the required two expansions in parts (a).

In part (b)(i), candidates generally used the chain rule to find $\frac{dy}{dx}$ and then applied the product rule to find $\frac{d^2y}{dx^2}$. Although a common error was to differentiate $\sec^2 x$ as 2sec*x* tan*x*, those who did find $\frac{d^2y}{dx^2}$ correctly often produced a convincing solution to reach the printed answer. Some very good solutions were seen for parts (b)(ii) and (b)(iii) but it was disturbing to see some other candidates make no attempt to find the third derivative in part (b)(ii), yet claim that f′′′(0) = 3 and write down the printed result in part (b)(iii). Clearly in such cases marks cannot be awarded.

Candidates who attempted part (c) generally showed a thorough understanding of the process required, including the need for explicitly reaching the stage of a constant term in both the numerator and denominator before taking the limit as $x\to 0$ .

**Question 8:**

a) $x = e^t \quad \frac{dx}{dt} = e^t = x \;\; and \;\; \frac{dt}{dx} = \frac{1}{e^t} = \frac{1}{x}$

$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} = x\frac{dy}{dx}$

b) $\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{dx}{dt} \times \frac{d}{dx}\left(x\frac{dy}{dx}\right)$

$\frac{d^2y}{dt^2} = x\left(\frac{dy}{dx} + x\frac{d^2y}{dx^2}\right) = x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2}$

$x\frac{dy}{dx} = \frac{dy}{dt}$

$so \;\; x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$

The equation $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 2\ln x$

$becomes: \quad \frac{d^2y}{dt^2} - \frac{dy}{dt} - 3\frac{dy}{dt} + 4y = 2t$

$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 2t$

c) • The auxiliary equation :

$\lambda^2 - 4\lambda + 4 = 0$

$(\lambda - 2)^2 = 0$

*repeated root* $\lambda = 2$

The complementary function is

$y = (At + B)e^{2t}$

• The particular integral $y = at + b$

$\frac{dy}{dt} = a \;,\; \frac{d^2y}{dt^2} = 0$

$0 - 4a + 4at + 4b = 2t$

This gives $\begin{cases} 4a = 2 \\ 4b - 4a = 0 \end{cases}$ $so \;\; a = \frac{1}{2} \;\; and \;\; b = \frac{1}{2}$

The general solution is $y = \frac{1}{2}t + \frac{1}{2} + (At + B)e^{2t}$

d) $y = \frac{1}{2}\ln x + \frac{1}{2} + (A\ln x + B)x^2$

When $x = 1, \; y = \frac{3}{2} \;\; so \;\; \frac{3}{2} = 0 + \frac{1}{2} + B \quad B = 1$

$y = \frac{1}{2}\ln x + \frac{1}{2} + Ax^2\ln x + x^2$

$\frac{dy}{dx} = \frac{1}{2x} + 2Ax\ln x + Ax^2 \times \frac{1}{x} + 2x$

When $x = 1, \frac{dy}{dx} = \frac{1}{2} \quad so \;\; \frac{1}{2} = \frac{1}{2} + A + 2$

$A = -2$

*Conclusion*: $y = \frac{1}{2}\ln x + \frac{1}{2} + (-2\ln x + 1)x^2$

**Exam report**

In part (a), most candidates were able to convincingly show, by use of the chain rule, that $x\frac{dy}{dx} = \frac{dy}{dt}$.

Part (b), as expected, was not answered well by the average candidate. Those more able candidates who differentiated the result in part (a) — either with respect to *x*, writing $\frac{d}{dx}\left(\frac{dy}{dt}\right)$ as $\frac{dt}{dx}\frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{dt}{dx}\frac{d^2y}{dt^2}$, or with respect to *t*, writing $\frac{d}{dt}\left(x\frac{dy}{dx}\right) = \frac{dx}{dt}\frac{d}{dx}\left(x\frac{dy}{dx}\right)$ — normally scored all or most of the marks in part (b).

In part (c), although most candidates realised what was required, it was not uncommon to see *x* appear in general solutions. Many candidates correctly tried a particular integral of the form *at* + *b*, but a common error was to write 4*y* as 4*at* + *b* , which led to an incorrect particular integral. However, this error was classed as a slip and so some follow through was applied in marking part (d). In part (d), those candidates who had a general solution in part (c) that was entirely in terms of *t* were generally able to pick up at least the method marks in the final part of the question. A common error was to differentiate $\frac{1}{2}\ln x + \frac{1}{2}$ *as* $\frac{1}{2x} + \frac{1}{2}$, which led to the loss of the last two accuracy marks.


**Grade boundaries**

| Grade | | | A | B | C | D | E |
|---|---|---|---|---|---|---|---|
| Mark | Max 75 | | 66 | 59 | 52 | 45 | 38 |

**MFP3**

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| 1 | $k_1 = 0.1\times\left(3+\sqrt{4}\right)$ (=0.5) | M1 | | |
| | $k_2 = 0.1\,\mathrm{f}\,(3.1, 4.5)$ | M1 | | |
| | $k_2 = 0.1\times\left(3.1+\sqrt{4.5}\right) = 0.522132...$ | A1 | | PI accept 3dp or better |
| | $y(3.1) = y(3) + \frac{1}{2}[k_1 + k_2]$ $= 4 + 0.5 \times 1.022132....$ | m1 | | Dep on previous two Ms and numerical values for $k$'s |
| | $y(3.1) = 4.511$ | A1 | 5 | Must be 4.511 |
| | **Total** | | **5** | |
| 2(a) | $p\cos x - q\sin x + 5p\sin x + 5q\cos x = 13\cos x$ | M1 | | Differentiation and subst. into DE |
| | $p + 5q = 13;\quad 5p - q = 0$ | m1 | | Equating coeffs. |
| | $p = \frac{1}{2};\quad q = \frac{5}{2}$ | A1 | 3 | OE Need both |
| (b) | Aux. eqn. $m + 5 = 0$ | M1 | | PI. Or solving $y'(x)+5y=0$ as far as $y=$ |
| | $(y_{CF} =)Ae^{-5x}$ | A1 | | OE |
| | $(y_{GS} =)Ae^{-5x} + \frac{1}{2}\sin x + \frac{5}{2}\cos x$ | B1F | 3 | c's CF + c's PI with exactly one arbitrary constant OE |
| | **Total** | | **6** | |
| 3(a) | $r + r\cos\theta = 2$ | M1 | | |
| | $r + x = 2$ | B1 | | $r\cos\theta = x$ stated or used |
| | $r = 2 - x$ | A1 | | |
| | $x^2 + y^2 = (2-x)^2$ | M1 | | $r^2 = x^2 + y^2$ used |
| | $y^2 = 4 - 4x$ | A1 | 5 | Must be in the form $y^2 = \mathrm{f}(x)$ but accept ACF for f($x$). |
| (b) | Equation of line: $r\cos\theta = \frac{3}{4} \Rightarrow x = \frac{3}{4}$ | M1 | | Use of $r\cos\theta = x$ |
| | | A1 | | $4x=3$ OE |
| | $y^2 = 4 - 4\left(\frac{3}{4}\right) = 1 \Rightarrow y = \pm 1$; [Pts $\left(\frac{3}{4}, \pm 1\right)$] | M1 | | |
| | Distance between pts (0.75, 1) and (0.75, −1) is 2 | A1 | 4 | |
| | **Altn:** | | | |
| | At pts of intersection, $r = \frac{5}{4}$ and $\cos\theta = \frac{3}{5}$ OE | (M1A1) | | (M1 elimination of either $r$ or $\theta$) (For A condone slight prem approx.) |
| | Distance $PQ = 2r\sin\theta$ | (M1) | | Or use of cosine rule or Pythag. |
| | $= 2\times\frac{5}{4}\times\frac{4}{5} = 2$ | (A1) | | Must be from exact values. |
| | **Total** | | **9** | |

**MFP3(cont)**

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| 4 | IF is $e^{\int -\frac{2}{x}\,dx}$ | M1 | | Award even if negative sign missing |
| | $= e^{-2\ln(x)\ (+c)} = e^{\ln(x)^{-2}\ (+c)}$ | A1 | | OE Condone missing $c$ |
| | $= (k)x^{-2}$ | A1F | | Ft earlier sign error |
| | $x^{-2}\frac{dy}{dx} - 2x^{-3}y = 2xe^{2x}$ | | | |
| | $\frac{d}{dx}\left(x^{-2}y\right) = 2x\,e^{2x}$ | M1 | | LHS as d/d$x$($y$×IF) PI |
| | $x^{-2}y = \int 2x\,e^{2x}\,dx$ | | | |
| | $= \int x\ d(e^{2x}) = x\,e^{2x} - \int e^{2x}\,dx$ | M1 A1 | | Integration by parts in correct dirn |
| | $x^{-2}y = xe^{2x} - \frac{1}{2}e^{2x}\ (+c)$ | A1 | | ACF |
| | When $x = 2$, $y = e^4$ so $\frac{1}{4}e^4 = 2e^4 - \frac{1}{2}e^4 + c$ | m1 | | Boundary condition used to find $c$ after integration. |
| | $c = -\frac{5}{4}e^4$ | | | |
| | $y = x^3e^{2x} - \frac{1}{2}x^2e^{2x} - \frac{5}{4}x^2e^4$ | A1 | 9 | Must be in the form $y = \mathrm{f}(x)$ |
| | **Total** | | **9** | |

MFP3(cont)

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| 5(a) | $\frac{12x+8-12x-3}{(4x+1)(3x+2)}=\frac{5}{(4x+1)(3x+2)}$ | B1 | 1 | Accept $C=5$ |
| (b) | $\int\frac{10}{(4x+1)(3x+2)}\,dx=2\int\left(\frac{4}{4x+1}-\frac{3}{3x+2}\right)dx$ | M1 | | |
| | $=2\left[\ln(4x+1)-\ln(3x+2)\right]\ (+c)$ | A1 | | OE |
| | $I=\lim_{a\to\infty}\int_1^a\left(\frac{10}{(4x+1)(3x+2)}\right)dx$ | M1 | | $\infty$ replaced by $a$ and $\lim_{a\to\infty}$ (OE) |
| | $=2\lim_{a\to\infty}\left[\ln(4a+1)-\ln(3a+2)\right]-(\ln 5-\ln 5)$ | | | |
| | $=2\lim_{a\to\infty}\left[\ln\left(\frac{4a+1}{3a+2}\right)\right]=2\lim_{a\to\infty}\left[\ln\left(\frac{4+\frac{1}{a}}{3+\frac{2}{a}}\right)\right]$ | m1,m1 | | Limiting process shown. Dependent on the previous M1M1 |
| | $=2\ln\frac{4}{3}=\ln\frac{16}{9}$ | A1 | 6 | CSO |
| | **Total** | | 7 | |

MFP3(cont)

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| 6 | Area $=\frac{1}{2}\int\left(2\sin 2\theta\sqrt{\cos\theta}\right)^2\ d\theta$ | M1 | | Use of $\frac{1}{2}\int r^2\,d\theta$ |
| | $=\frac{1}{2}\int_0^{\frac{\pi}{2}}\left(4\cos\theta\sin^2 2\theta\right)\ d\theta$ | B1 | | $r^2=4\cos\theta\sin^2 2\theta$ or better |
| | | B1 | | Correct limits |
| | $=\frac{1}{2}\int_0^{\frac{\pi}{2}}\left(16\sin^2\theta\cos^3\theta\right)\ d\theta$ | M1 | | $\sin^2 2\theta=k\sin^2\theta\cos^2\theta\quad(k>0)$ |
| | $=\int_0^{\frac{\pi}{2}}\left(8\sin^2\theta\,(1-\sin^2\theta)\right)\ d\sin\theta$ | m1 | | Substitution or another valid method to integrate $\sin^2\theta\cos^3\theta$ |
| | $=\left[\frac{8\sin^3\theta}{3}-\frac{8\sin^5\theta}{5}\right]_0^{\frac{\pi}{2}}$ | A1F | | Correct integration of $p\sin^2\theta\cos^3\theta$ |
| | $=\left(\frac{8}{3}-\frac{8}{5}\right)-0=\frac{16}{15}$ | A1 | 7 | CSO AG |
| | **Alternatives for the last four marks** | | | |
| | Area $=\int_0^{\frac{\pi}{2}}\left(\cos\theta-\cos 4\theta\cos\theta\right)\ d\theta$ | (M1) | | $2\cos\theta\sin^2 2\theta=\lambda\cos\theta+\mu\cos 4\theta\cos\theta$ $(\lambda,\mu\neq 0)$ |
| | $\int\left(\cos 4\theta\cos\theta\right)\ d\theta$ | (m1) | | Integration by parts twice or use of $\cos 4\theta\cos\theta=\frac{1}{2}(\cos 5\theta+\cos 3\theta)$ |
| | $=-\frac{1}{15}(\cos 4\theta\sin\theta-4\sin 4\theta\cos\theta)$ | (A1F) | | Correct integration of $p\cos 4\theta\cos\theta$ [eg $p\left[\frac{1}{10}\sin 5\theta+\frac{1}{6}\sin 3\theta\right]$] |
| | Area $=(1-0)+\frac{1}{15}[(1-0)-(0)]=\frac{16}{15}$ | (A1) | | CSO AG $\{1-\frac{1}{10}+\frac{1}{6}=\frac{16}{15}\}$ |
| | **Total** | | 7 | |

MFP3(cont)

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| 7(a)(i) | $\cos x+\sin x=1+x-\frac{1}{2}x^2-\frac{1}{6}x^3$ | B1 | 1 | Accept coeffs unsimplified, even 3! for 6. |
| (ii) | $\ln(1+3x)=3x-\frac{1}{2}(3x)^2+\frac{1}{3}(3x)^3=3x-\frac{9}{2}x^2+9x^3$ | B1 | 1 | Accept coeffs unsimplified |
| (b)(i) | $y=e^{\tan x}$, $\frac{dy}{dx}=\sec^2 x\, e^{\tan x}$ | M1<br>A1 | | Chain rule<br>ACF eg $y\sec^2 x$ |
| | $\frac{d^2y}{dx^2}=2\sec^2 x\tan x\, e^{\tan x}+\sec^4 x\, e^{\tan x}$ | m1<br>A1 | | Product rule OE<br>ACF |
| | $=\sec^2 x\, e^{\tan x}(2\tan x+\sec^2 x)$<br>$-\frac{dy}{dx}(2\tan x+1+\tan^2 x)$ | | | |
| | $\frac{d^2y}{dx^2}=(1+\tan x)^2\frac{dy}{dx}$ | A1 | 5 | AG Completion; CSO any valid method. |
| (ii) | $\frac{d^3y}{dx^3}=2(1+\tan x)\sec^2 x\frac{dy}{dx}+(1+\tan x)^2\frac{d^2y}{dx^2}$ | M1 | | |
| | When $x=0$, $\frac{d^3y}{dx^3}=2(1)(1)(1)+(1)(1)=3$ | A1 | 2 | CSO |
| (iii) | $y(0)=1;\ y'(0)=1;\ y''(0)=1;\ y'''(0)=3;$<br>$y(x)\approx y(0)+xy'(0)+\frac{1}{2}x^2y''(0)+\frac{1}{3!}x^3y'''(0)$ | M1 | | |
| | $e^{\tan x}\approx 1+x+\frac{1}{2}x^2+\frac{1}{2}x^3$ | A1 | 2 | CSO AG |
| (c) | $\lim_{x\to 0}\left[\frac{e^{\tan x}-(\cos x+\sin x)}{x\ln(1+3x)}\right]$ | | | |
| | $=\lim_{x\to 0}\frac{1+x+\frac{x^2}{2}+\frac{x^3}{2}-1-x+\frac{x^2}{2}+\frac{x^3}{6}}{x\left(3x-\frac{9}{2}x^2+\ldots\right)}$ | M1 | | Using series expns. |
| | $=\lim_{x\to 0}\left[\frac{x^2+\frac{2}{3}x^3+..}{3x^2-\frac{9}{2}x^3...}\right]=\lim_{x\to 0}\left[\frac{1+\frac{2}{3}x+..}{3-\frac{9}{2}x...}\right]$ | m1 | | Dividing numerator and denominator by $x^2$ to get constant terms. OE following a slip. |
| | $=\frac{1}{3}$ | A1 | 3 | |
| | Total | | 14 | |

MFP3(cont)

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| 8(a) | $\frac{dx}{dt}\frac{dy}{dx}=\frac{dy}{dt}$ | M1 | | Chain rule |
| | $e^t\frac{dy}{dx}=\frac{dy}{dt}\Rightarrow x\frac{dy}{dx}=\frac{dy}{dt}$ | A1 | 2 | CSO AG |
| (b) | $\frac{d}{dt}\left(x\frac{dy}{dx}\right)=\frac{d^2y}{dt^2}$; $\frac{dx}{dt}\frac{d}{dx}\left(x\frac{dy}{dx}\right)=\frac{d^2y}{dt^2}$ | M1 | | OE $\frac{d}{dx}\left(x\frac{dy}{dx}\right)=\frac{dt}{dx}\frac{d^2y}{dt^2}$ |
| | $\frac{dx}{dt}\left(\frac{dy}{dx}+x\frac{d^2y}{dx^2}\right)=\frac{d^2y}{dt^2}$ | m1 | | Product rule (dep on previous M) |
| | $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}=\frac{d^2y}{dt^2}$ | A1 | | OE |
| | $x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+4y=2\ln x$ becomes | | | |
| | $\frac{d^2y}{dt^2}-x\frac{dy}{dx}-3x\frac{dy}{dx}+4y=2\ln x$ | | | |
| | $\Rightarrow\frac{d^2y}{dt^2}-4\frac{dy}{dt}+4y=2\ln e^t$ (using (a) | m1 | | |
| | $\Rightarrow\frac{d^2y}{dt^2}-4\frac{dy}{dt}+4y=2t$ | A1 | 5 | CSO AG |
| (c) | Auxl eqn $m^2-4m+4=0$ | M1 | | PI |
| | $(m-2)^2=0$, $m=2$ | A1 | | PI |
| | CF: $(y_C=)$ $(At+B)e^{2t}$ | M1 | | Ft wrong value of $m$ provided equal roots and 2 arb. constants in CF. Condone $x$ for $t$ here |
| | PI Try $(y_P=)$ $at+b$ | M1 | | If extras, coeffs. must be shown to be 0. |
| | $-4a+4at+4b=2t\Rightarrow a=b=\frac{1}{2}$ | A1 | | Correct PI. Condone $x$ for $t$ here |
| | GS $\{y\}=(At+B)e^{2t}+0.5(t+1)$ | B1F | 6 | Ft on c's CF + PI, provided PI is non-zero and CF has two arbitrary constants and RHS is fn of $t$ only |
| (d) | $\Rightarrow y=(A\ln x+B)x^2+0.5(\ln x+1)$ | M1 | | |
| | $y=1.5$ when $x=1\Rightarrow B=1$ | A1F | | Ft one earlier slip |
| | $y'(x)=(A\ln x+B)\,2x+Ax+0.5x^{-1}$ | m1 | | Product rule |
| | $y'(1)=0.5\Rightarrow A--2$ | A1F | | Ft one earlier slip |
| | $y=(1-2\ln x)x^2+\frac{1}{2}(\ln x+1)$ | A1 | 5 | ACF |
| | Total | | 18 | |
| | TOTAL | | 75 | |

# Notes about Jan 2011:

General Certificate of Education
Advanced Level Examination
June 2011

# Mathematics — MFP3

**Unit Further Pure 3**

**Thursday 16 June 2011 1.30 pm to 3.00 pm**

**For this paper you must have:**
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

**Time allowed**
- 1 hour 30 minutes

**Instructions**
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- Write the question part reference (eg (a), (b)(i) etc) in the left-hand margin.
- You must answer the questions in the spaces provided. Do not write outside the box around each page.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work that you do not want to be marked.

**Information**
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 75.

**Advice**
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

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**1** The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = f(x, y)$$

where $f(x, y) = x + \ln(1 + y)$

and $y(2) = 1$

Use the improved Euler formula

$$y_{r+1} = y_r + \frac{1}{2}(k_1 + k_2)$$

where $k_1 = hf(x_r, y_r)$ and $k_2 = hf(x_r + h, y_r + k_1)$ and $h = 0.2$, to obtain an approximation to $y(2.2)$, giving your answer to four decimal places. *(5 marks)*

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**2 (a)** Find the values of the constants $p$ and $q$ for which $p + qxe^{-2x}$ is a particular integral of the differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 4 - 9e^{-2x}$$ *(5 marks)*

**(b)** Hence find the general solution of this differential equation. *(3 marks)*

**(c)** Hence express $y$ in terms of $x$, given that $y = 4$ when $x = 0$ and that $\frac{dy}{dx} \to 0$ as $x \to \infty$. *(4 marks)*

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**3 (a)** Find $\int x^2 \ln x \, dx$. *(3 marks)*

**(b)** Explain why $\int_0^e x^2 \ln x \, dx$ is an improper integral. *(1 mark)*

**(c)** Evaluate $\int_0^e x^2 \ln x \, dx$, showing the limiting process used. *(3 marks)*

**4** By using an integrating factor, find the solution of the differential equation

$$\frac{dy}{dx} + (\cot x)y = \sin 2x, \quad 0 < x < \frac{\pi}{2}$$

given that $y = \frac{1}{2}$ when $x = \frac{\pi}{6}$. *(10 marks)*

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**5 (a)** Given that $y = \ln(1 + 2\tan x)$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

(You may leave your expression for $\frac{d^2y}{dx^2}$ unsimplified.) *(4 marks)*

**(b)** Hence, using Maclaurin's theorem, find the first two non-zero terms in the expansion, in ascending powers of $x$, of $\ln(1 + 2\tan x)$. *(2 marks)*

**(c)** Find

$$\lim_{x \to 0} \left[ \frac{\ln(1 + 2\tan x)}{\ln(1 - x)} \right]$$ *(4 marks)*

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**6** A differential equation is given by

$$(x^3 + 1)\frac{d^2y}{dx^2} - 3x^2\frac{dy}{dx} = 2 - 4x^3$$

**(a)** Show that the substitution

$$u = \frac{dy}{dx} - 2x$$

transforms this differential equation into

$$(x^3 + 1)\frac{du}{dx} = 3x^2 u$$ *(4 marks)*

**(b)** Hence find the general solution of the differential equation

$$(x^3 + 1)\frac{d^2y}{dx^2} - 3x^2\frac{dy}{dx} = 2 - 4x^3$$

giving your answer in the form $y = f(x)$. *(8 marks)*

**7** The curve $C_1$ is defined by $r = 2\sin\theta, \quad 0 \leqslant \theta < \frac{\pi}{2}$.

The curve $C_2$ is defined by $r = \tan\theta, \quad 0 \leqslant \theta < \frac{\pi}{2}$.

**(a)** Find a cartesian equation of $C_1$. *(3 marks)*

**(b) (i)** Prove that the curves $C_1$ and $C_2$ meet at the pole $O$ and at one other point, $P$, in the given domain. State the polar coordinates of $P$. *(4 marks)*

**(ii)** The point $A$ is the point on the curve $C_1$ at which $\theta = \frac{\pi}{4}$.

The point $B$ is the point on the curve $C_2$ at which $\theta = \frac{\pi}{4}$.

Determine which of the points $A$ or $B$ is further away from the pole $O$, justifying your answer. *(2 marks)*

**(iii)** Show that the area of the region bounded by the arc $OP$ of $C_1$ and the arc $OP$ of $C_2$ is $a\pi + b\sqrt{3}$, where $a$ and $b$ are rational numbers. *(10 marks)*

**END OF QUESTIONS**



AQA - FP3

# AQA – Further pure 3 – Jun 2011 – Answers

| Question 1: | Exam report |
|---|---|
| $\frac{dy}{dx} = f(x,y) = x + \ln(1+y) \quad and \quad y(2) = 1$<br><br>$k_1 = hf(x_r, y_r) = 0.2 \times (2 + \ln(1+1)) = 0.53863$<br>$y_0 + k_1 = 1 + 0.53863 = 1.53863$<br>$k_2 = 0.2 \times (2.2 + \ln(1 + 1.53863)) = 0.6263$<br><br>$y(2.2) = 1 + \frac{1}{2}(0.53863 + 0.62632) = 1.5825 \text{ to } 4d.p.$ | Numerical solutions of first order differential equations continue to be a good source of marks with most candidates showing full working with clear substitutions into relevant formulae. Unfortunately, more candidates than usual made arithmetical errors in calculating the value of $k_2$. Very few candidates failed to give their final answer to the specified degree of accuracy. |

| Question 2: | Exam report |
|---|---|
| a) A particular integral $y = p + qxe^{-2x}$<br>$\frac{dy}{dx} = qe^{-2x} - 2qxe^{-2x} = (q - 2qx)e^{-2x}$<br>$\frac{d^2y}{dx^2} = -2qe^{-2x} - 2qe^{-2x} + 4qxe^{-2x}$<br>$= (-4q + 4qx)e^{-2x}$<br>$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 4 - 9e^{-2x}$<br>$(-4q + 4qx)e^{-2x} + (q - 2qx)e^{-2x} - 2p - 2qxe^{-2x} = 4 - 9e^{-2x}$<br>$(-3q)e^{-2x} - 2p = 4 - 9e^{-2x}$<br>$so \ -3q = -9 \ and \ -2p = 4$<br>$q = 3 \ and \ p = -2$<br>$y_{PI} = -2 + 3xe^{-2x}$<br>• The auxiliary equation is<br>$\lambda^2 + \lambda - 2 = 0$<br>$(\lambda + 2)(\lambda - 1) = 0 \qquad \lambda = -2 \ or \ \lambda = 1$<br>$y_c = Ae^{-2x} + Be^x$<br>• The general solution is $y = (3x + A)e^{-2x} + Be^x - 2$<br>c) When $x = 0, y = 4$ so $4 = A + B - 2 \qquad A + B = 6$<br>$\frac{dy}{dx} = 2e^{-2x} - 2(3x + A)e^{-2x} + Be^x$<br>When $x \to \infty, \frac{dy}{dx} \to 0$. This means that $B = 0$ $(and \ A = 6)$<br>*because* if $B \neq 0, Be^x \xrightarrow[x \to \infty]{} \infty$<br>*Note that* $(3x + A)e^{-2x} = 3xe^{-2x} + Ae^{-2x} \xrightarrow[x \to \infty]{} 0$<br>*Solution*: $y = (3x + 6)e^{-2x} - 2$ | Most candidates were able to find the correct values of the constants *p* and *q* so that the given expression was a particular integral of the differential equation. An even higher proportion of candidates scored full marks for their general solution in part (b) as examiners allowed follow through on incorrect values for *p* and *q*. However, less than 20% of the candidates scored full marks for their answers to part (c). This was almost entirely due to no explicit consideration being given to the value of $xe^{-2x}$ as $x \to \infty$. More surprisingly, a significant minority of candidates could not correctly deal with the boundary condition $\frac{dy}{dx} \to 0 \ as \ x \to \infty$ as they had an $e^x$ term in their final answer. |

| Question 3: | Exam report |
| --- | --- |
| a) $\int x^2 \ln x\,dx = \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^3 \times \frac{1}{x}dx$ <br> $= \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^2 dx$ <br> $= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + c$ <br> b) $\int_0^e x^2 \ln x dx$ is an improper integral because the function $f(x) = x^2 \ln x$ is not defined for $x = 0$. <br> c) $\int_a^e x^2 \ln x\,dx = \left[\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3\right]_a^e =$ <br> $= \left(\frac{1}{3}e^3 \times 1 - \frac{1}{9}e^3\right) - \left(\frac{1}{3}a^3 \ln a - \frac{1}{9}a^3\right)$ <br> $= \frac{2}{9}e^3 - \left(\frac{1}{3}a^3 \ln a - \frac{1}{9}a^3\right)$ <br> $\lim_{a\to 0} a^3 \ln a = 0$ *and* $\lim_{a\to 0} a^3 = 0$ *so* <br> $\int_0^{e^2} x^2 \ln x dx$ *exists and* $\int_0^{e^2} x^2 \ln x dx = \frac{2}{9}e^3$ | Part (a) was, as expected, very well answered with candidates clearly showing the method of integration by parts. In part (b), a significant number of candidates were not explicit in explaining that the integrand was not defined at $x = 0$. It was pleasing to see in part (c) that more candidates than usual replaced the lower limit by, for example, $a$, carried out the integration and then considered the limiting process as $a \to 0$. However, a significant number of candidates did not specifically consider $\lim_{a\to 0} a^3 \ln a$. |

| Question 4: | Exam report |
| --- | --- |
| $\frac{dy}{dx} + Cot(x) \times y = Sin2x \quad 0 < x < \frac{\pi}{2}$ <br> An integrating factor is <br> $I = e^{\int Cot(x)dx} = e^{\ln\|Sinx\|} = Sinx$ <br> $(because \text{ for } 0 < x < \frac{\pi}{2}, \sin x > 0)$ <br> The equation becomes: <br> $Sin(x)\frac{dy}{dx} + Cos(x)y = Sin(2x)Sin(x)$ <br> $\frac{d}{dx}(y \times Sinx) = 2Cos(x)Sin^2(x)$ <br> $y \times Sinx = 2\int Cos(x)Sin^2(x)dx$ <br> $\int f' \times f^2 = \frac{1}{3}f^3$ <br> $y \times Sin(x) = 2 \times \frac{1}{3}Sin^3 x + c$ <br> $y = \frac{2}{3}Sin^2 x + \frac{c}{Sin(x)}$ <br> When $x = \frac{\pi}{6}, y = \frac{1}{2}$ *so* $\frac{1}{2} = \frac{2}{3}\left(\frac{1}{2}\right)^2 + 2c$ <br> $\frac{1}{2} = \frac{1}{6} + 2c \qquad c = \frac{1}{6}$ <br> *Solution*: $y = \frac{2}{3}Sin^2 x + \frac{1}{6Sin(x)}$ | This question was answered well with most candidates able to show that they knew how to find and use a correct integrating factor to solve a first order differential equation. The majority of candidates then correctly used substitution or inspection to complete the integration. Those candidates who tried to integrate by parts often made errors or did not appreciate that two applications were required. The minority of candidates who correctly expressed sin2$x$ sin$x$ as the difference of two cosines generally had less difficulty in completing the integration correctly. |

**Question 5:**

$a)\ y=\ln(1+2Tanx)\quad \left(if\ \ y=\ln(u)\ then\ \frac{dy}{dx}=\frac{u'}{u}\right)$

$$\frac{dy}{dx}=\frac{2Sec^2x}{1+2Tanx}$$

Let's call $v=2Sec^2x=2Cos^{-2}(x)$ $then$

$$\frac{dv}{dx}=2\times-2\times-Sin(x)\times Cos^{-3}(x)$$

$$\frac{d^2y}{dx^2}=\frac{\frac{4Sinx}{Cos^3x}(1+2Tanx)-2Sec^2x\left(2Sec^2x\right)}{(1+2Tanx)^2}$$

$b)\ y(0)=\ln(1+0)=0$

$$y'(0)=\frac{2}{1}=2\ \ and\ \ y''(0)=\frac{0-4}{1}=-4$$

$$\ln(1+2Tanx)=0+2x-\frac{4}{2}x^2+\ldots$$

$$\ln(1+2Tanx)=0+2x-2x^2+\ldots$$

$c)\ \ln(1-x)=-x-\frac{1}{2}x^2+\ldots$

$$\frac{\ln(1+2Tanx)}{\ln(1-x)}=\frac{2x-2x^2+\ldots}{-x-\frac{1}{2}x^2+\ldots}=\frac{2-2x+\ldots}{-1-\frac{1}{2}x+\ldots}$$

$$so\ \lim_{x\to 0}\frac{\ln(1+2Tanx)}{\ln(1-x)}=-2$$

**Exam report**

A majority of candidates were able to differentiate the given function correctly to find the first and second differentials although some made the second differentiation more complicated than was needed by first expressing their answer for the first derivative in terms of sin2*x* and cos2*x*. Candidates should be aware that the expression for the derivative of sec *x* is in the formulae booklet.

In part (b), most candidates displayed good knowledge of Maclaurin's theorem but only those who had made no errors in earlier differentiations could score both marks for showing the printed result. It is pleasing to report that there was an improvement in candidates explicitly reaching the stage of a constant term with higher order terms in each of the numerator and denominator before taking the limit as $x\to 0$ in the final part of the question.

**Question 6:**

$a)\ u=\frac{dy}{dx}-2x\ \ so\ \frac{dy}{dx}=u+2x$

$$\frac{du}{dx}=\frac{d^2y}{dx^2}-2\ \ so\ \frac{d^2y}{dx^2}=\frac{du}{dx}+2$$

By substituting in the equation:

$$\left(x^3+1\right)\left(\frac{du}{dx}+2\right)-3x^2\left(u+2x\right)=2-4x^3$$

$$x^3\frac{du}{dx}+2x^3+\frac{du}{dx}+2-3x^2u-6x^3=2-4x^3$$

$$\left(x^3+1\right)\frac{du}{dx}-3x^2u=0\ \ or\ \ \left(x^3+1\right)\frac{du}{dx}=3x^2u$$

$b)$ Separating the variables:

$$\frac{1}{u}\frac{du}{dx}=\frac{3x^2}{(x^3+1)}\qquad \int\frac{1}{u}du=\int\frac{3x^2}{(x^3+1)}dx$$

$$\ln u=\ln\left(x^3+1\right)+c$$

Let's Write $c=\ln A$

$$\ln u=\ln(A(x^3+1))\ \ and\qquad u=A(x^3+1)$$

$$u=\frac{dy}{dx}-2x=A(x^3+1)$$

$$\frac{dy}{dx}=Ax^3+A+2x$$

$$y=\frac{1}{4}Ax^4+Ax+x^2+B$$

**Exam report**

Part (a) was generally answered correctly, with most candidates showing sufficient detail in reaching the printed result. Those candidates who attempted to integrate using the method of separation of variables were generally more successful than those who attempted to use an integrating factor; a missing negative sign in the initial statement of the integrating factor involving an integral was the most common error. However, it was disappointing to see that a large number of candidates could not deal correctly with the constant of integration when simplifying to *u* = g(*x*). A number of candidates also then did not include a second arbitrary constant when integrating to find *y* = f(*x*) and so it was relatively common to see general solutions to the second order differential equation not containing two arbitrary constants.

**Question 7:**

$C_1 : r = 2Sin\theta \qquad 0 \le \theta < \frac{\pi}{2}$

$C_2 : r = Tan\theta \qquad 0 \le \theta < \frac{\pi}{2}$

$a)\ r = 2Sin\theta$

$r^2 = 2rSin\theta$

$x^2 + y^2 = 2y \ \ or \ \ x^2 = 2y - y^2$

$b)i)$ Solving simultaneously:

$r = 2Sin\theta = Tan\theta$

$$2Sin\theta - \frac{Sin\theta}{Cos\theta} = 0$$

$$Sin\theta(2 - \frac{1}{Cos\theta}) = 0$$

$so \ \ Sin\theta = 0 \ or \ Cos\theta = \frac{1}{2}$

$\theta = 0 \ \ or \ \ \theta = \frac{\pi}{3}$

When $\theta = 0,\ r = Tan\theta = 0 \qquad O(0,0)$

When $\theta = \frac{\pi}{3}, r = Tan\theta = \sqrt{3} \qquad P(\sqrt{3}, \frac{\pi}{3})$

$ii)$ When $\theta = \frac{\pi}{4}, r = 2Sin\theta = \sqrt{2} \qquad A(\sqrt{2}, \frac{\pi}{4})$

When $\theta = \frac{\pi}{4}, r = Tan\theta = 1 \qquad B(1, \frac{\pi}{4})$

$\sqrt{2} > 1$, $A$ is further away from the pole.

$iii)$ • Area bounded by the arc OP of $C_1$ *is*

$$A_1 = \int_0^{\frac{\pi}{3}} \frac{1}{2}(2Sin\theta)^2 d\theta = \int_0^{\frac{\pi}{3}} 2Sin^2\theta d\theta = \int_0^{\frac{\pi}{3}} 1 - Cos2\theta d\theta$$

$$A_1 = \left[\theta - \frac{1}{2}Sin2\theta\right]_0^{\frac{\pi}{3}} = \left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right) - (0) = \frac{\pi}{3} - \frac{\sqrt{3}}{4}$$

• Area bounded by the arc OP of $C_2$ *is*

$$A_2 = \int_0^{\frac{\pi}{3}} \frac{1}{2} Tan^2\theta d\theta = \frac{1}{2}\int_0^{\frac{\pi}{3}} 1 + Tan^2\theta - 1 d\theta$$

$$A_2 = \frac{1}{2}[Tan\theta - \theta]_0^{\frac{\pi}{3}} = \frac{\sqrt{3}}{2} - \frac{\pi}{6}$$

The area between the two curves is the difference between $A_1 and\ A_2$

$$A = A_1 - A_2 = \frac{\pi}{3} - \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{2} + \frac{\pi}{6} = \frac{\pi}{2} - \frac{3\sqrt{3}}{4}$$

**Exam report**

This final question tested polar coordinates and proved to be the most demanding question on the paper. In part (a), it was pleasing to see that the majority of candidates were able to use a correct method to find the cartesian equation. Those candidates who attempted to square both sides of the polar equation were generally less successful than those who started by multiplying both sides by *r*. In part (b)(i), a large number of candidates only considered the solution to cosθ = 0.5 when solving 2sinθ = tanθ and so failed to prove that the curves only met at the two points. Those candidates who attempted to verify that the curves intersected at the pole often substituted θ = 0 rather than *r* = 0 into the two equations, but then often failed to give a final statement. In part (b)(ii), candidates generally substituted $\theta = \frac{\pi}{4}$ correctly into the two equations but a significant minority then did not give a full justification of whether *A* or *B* was further away from the pole. Although only the better candidates understood the relevance of part (b)(ii) in finding the required area in part (b)(iii), many candidates were able to make good progress in this part of the question. It was, however, disappointing to see some solutions that did not even use the correct formula for the area of a sector in polar coordinates. Those candidates who applied the correct formula were often able to write $\sin^2\theta$ in terms of cos2θ and carried out the subsequent integration correctly to find the area bounded by $C_1$ and the line segment *OP*. A significant number of candidates, however, could not find a correct method to integrate $\tan^2\theta$. The most successful method was to use the identity $\tan^2\theta = \sec^2\theta - 1$ although some other candidates found the correct area bounded by $C_2$ and the line segment *OP* by using integration by parts. Although a significant number of those candidates who obtained the correct values for the two areas went on to subtract these values to obtain the correct final answer, some others added their correct values and so failed to gain the final two marks.


**Grade boundaries**

| Grade | | A* | A | B | C | D | E |
|---|---|---|---|---|---|---|---|
| Mark | Max 75 | 69 | 64 | 55 | 46 | 38 | 30 |

MFP3

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| 1 | $k_1 = 0.2\times[2+\ln(1+1)]$ | M1 | | PI. May be seen within given formula |
| | $= 0.5386(29\ldots)$ (= *) | | | Accept 3sf rounded or truncated or better as evidence of the M1 line |
| | $k_2 = 0.2\times f(2.2,\ 1+*\ldots)$ | | | |
| | $\ldots = 0.2\times[2.2 + \ln(1+1.5386\ldots)]$ | M1 | | $0.2\times[2.2+\ln(1+1+\text{c's } k_1)]$. PI May be seen within given formula |
| | $\ldots. = 0.6263(248\ldots.)$ | A1 | | 4dp or better. PI by later work |
| | $y(2.2) = y(2) + \frac{1}{2}[k_1+k_2]$ | | | |
| | $= 1 + 0.5\times[0.5386\ldots + 0.6263\ldots]$ | m1 | | Dep on previous two Ms but ft on c's numerical values (or numerical expressions) for *k*'s following evaluation of these. |
| | $= 1 + 0.5\times1.16495\ldots.$ | | | |
| | $(= 1.582477\ldots) = 1.5825$ to 4dp | A1 | 5 | CAO Must be 1.5825 |
| | | | | **SC** For those scoring M1M0 who have $k_2$=0.5261(78..), and final answer 1.5324 (ie 4 dp) for y(2.2) award a total of 2 marks [M1B1] |
| | Total | | 5 | |

MFP3 (cont)

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| 2(a) | PI: $y_{PI} = p + qx\mathrm{e}^{-2x}$ | | | |
| | $y'_{PI} = q\mathrm{e}^{-2x} - 2qx\mathrm{e}^{-2x}$ | M1 | | Product rule used |
| | $y''_{PI} = -4q\mathrm{e}^{-2x} + 4qx\mathrm{e}^{-2x}$ | | | |
| | $-4q\mathrm{e}^{-2x} + 4qx\mathrm{e}^{-2x} + q\mathrm{e}^{-2x} - 2qx\mathrm{e}^{-2x}$ $-2p - 2qx\mathrm{e}^{-2x} -\ 4 - 9\mathrm{e}^{-2x}$ | M1 | | Subst. into DE |
| | $-3q = -9$ and $-2p = 4$ | m1 | | Equating coefficients |
| | $-3q = -9$ so $q = 3$; | A1 | | |
| | $-2p = 4$ so $p = -2$; | B1 | 5 | |
| | $[\,y_{PI} = 3x\mathrm{e}^{-2x} - 2\,]$ | | | |
| (b) | Aux. eqn. $m^2 + m - 2 = 0$ | | | |
| | $(m-1)(m+2) = 0$ | M1 | | Factorising or using quadratic formula OE PI by correct two values of '*m*' seen/used |
| | $y_{CF} = A\mathrm{e}^{x} + B\mathrm{e}^{-2x}$ | A1 | | |
| | $y_{GS} = A\mathrm{e}^{x} + B\mathrm{e}^{-2x} + 3x\mathrm{e}^{-2x} - 2$ | B1F | 3 | $(y_{GS})$ = c's CF + c's PI, provided 2 arbitrary constants |
| (c) | $x = 0,\ y = 4 \Rightarrow 4 = A + B - 2$ | B1F | | Only ft if exponentials in GS |
| | $\frac{\mathrm{d}y}{\mathrm{d}x} = A\mathrm{e}^{x} - 2B\mathrm{e}^{-2x} + 3\mathrm{e}^{-2x} - 6x\mathrm{e}^{-2x}$ | | | |
| | As $x \to \infty$, $(\mathrm{e}^{-2x} \to 0$ and) $x\mathrm{e}^{-2x} \to 0$ | E1 | | |
| | As $x \to \infty$, $\frac{\mathrm{d}y}{\mathrm{d}x} \to 0$ so $A = 0$ | B1 | | |
| | When $A = 0$, $4 = 0 + B - 2 \Rightarrow B = 6$ | | | |
| | $y = 6\mathrm{e}^{-2x} + 3x\mathrm{e}^{-2x} - 2$ | B1 | 4 | $y = 6\mathrm{e}^{-2x} + 3x\mathrm{e}^{-2x} - 2$ OE |
| | Total | | 12 | |

MFP3 (cont)

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| 3(a) | $\int x^2 \ln x \, dx = \frac{x^3}{3}\ln x - \int \frac{x^3}{3}\left(\frac{1}{x}\right) dx$ | M1 | | $\ldots - kx^3 \ln x \pm \int f(x)$, with f(x) not involving the 'original' $\ln x$ |
| | | A1 | | |
| | $\ldots\ldots - \frac{x^3}{3}\ln x - \frac{x^3}{9}\ (+c)$ | A1 | 3 | Condone absence of '$+c$' |
| (b) | Integrand is not defined at $x = 0$ | E1 | 1 | OE |
| (c) | $\int_0^e x^2 \ln x \, dx = \left\{\lim_{a\to 0} \int_a^e x^2 \ln x \, dx\right\}$ | | | OE |
| | $= \left(\frac{e^3}{3}\ln e - \frac{e^3}{9}\right) - \lim_{a\to 0}\left[\frac{a^3}{3}\ln a - \frac{a^3}{9}\right]$ | M1 | | $F(e) - \lim_{a\to 0}[F(a)]$ |
| | But $\lim_{a\to 0} a^3 \ln a = 0$ | E1 | | Accept a general form eg $\lim_{x\to 0} x^k \ln x = 0$ |
| | So $\int_0^e x^2 \ln x \, dx = \frac{2e^3}{9}$ | A1 | 3 | CSO |
| | **Total** | | **7** | |
| 4 | $\frac{dy}{dx} + (\cot x)y = \sin 2x$ | | | |
| | IF is $\exp\left(\int \cot x \, dx\right)$ | M1 | | and with integration attempted |
| | $= e^{\ln(\sin x)\ (+c)}$ | A1 | | OE Condone missing '$+c$' |
| | $= (k)\sin x$ | A1 | | 'IF = $\sin x$' scores M1A1A1 |
| | $\sin x\frac{dy}{dx} + (\cos x)y = \sin 2x \sin x$ | | | |
| | $\frac{d}{dx}[y\sin x] = \sin 2x \sin x$ | M1 | | LHS as differential of $y\times$IF PI |
| | $y\sin x = \int \sin 2x \sin x \, dx$ | A1F | | Ft on c's IF provided no exp. or logs |
| | $\Rightarrow y\sin x = \int 2\sin^2 x \cos x \, dx$ | B1 | | $\sin 2x = 2\sin x\cos x$ used |
| | $\Rightarrow y\sin x - \int 2\sin^2 x \, d(\sin x)$ | m1 | | dep on both Ms<br>Use of relevant substitution to stage $\int 2s^2 ds$ or further or by inspection to $k\sin^3 x$ |
| | $y\sin x - \frac{2}{3}\sin^3 x\ (+c)$ | A1 | | ACF |
| | $\frac{1}{2}\sin\frac{\pi}{6} = \frac{2}{3}\sin^3\frac{\pi}{6} + c$ | m1 | | dep on both Ms<br>Boundary condition used in attempt to find value of $c$ after integration |
| | $c = \frac{1}{6}$ so $y\sin x = \frac{2}{3}\sin^3 x + \frac{1}{6}$ | A1 | 10 | CSO – no errors seen – accept equivalent forms |
| | **Total** | | **10** | |

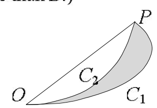
MFP3 (cont)

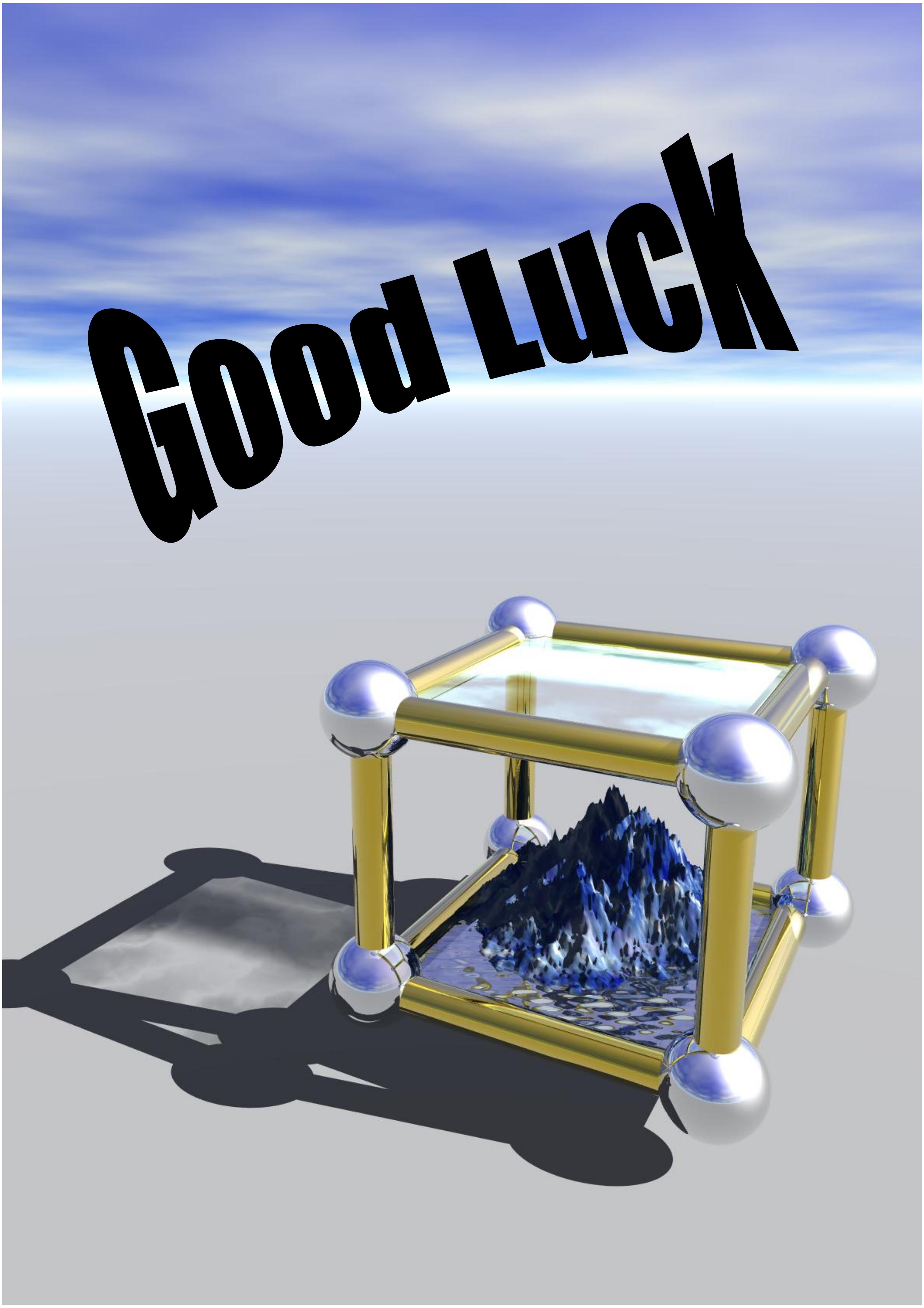
| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| 5(a) | $\frac{dy}{dx} = \frac{2\sec^2 x}{1 + 2\tan x}$ | M1<br>A1 | | Chain rule<br>ACF for $y'(x)$ |
| | $\frac{d^2y}{dx^2} = \frac{(1+2\tan x)(4\sec^2 x\tan x) - 2\sec^2 x(2\sec^2 x)}{(1+2\tan x)^2}$ | M1<br>A1 | 4 | Quotient rule OE in which both $u$ and $v$ are not const. or applied to a correct form of $y'$<br>ACF for $y''(x)$ |
| (b) | McC. Thm: $y(0) + xy'(0) + \frac{x^2}{2}y''(0)$ | | | |
| | $(y(0) = 0)$; $y'(0) = 2$; $y''(0) = -4$ | M1 | | Attempt to evaluate at least $y'(0)$ and $y''(0)$. PI |
| | $\ln(1 + 2\tan x) \approx 2x - 2x^2$ | A1 | 2 | Dep on previous 5 marks |
| (c) | $\ln(1-x) = -x - \frac{1}{2}x^2\ldots$ | B1 | | Ignore higher power terms |
| | $\left[\frac{\ln(1+2\tan x)}{\ln(1-x)}\right] \approx \frac{2x - 2x^2\ldots}{-x - \frac{1}{2}x^2\ldots}$ | M1 | | Expansions used |
| | $= \frac{2 - 2x\ldots}{-1 - \frac{1}{2}x\ldots}$ | m1 | | Dividing num. and den. by $x$ to get constant term in each and non-const term in at least num. or den. |
| | So $\lim_{x\to 0}\left[\frac{\ln(1+2\tan x)}{\ln(1-x)}\right] = \frac{2}{-1} = -2$ | A1F | 4 | ft c's answer to (b) provided answer (b) is in the form $\pm px \pm qx^2\ldots$ and B1 awarded |
| | **Total** | | **10** | |

MFP3 (cont)

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| 6(a) | $u=\frac{dy}{dx}-2x \Rightarrow \frac{du}{dx}=\frac{d^2y}{dx^2}-2$ | M1<br>A1 | | Differentiating subst wrt $x$, $\geq$ two terms correct |
| | DE becomes | | | |
| | $(x^3+1)(\frac{du}{dx}+2)-3x^2(u+2x)=2-4x^3$ | M1 | | Substitute into LHS of DE as far as no $y$s |
| | $(x^3+1)\frac{du}{dx}+2x^3+2-3x^2u-6x^3=2-4x^3$ | | | |
| | DE becomes $(x^3+1)\frac{du}{dx}=3x^2u$ | A1 | 4 | CSO AG |
| (b) | $\int \frac{1}{u}\,du=\int \frac{3x^2}{x^3+1}\,dx$ | M1 | | Separate variables OE PI |
| | $\ln u=\ln(x^3+1)+\ln A$ | A1;A1 | | $\ln u$; $\ln(x^3+1)$ |
| | | A1F | | Applying law of logs to correctly combine two log terms or better |
| | $u=A(x^3+1)$ | A1 | | OE RHS |
| | $\frac{dy}{dx}=A(x^3+1)+2x$ | m1 | | $u=f(x)$ to $\frac{dy}{dx}=\pm f(x)\pm 2x$ |
| | $y=A\left(\frac{x^4}{4}+x\right)+x^2+B$ | m1<br>A1 | 8 | Solution with two arbitrary constants and both previous M and m scored<br>OE RHS |
| | Total | | 12 | |

MFP3 (cont)

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| 7(a) | $r=2\sin\theta \Rightarrow r^2=2r\sin\theta$<br>$x^2+y^2=2y$ | M1<br>A2,1 | 3 | OE (A1) either for $r^2=x^2+y^2$ or for $r\sin\theta=y$<br>**SC** If M0 give B1 for $r^2=x^2+y^2$ or for $r\sin\theta=y$ used |
| (b)(i) | $2\sin\theta=\tan\theta$<br>$2\sin\theta\cos\theta=\sin\theta$<br>$\sin\theta(2\cos\theta-1)=0$ | M1<br>m1 | | Equating $r$s<br>Both solutions have to be considered if not in factorised form |
| | $\sin\theta=0\Rightarrow\theta=0$; $\cos\theta=\frac{1}{2}\Rightarrow\theta=\frac{\pi}{3}$ | | | Alternative: $\sin 2\theta=\sin\theta\Rightarrow\theta=0,\frac{\pi}{3}$ |
| | $\theta=0\Rightarrow r=0$ ie pole $O$ (0,0) | B1 | | Indep. Can just verify using both eqns +statement. |
| | $\theta=\frac{\pi}{3}\Rightarrow r=\sqrt{3}$ $\left(P\left(\sqrt{3},\frac{\pi}{3}\right)\right)$ | A1 | 4 | CSO |
| (ii) | At $A$, $\theta=\frac{\pi}{4}$, $r=2\sin\frac{\pi}{4}=\sqrt{2}$<br>At $B$, $\theta=\frac{\pi}{4}$, $r=\tan\frac{\pi}{4}=1$ | M1 | | Substitute $\theta=\frac{\pi}{4}$ into the equations of both curves. |
| | Since $\sqrt{2}>1$, $A$ is further away (from the pole than $B$.) | E1 | 2 | CSO |
| (iii) |  | | | |
| | Area bounded by line $OP$ and curve $C_1$ | | | |
| | $=\frac{1}{2}\int_0^{\frac{\pi}{3}} 4\sin^2\theta\,d\theta$ | M1 | | Use of $\frac{1}{2}\int r^2\,d\theta$; ignore limits here |
| | $=\int(1-\cos 2\theta)\,d\theta$ | m1 | | Attempt to write $\sin^2\theta$ in terms of $\cos2\theta$ only |
| | $-\left[\theta-\frac{1}{2}\sin 2\theta\right]$ | A1 | | Ignore limits here |
| | $-\left(\frac{\pi}{3}-\frac{1}{2}\times\frac{\sqrt{3}}{2}\right)-0-\frac{\pi}{3}-\frac{\sqrt{3}}{4}$ | A1 | | PI |
| | Area bounded by line $OP$ and curve $C_2$ | | | |
| | $=\frac{1}{2}\int_0^{\frac{\pi}{3}} \tan^2\theta\,d\theta$ | M1 | | Use of $\frac{1}{2}\int r^2\,d\theta$; ignore limits here |
| | $-\frac{1}{2}\int(\sec^2\theta-1)\,d\theta$ | m1 | | Using $\tan^2\theta=\pm\sec^2\theta\pm1$ PI |
| | $=\frac{1}{2}[\tan\theta-\theta]$ | A1 | | Ignore limits here |
| | $=\frac{1}{2}\left(\sqrt{3}-\frac{\pi}{3}\right)-0=\frac{\sqrt{3}}{2}-\frac{\pi}{6}$ | A1 | | PI |
| | Required area $=\left(\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right)-\left(\frac{\sqrt{3}}{2}-\frac{\pi}{6}\right)$ | M1 | | Can award earlier eg if we see $\frac{1}{2}\int_0^{\frac{\pi}{3}} 4\sin^2\theta\,d\theta-\frac{1}{2}\int_0^{\frac{\pi}{3}}\tan^2\theta\,d\theta$ |
| | $=\frac{1}{2}\pi-\frac{3}{4}\sqrt{3}$ $\left(a=\frac{1}{2},\ b=-\frac{3}{4}\right)$ | A1 | 10 | CSO |
| | Total | | 19 | |
| | TOTAL | | 75 | |


Good Luck