De Moivre's theorem

Abraham de Moivre (1667 - 1754)

Specifications:

De Moivre's Theorem

De Moivre's theorem for integral n.	Use of $z + \frac{1}{z} = 2\cos\theta$ and $z - \frac{1}{z} = 2\,\mathrm{i}\sin\theta$, leading to, for example, expressing $\sin^5\theta$ in terms of multiple angles and $\tan 5\theta$ in term of powers of $\tan\theta$. Applications in evaluating integrals, for example, $\int \sin^5\theta\,\mathrm{d}\theta$.
De Moivre's theorem; the nth roots of unity, the exponential form of a complex number.	The use, without justification, of the identity $e^{\mathrm{i}x} = \cos x + \mathrm{i}\sin x$.
Solutions of equations of the form $z^n = a + \mathrm{i}b$.	To include geometric interpretation and use, for example, in expressing $\cos\frac{5\pi}{12}$ in surd form.

De Moivre's theorem

Consider a complex number z with modulus 1 in its trigonometric form:
$z = Cos\theta + iSin\theta$

De Moivre's theorem :

$$(Cos\theta + iSin\theta)^n = Cos(n\theta) + iSin(n\theta) \text{ for all n positive integers}$$

Prove this theorem by induction:

Generalisation:

Given $z = r\left(Cos\theta + iSin\theta\right)$

then for all n integers, $z^n = r^n\left(Cos(n\theta) + iSin(n\theta)\right)$

Powers of a complex number

It is given that $z = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$ *and* $w = 1 + i$

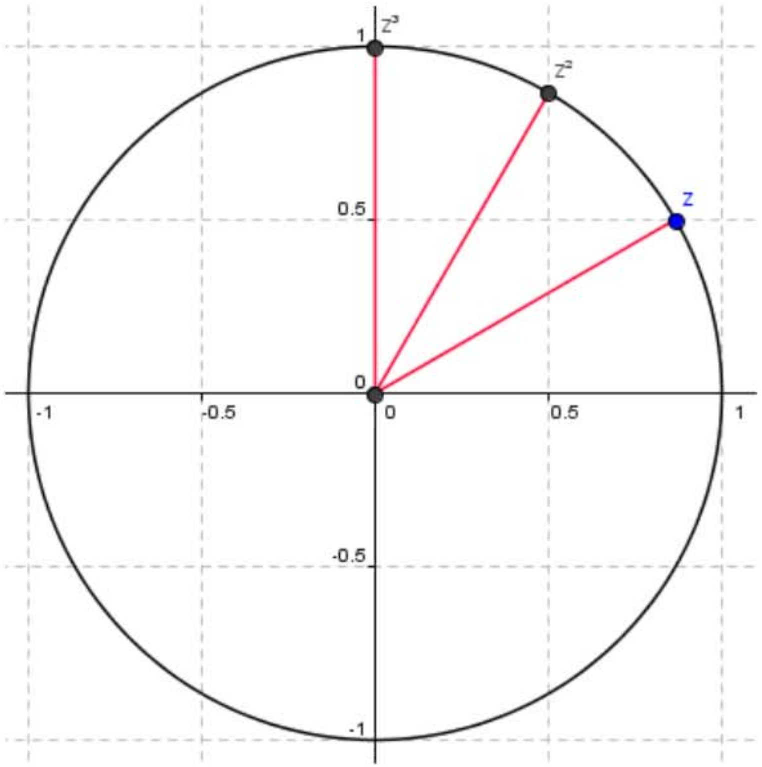
Using the De Moivre's theorem, work out in the form a+ib the following complex numbers

1) z^2 , z^3 , z^{10} , z^{600}

2) w^2 , w^3 , w^8

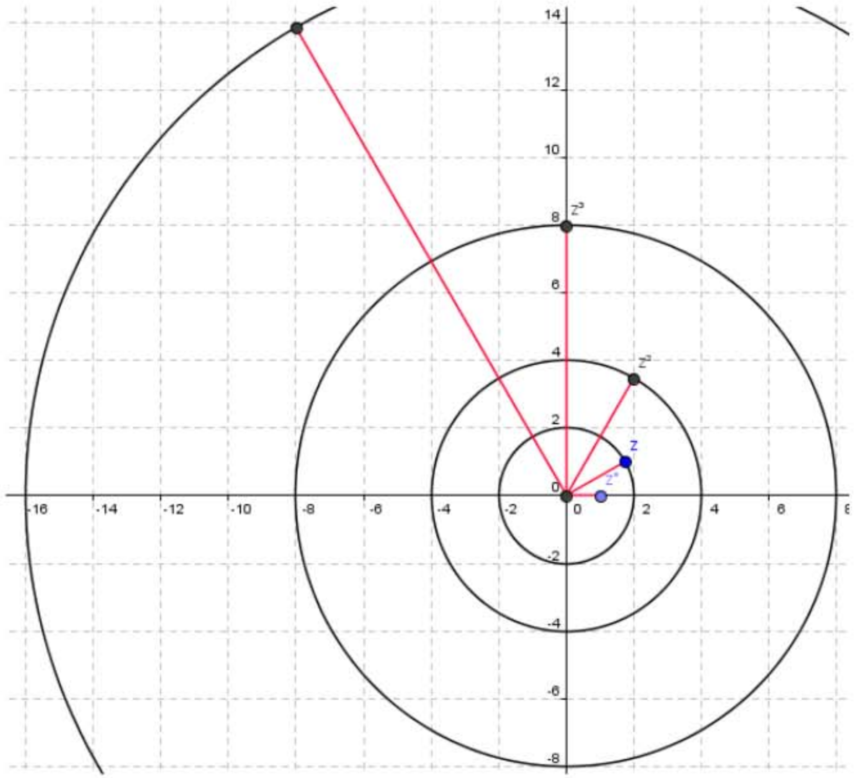
Illustration in the Argand diagram:

Power of z, with | z | = 1



Power of z, with |z| ≠ 1

Equiangular spiral



Exercises:

1 Use de Moivre's theorem to simplify each of the following:

a $(\cos\theta + i\sin\theta)^6$ **b** $(\cos 3\theta + i\sin 3\theta)^4$

c $\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^5$ **d** $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^8$

e $\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)^5$ **f** $\left(\cos\frac{\pi}{10} - i\sin\frac{\pi}{10}\right)^{15}$

g $\dfrac{\cos 5\theta + i\sin 5\theta}{(\cos 2\theta + i\sin 2\theta)^2}$ **h** $\dfrac{(\cos 2\theta + i\sin 2\theta)^7}{(\cos 4\theta + i\sin 4\theta)^3}$

i $\dfrac{1}{(\cos 2\theta + i\sin 2\theta)^3}$ **j** $\dfrac{(\cos 2\theta + i\sin 2\theta)^4}{(\cos 3\theta + i\sin 3\theta)^3}$

k $\dfrac{\cos 5\theta + i\sin 5\theta}{(\cos 3\theta - i\sin 3\theta)^2}$ **l** $\dfrac{\cos\theta - i\sin\theta}{(\cos 2\theta - i\sin 2\theta)^3}$

2 Evaluate $\dfrac{\left(\cos\frac{7\pi}{13} + i\sin\frac{7\pi}{13}\right)^4}{\left(\cos\frac{4\pi}{13} - i\sin\frac{4\pi}{13}\right)^6}$.

3 Express the following in the form $x + iy$ where $x \in \mathbb{R}$ and $y \in \mathbb{R}$.

a $(1 + i)^5$ **b** $(-2 + 2i)^8$ **c** $(1 - i)^6$

d $(1 - \sqrt{3}i)^6$ **e** $\left(\frac{3}{2} - \frac{1}{2}\sqrt{3}i\right)^9$ **f** $\left(-2\sqrt{3} - 2i\right)^5$

4 Express $(3 + \sqrt{3}i)^5$ in the form $a + b\sqrt{3}i$ where a and b are integers.

1 **a** $\cos 6\theta + i\sin 6\theta$ **b** $\cos 12\theta + i\sin 22\theta$
c $-\frac{\sqrt{3}}{2} + \frac{1}{2}i$ **d** $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$
e 1 **f** i
g $\cos\theta + i\sin\theta$ **h** $\cos 2\theta + i\sin 2\theta$
i $\cos 6\theta - i\sin 6\theta$ **j** $\cos\theta - i\sin\theta$
k $\cos 11\theta - i\sin 11\theta$ **l** $\cos 5\theta - i\sin 5\theta$
2 1
3 **a** $(1 + i)^5 = -4 - 4i$
b $(-2 + 2i)^8 = 4096$
c $(1 - i)^6 = 8i$
d $(1 - \sqrt{3}i)^6 = 64$
e $\left(\frac{3}{2} - \frac{1}{2}\sqrt{3}i\right)^9 = 81\sqrt{3}i$
f $(-2\sqrt{3} - 2i)^5 = 512\sqrt{3} - 512i$
4 $(3 + \sqrt{3}i)^5 = -432 + 144\sqrt{3}i$

Can we extend the De Moivre's theorem for n negative or fractional?

Case 1: n = -1

$$z = Cos\theta + iSin\theta$$

$$z^{-1} = \frac{1}{Cos\theta + iSin\theta} = \frac{1}{Cos\theta + iSin\theta} \times \frac{Cos\theta - iSin\theta}{Cos\theta - iSin\theta}$$

$$z^{-1} = \frac{Cos\theta - iSin\theta}{Cos^2\theta + Sin^2\theta} = Cos\theta - iSin\theta$$

$$z^{-1} = Cos(-\theta) + iSin(-\theta)$$

Case 2: $n \in \mathbb{Z}$

$z = Cos\theta + iSin\theta$ and n is a positive integer

$z^{-n} = (z^{-1})^n = \left(Cos(-\theta) + iSin(-\theta)\right)^n \quad (see\ case 1)$

n being positive, we can apply de moivre's theorem

$$z^{-n} = \left(Cos(-\theta) + iSin(-\theta)\right)^n = Cos(-n\theta) + Sin(-n\theta)$$

$$z^{-n} = Cos(n\theta) - Sin(n\theta)$$

$DeMoivre's\ theorem$ can be extended to all $n \in \mathbb{Z}$

Case 3: $n \in \mathbb{Q}$

If n is a fraction, say $\frac{p}{q}$ where p and q are integers, then $\left(Cos\frac{p}{q}\theta + iSin\frac{p}{q}\theta\right)^q = \left(Cos(p\theta) + iSin(p\theta)\right) = \left(Cos\theta + iSin\theta\right)^p$

We now compose by the q^{th} $root$:

$$\left(\left(Cos\frac{p}{q}\theta + iSin\frac{p}{q}\theta\right)^q\right)^{\frac{1}{q}} = \left(\left(Cos\theta + iSin\theta\right)^p\right)^{\frac{1}{q}} \quad this\ gives:$$

$$\left(Cos\frac{p}{q}\theta + iSin\frac{p}{q}\theta\right) = \left(Cos\theta + iSin\theta\right)^{\frac{p}{q}}$$

The De Moivre's theorem and trigonometric identities

$$(a + b)^n = a^n + {}^nC_1\, a^{n-1}\, b + {}^nC_2\, a^{n-2}\, b^2 + {}^nC_3\, a^{n-3}\, b^3 + \ldots + b^n$$

Part A: Expressing cos(nθ) or sin(nθ) in terms of powers of cosθ and sinθ

$$(\cos\theta + i\sin\theta)^2 =$$

$$(\cos\theta + i\sin\theta)^3 =$$

Part B: Expressions of $\tan(n\theta)$

i) Express $\text{Sin}3\theta$ and $\text{Cos}3\theta$ in terms of $\cos\theta$ and $\sin\theta$
ii) Hence, express $\text{Tan}3\theta$ in terms of $\cos\theta$ and $\sin\theta$
iii) Express $\text{Tan}3\theta$ in terms of $\text{Tan}\theta$

The De Moivre's theorem and trigonometric identities

Part C: Linearising powers of cosθ and sinθ

$z = \cos\theta + i\sin\theta$

Work out

$$i)\ z + \frac{1}{z} =$$

$$ii)\ z - \frac{1}{z} =$$

To Remember:

If $z = \cos\theta + \mathrm{i}\sin\theta$

$$z + \frac{1}{z} = 2\cos\theta$$

$$z - \frac{1}{z} = 2\mathrm{i}\sin\theta$$

and

If $z = \cos\theta + \mathrm{i}\sin\theta$,

$$z^n + \frac{1}{z^n} = 2\cos n\theta$$

$$z^n - \frac{1}{z^n} = 2\mathrm{i}\sin n\theta$$

Application : Linearise $\cos^2\theta$

Solved exercises

Example 13

Express $\cos 3\theta$ in terms of powers of $\cos\theta$.

$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$ — de Moivre's theorem.

$= \cos^3\theta + {}^3C_1\cos^2\theta\,(i\sin\theta) + {}^3C_2\cos\theta\,(i\sin\theta)^2 + (i\sin\theta)^3$ — Applying the binomial expansion to $(\cos\theta + i\sin\theta)^3$ where $a = \cos\theta$ and $b = i\sin\theta$.

$= \cos^3\theta + 3i\cos^2\theta\sin\theta + 3i^2\cos\theta\sin^2\theta + i^3\sin^3\theta$ — Simplify.

$= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$ — Applying $i^2 = -1$ and $i^3 = -i$.

Equating the real parts gives

$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ — Note for the LHS that the real part of $\cos 3\theta + i\sin 3\theta$ is $\cos 3\theta$.

$= \cos^3\theta - 3\cos\theta\,(1 - \cos^2\theta)$ — Apply $\sin^2\theta = 1 - \cos^2\theta$.

$= \cos^3\theta - 3\cos\theta + 3\cos^3\theta$ — Multiplying out brackets.

$= 4\cos^3\theta - 3\cos\theta$ — Simplify.

Therefore, $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

Example 15

Express $\cos^5\theta$ in the form $a\cos 5\theta + b\cos 3\theta + c\cos\theta$, where a, b and c are constants.

$\left(z + \frac{1}{z}\right)^5 = (2\cos\theta)^5 = 32\cos^5\theta$ — Applying $z + \frac{1}{z} = 2\cos\theta$.

$= z^5 + {}^5C_1 z^4\left(\frac{1}{z}\right) + {}^5C_2 z^3\left(\frac{1}{z}\right)^2 + {}^5C_3 z^2\left(\frac{1}{z}\right)^3 + {}^5C_4 z\left(\frac{1}{z}\right)^4 + \left(\frac{1}{z}\right)^5$ — Applying the binomial expansion to $\left(z + \frac{1}{z}\right)^5$ where $a = z$ and $b = \frac{1}{z}$.

$= z^5 + 5z^4\left(\frac{1}{z}\right) + 10z^3\left(\frac{1}{z^2}\right) + 10z^2\left(\frac{1}{z^3}\right) + 5z\left(\frac{1}{z^4}\right) + \left(\frac{1}{z^5}\right)$ — Simplify.

$= z^5 + 5z^3 + 10z + \frac{10}{z} + \frac{5}{z^3} + \frac{1}{z^5}$ — Simplify further.

$= \left(z^5 + \frac{1}{z^5}\right) + 5\left(z^3 + \frac{1}{z^3}\right) + 10\left(z + \frac{1}{z}\right)$ — Group z^n and $\frac{1}{z^n}$ terms.

$= 2\cos 5\theta + 5(2\cos 3\theta) + 10(2\cos\theta)$ — Applying $z^n + \frac{1}{z^n} = 2\cos n\theta$.

So, $32\cos^5\theta = 2\cos 5\theta + 10\cos 3\theta + 20\cos\theta$ — Put LHS $= 32\cos^5\theta =$ RHS.

and $\cos^5\theta = \frac{1}{16}\cos 5\theta + \frac{5}{16}\cos 3\theta + \frac{5}{8}\cos\theta$ — $a = \frac{1}{16}$, $b = \frac{5}{16}$ and $c = \frac{5}{8}$.

Example 16

Prove that $\sin^3\theta = -\frac{1}{4}\sin 3\theta + \frac{3}{4}\sin\theta$.

$\left(z - \frac{1}{z}\right)^3 = (2i\sin\theta)^3 = 8i^3\sin^3\theta = -8i\sin^3\theta$ — Applying $z - \frac{1}{z} = 2i\sin\theta$.

$= z^3 + {}^3C_1 z^2\left(-\frac{1}{z}\right) + {}^3C_2 z\left(-\frac{1}{z}\right)^2 + \left(-\frac{1}{z}\right)^3$ — Applying the binomial expansion to $\left(z - \frac{1}{z}\right)^3$ where $a = z$ and $b = -\frac{1}{z}$.

$= z^3 + 3z^2\left(-\frac{1}{z}\right) + 3z\left(\frac{1}{z^2}\right) + \left(-\frac{1}{z^3}\right)$ — Simplify.

$= z^3 - 3z + \frac{3}{z} - \frac{1}{z^3}$ — Simplify further.

$= \left(z^3 - \frac{1}{z^3}\right) - 3\left(z - \frac{1}{z}\right)$ — Group z^n and $\frac{1}{z^n}$ terms.

$= 2i\sin 3\theta - 3(2i\sin\theta)$ — Applying $z^n - \frac{1}{z^n} = 2i\sin n\theta$.

So, $-8i\sin^3\theta = 2i\sin 3\theta - 6i\sin\theta$ — Put LHS $= -8i\sin^3\theta =$ RHS.

and $\sin^3\theta = -\frac{1}{4}\sin 3\theta + \frac{3}{4}\sin\theta$ — Divide both sides by -8

Exercises:

Use applications of de Moivre's theorem to prove the following trigonometric identities:

1 $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

2 $\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

3 $\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$

4 $\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$

5 $\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$

6 **a** Show that $32\cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$.

b Hence find $\int_0^{\frac{\pi}{6}} \cos^6\theta \, d\theta$ in the form $a\pi + b\sqrt{3}$ where a and b are constants.

7 **a** Use de Moivre's theorem to show that $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$.

b Hence, or otherwise, show that $\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$.

c Use your answer to part **b** to find, to 2 d.p., the four solutions of the equation $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$.

6 **a** $32\cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$

b $\int_0^{\frac{\pi}{6}} \cos^6\theta = \frac{5\pi}{96} + \frac{9}{64}\sqrt{3}$

7 **a** $\sin^4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

b $\tan^4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$

c $x = 0.20, 1.50, -5.03, -0.67$ (2 d.p.)

Exponential form of a complex number

Here are some properties we know about complex numbers

$z_1(r_1, \theta_1)$ *and* $z_2(r_2, \theta_2)$ *then* $z_1 z_2(\ldots\ldots\ ,\ \ldots\ldots)$

$$\frac{z_1}{z_2}(\ldots\ldots\ ,\ \ldots\ldots)$$

$$z_1^{\ n}(\ldots\ldots\ ,\ \ldots\ldots)$$

We could say that the moduli "behave" like real numbers but the arguments "behave" like indices.

We use the following notation to illustrate these properties:

$\cos\theta + i\sin\theta$ *is noted* $e^{i\theta}$

and $r(\cos\theta + i\sin\theta)$ *is noted as* $re^{i\theta}$.

Using this notation, the De Moivre's theorem becomes an extension of the rules of indices/exponential we know since year 9!

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

becomes $\left(e^{i\theta}\right)^n = e^{in\theta}$

Consequences

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Examples:

Express the following in the form $re^{i\theta}$:

(a) $1+i$ b) $\sqrt{3}-i$ (c) $3+\sqrt{3}i$ (d) $-2\sqrt{3}+2i$

Work out

e $\left(\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}\right)^5$

f $\left(\cos\frac{\pi}{10}-i\sin\frac{\pi}{10}\right)^{15}$

g $\dfrac{\cos 5\theta + i\sin 5\theta}{(\cos 2\theta + i\sin 2\theta)^2}$

h $\dfrac{(\cos 2\theta + i\sin 2\theta)^7}{(\cos 4\theta + i\sin 4\theta)^3}$

$$Exponential:(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})=$$

$$Algebraic:(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})=$$

n^{th} roots of the unity or Solving $z^n = 1$

Case 1: n = 3

$$z^3 = 1$$
$$\left(re^{i\theta}\right)^3 = e^{i0} \quad and \quad r^3 e^{i3\theta} = e^{i0}$$
This gives
$$\mathrm{r}^3 = 1 \quad and \quad 3\theta = 0 + k \times 2\pi$$
$$r = 1 \quad and \quad \theta = k \times \frac{2\pi}{3} \quad k = 0,1,2$$
The 3rd roots of the unity are
$$1\mathrm{e}^{i0} = 1 \ , \ 1e^{i\frac{2\pi}{3}} = e^{i\frac{2\pi}{3}} \ , \ 1e^{i\frac{4\pi}{3}} = e^{i\frac{4\pi}{3}}$$

General case:

$$z^n = 1$$
$$\left(re^{i\theta}\right)^n = e^{i0} \quad and \quad r^n e^{in\theta} = e^{i0}$$
This gives
$$r^n = 1 \quad and \quad n\theta = 0 + k \times 2\pi$$
$$r = 1 \quad and \quad \theta = k \times \frac{2\pi}{n} \quad k = 0,1,2,\ldots,n-1$$

The n^{th} *roots of unity* are $1, \omega, \omega^2, \omega^3, \ldots, \omega^{n-1}$ *where* $\omega = e^{i\frac{2\pi}{n}}$

or

The n^{th} *roots of unity* are $e^{i\frac{2r\pi}{n}}$ *for* $0 \le r \le n-1$

Note: Any set of n consecutive values for r will produce an equivalent set of solutions.
We sometime prefer sets symmetrical around "0" in order to have arguments between $-\pi$ and π.

Sum the the n^th roots of unity

The n^{th} *roots of unity* are $1, \omega, \omega^2, \omega^3, ..., \omega^{n-1}$ *where* $\omega = e^{i\frac{2\pi}{n}}$

Let's work out :

$$1+\omega+\omega^2+\omega^3+...+\omega^{n-2}+\omega^{n-1}$$

$1+\omega+\omega^2+...+\omega^{n-1}$ is the sum of a geometric series with common ration ω

so $1+\omega+\omega^2+...+\omega^{n-1} = \dfrac{1-\omega^n}{1-\omega} = 0$ because by definition $\omega^n = 1$

For $n \in \mathbb{N}$, $n > 1$, the sum of the n^th roots of unity:

$$1+\omega+\omega^2+\omega^3+...+\omega^{n-2}+\omega^{n-1} = 0$$

Exercises:

1 If $\omega = e^{\frac{2}{3}\pi i}$, simplify the following (expressing your answer in terms of ω where appropriate).

(a) ω^5 (b) ω^{-3} (c) $1+\omega^2$ (d) $\omega + \dfrac{1}{\omega}$

(e) $(1-\omega)^2$ (f) $(1-\omega)(1-\omega^2)$ (g) $\dfrac{1}{(1+\omega)^2}$ (h) $\dfrac{1+\omega}{1+\omega^2}$

2 If $\omega = e^{\frac{2}{5}\pi i}$, simplify the following.

(a) ω^5 (b) ω^{-4}

(c) $(1+\omega)(1+\omega^2)$ (d) $(1-\omega)(1-\omega^2)(1-\omega^3)(1-\omega^4)$

3 If $\omega = e^{\frac{2}{9}\pi i}$, show that $(1+\omega)(1+\omega^2)(1+\omega^4) = -\omega^8$.

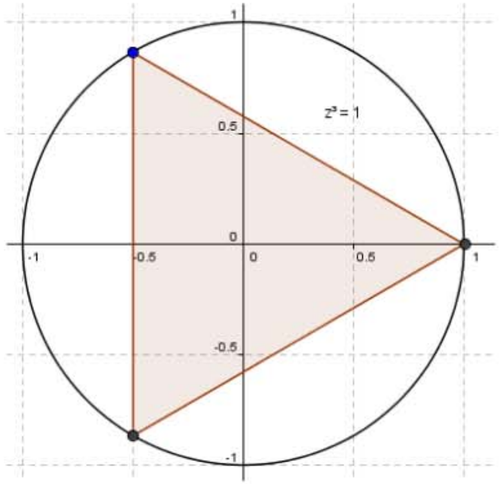
4 If $\omega = e^{\frac{2}{3}\pi i}$, and if $x+y+z=a$, $x+\omega y+\omega^2 z = b$ and $x+\omega^2 y+\omega z = c$, express $a+b+c$, $a+\omega^2 b+\omega c$ and $a+\omega b+\omega^2 c$ in terms of x, y and z.

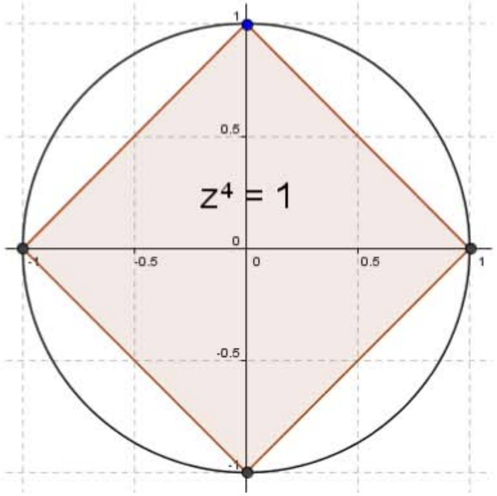
1 (a) ω^2 (b) 1 (c) $-\omega$ (d) -1
(e) -3ω (f) 3 (g) ω^2 (h) ω

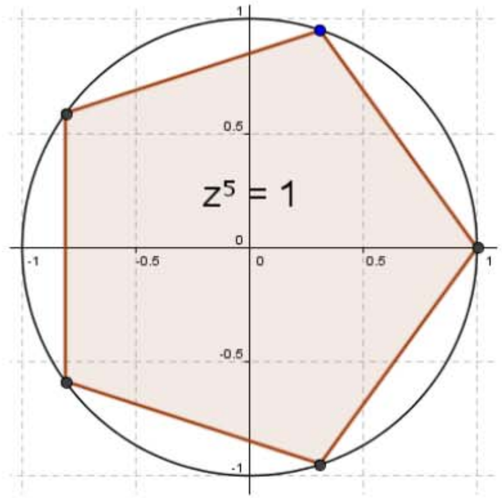
2 (a) 1 (b) ω (c) $-\omega^4$ (d) 5

4 $3x, 3y, 3z$; $x = 2$, $y = -\frac{1}{3} + \frac{1}{3}\omega$, $z = -\frac{2}{3} - \frac{1}{3}\omega$

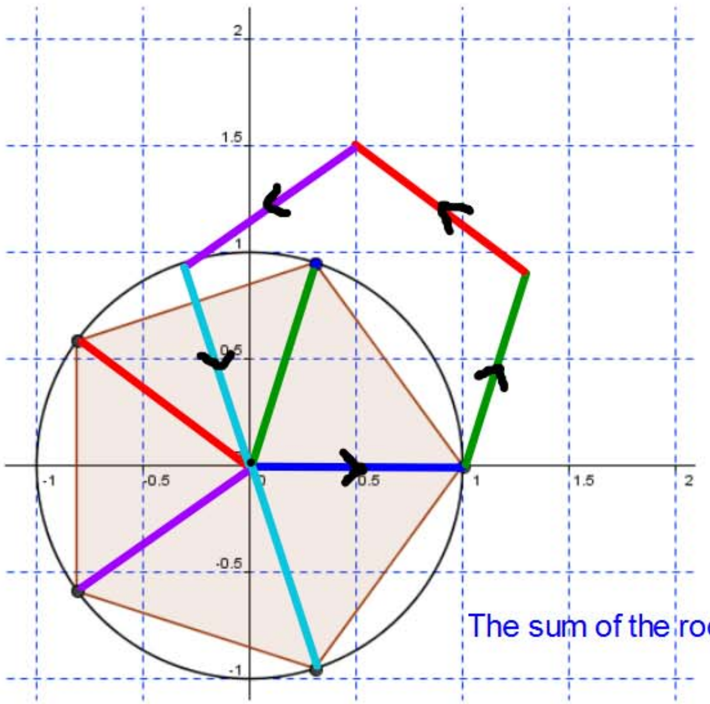
n^{th} roots of unity and the Argand diagram







The n^{th} roots of unity plotted in the Argand diagram are the vertices of a regular polygon.



The sum of the roots add up to 0

Solving equations of the form $z^n = a+ib$

The principal:

$z^n = a+ib$

Write z as $z = re^{i\theta}$

Find the modulus and argument of $a+ib$ and write $a+ib = r_1 e^{i\theta_1}$

With this notation, $z^n = a+ib$ becomes $r^n e^{in\theta} = r_1 e^{i\theta_1}$

This gives:

$r^n = r_1$ and $n\theta = \theta_1 + k \times 2\pi$

$r = \sqrt[n]{r_1}$ and $\theta = \dfrac{\theta_1}{n} + k \times \dfrac{2\pi}{n}$ $\quad k = 0, 1, 2, \ldots, n-1$

Example:

$z^4 = 16+16i$

$z^4 = 16+16i$

$(re^{i\theta})^4 = 16\sqrt{2}\left(\dfrac{1}{\sqrt{2}} + i\dfrac{1}{\sqrt{2}}\right)$

$r^4 e^{i4\theta} = 16\sqrt{2}e^{i\frac{\pi}{4}}$

This gives

$r = \sqrt[4]{16\sqrt{2}}$ and $4\theta = \dfrac{\pi}{4} + k \times 2\pi$

$r = 2^{\frac{9}{8}}$ and $\theta = \dfrac{\pi}{16} + k \times \dfrac{\pi}{2}$ $\quad k = 0, 1, 2, 3$

Exercises:

1 Solve the following equations, expressing your answers for z in the form $x + iy$, where $x \in \mathbb{R}$ and $y \in \mathbb{R}$.

a $z^4 - 1 = 0$ **b** $z^3 - i = 0$ **c** $z^3 = 27$

d $z^4 + 64 = 0$ **e** $z^4 + 4 = 0$ **f** $z^3 + 8i = 0$

2 Solve the following equations, expressing your answers for z in the form $r(\cos\theta + i\sin\theta)$, where $-\pi < \theta \leq \pi$.

a $z^7 = 1$ **b** $z^4 + 16i = 0$ **c** $z^5 + 32 = 0$

d $z^3 = 2 + 2i$ **e** $z^4 + 2\sqrt{3}i = 2$ **f** $z^3 + 32\sqrt{3} + 32i = 0$

3 Solve the following equations, expressing your answers for z in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. Give θ to 2 d.p.

a $z^4 = 3 + 4i$ **b** $z^3 = \sqrt{11} - 4i$ **c** $z^4 = -\sqrt{7} + 3i$

4 **a** Find the three roots of the equation $(z + 1)^3 = -1$.
Give your answers in the form $x + iy$, where $x \in \mathbb{R}$ and $y \in \mathbb{R}$.

b Plot the points representing these three roots on an Argand diagram.

c Given that these three points lie on a circle, find its centre and radius.

5 **a** Find the five roots of the equation $z^5 - 1 = 0$.
Give your answers in the form $r(\cos\theta + i\sin\theta)$, where $-\pi < \theta \leq \pi$.

b Given that the sum of all five roots of $z^5 - 1 = 0$ is zero, show that

$$\cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) = -\frac{1}{2}.$$

6 **a** Find the modulus and argument of $-2 - 2\sqrt{3}i$.

b Hence find all the solutions of the equation $z^4 + 2 + 2\sqrt{3}i = 0$.
Give your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$.

7 **a** Find the modulus and argument of $\sqrt{6} + \sqrt{2}i$.

b Solve the equation $z^{\frac{3}{4}} = \sqrt{6} + \sqrt{2}i$.
Give your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$.

Answers

1 **a** $z = 1, \mathrm{i}, -1, -\mathrm{i}$

b $z = \frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}, -\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}, -\mathrm{i}$

c $z = 3, -\frac{3}{2} + \frac{3\sqrt{3}}{2}\mathrm{i}, -\frac{3}{2} - \frac{3\sqrt{3}}{2}\mathrm{i}$

d $z = 2 + 2\mathrm{i}, -2 + 2\mathrm{i}, 2 - 2\mathrm{i}, -2 - 2\mathrm{i}$

e $z = 1 + \mathrm{i}, -1 + \mathrm{i}, 1 - \mathrm{i}, -1 - \mathrm{i}$

f $z = \sqrt{3} - \mathrm{i}, 2\mathrm{i}, -\sqrt{3} - \mathrm{i}$

2 **a** $z = \cos 0 + \mathrm{i}\sin 0, \cos\frac{2\pi}{7} + \mathrm{i}\sin\frac{2\pi}{7}$

$\cos\frac{4\pi}{7} + \mathrm{i}\sin\frac{4\pi}{7}, \cos\frac{6\pi}{7} + \mathrm{i}\sin\frac{6\pi}{7}$

$\cos\left(-\frac{2\pi}{7}\right) + \mathrm{i}\sin\left(-\frac{2\pi}{7}\right),$

$\cos\left(-\frac{4\pi}{7}\right) + \mathrm{i}\sin\left(-\frac{4\pi}{7}\right)$

$\cos\left(-\frac{6\pi}{7}\right) + \mathrm{i}\sin\left(-\frac{6\pi}{7}\right)$

b $z = 2\left(\cos\left(-\frac{\pi}{8}\right) + \mathrm{i}\sin\left(-\frac{\pi}{8}\right)\right),$

$2\left(\cos\left(\frac{3\pi}{8}\right) + \mathrm{i}\sin\left(\frac{3\pi}{8}\right)\right)$

$2\left(\cos\left(\frac{7\pi}{8}\right) + \mathrm{i}\sin\left(\frac{7\pi}{8}\right)\right),$

$2\left(\cos\left(-\frac{5\pi}{8}\right) + \mathrm{i}\sin\left(-\frac{5\pi}{8}\right)\right)$

c $z = 2\left(\cos\frac{\pi}{5} + \mathrm{i}\sin\frac{\pi}{5}\right), 2\left(\cos\frac{3\pi}{5} + \mathrm{i}\sin\frac{3\pi}{5}\right),$

$2(\cos\pi + \mathrm{i}\sin\pi), 2\left(\cos\left(-\frac{\pi}{5}\right) + \mathrm{i}\sin\left(-\frac{\pi}{5}\right)\right),$

$2\left(\cos\left(-\frac{3\pi}{5}\right) + \mathrm{i}\sin\left(-\frac{3\pi}{5}\right)\right)$

d $z = \sqrt{2}\left(\cos\frac{\pi}{12} + \mathrm{i}\sin\frac{\pi}{12}\right), \sqrt{2}\left(\cos\frac{3\pi}{4} + \mathrm{i}\sin\frac{3\pi}{4}\right),$

$\sqrt{2}\left(\cos\left(-\frac{7\pi}{12}\right) + \mathrm{i}\sin\left(\frac{-7\pi}{12}\right)\right)$

e $z = \sqrt{2}\left(\cos\left(-\frac{\pi}{12}\right) + \mathrm{i}\sin\left(-\frac{\pi}{12}\right)\right),$

$\sqrt{2}\left(\cos\left(\frac{5\pi}{12}\right) + \mathrm{i}\sin\left(\frac{5\pi}{12}\right)\right),$

$\sqrt{2}\left(\cos\left(\frac{11\pi}{12}\right) + \mathrm{i}\sin\left(\frac{11\pi}{12}\right)\right),$

$\sqrt{2}\left(\cos\left(-\frac{7\pi}{12}\right) + \mathrm{i}\sin\left(-\frac{7\pi}{12}\right)\right)$

f $z = 4\left(\cos\left(-\frac{5\pi}{18}\right) + \mathrm{i}\sin\left(-\frac{5\pi}{18}\right)\right),$

$4\left(\cos\left(\frac{7\pi}{18}\right) + \mathrm{i}\sin\left(\frac{7\pi}{18}\right)\right),$

$4\left(\cos\left(-\frac{17\pi}{18}\right) + \mathrm{i}\sin\left(-\frac{17\pi}{18}\right)\right)$

3 **a** $z = 5^{\frac{1}{4}}\mathrm{e}^{0.23\mathrm{i}}, 5^{\frac{1}{4}}\mathrm{e}^{1.80\mathrm{i}}, 5^{\frac{1}{4}}\mathrm{e}^{-1.34\mathrm{i}}, 5^{\frac{1}{4}}\mathrm{e}^{-2.91\mathrm{i}}$

b $z = \sqrt{3}\,\mathrm{e}^{-0.29\mathrm{i}}, \sqrt{3}\,\mathrm{e}^{1.80\mathrm{i}}, \sqrt{3}\,\mathrm{e}^{-2.39\mathrm{i}}$

c $z = \sqrt{2}\,\mathrm{e}^{0.57\mathrm{i}}, z = \sqrt{2}\,\mathrm{e}^{2.14\mathrm{i}}, z = \sqrt{2}\,\mathrm{e}^{-1.00\mathrm{i}}, z = \sqrt{2}\,\mathrm{e}^{-2.57\mathrm{i}}$

4 **a** $z = -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}, -2, \frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}$

5 **a** $z = 1, \cos\left(\frac{2\pi}{5}\right) + \mathrm{i}\sin\left(\frac{2\pi}{5}\right), \cos\left(\frac{4\pi}{5}\right) + \mathrm{i}\sin\left(\frac{4\pi}{5}\right),$

$\cos\left(-\frac{2\pi}{5}\right) + \mathrm{i}\sin\left(-\frac{2\pi}{5}\right),$

$\cos\left(-\frac{4\pi}{5}\right) + \mathrm{i}\sin\left(-\frac{4\pi}{5}\right)$

6 **a** $r = 4, \theta = -\frac{2\pi}{3}$

b $z = \sqrt{2}\,\mathrm{e}^{-\frac{\pi\mathrm{i}}{6}}, \sqrt{2}\,\mathrm{e}^{\frac{\pi\mathrm{i}}{3}}, \sqrt{2}\,\mathrm{e}^{\frac{5\pi\mathrm{i}}{6}}, \sqrt{2}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{3}}$

7 **a** $r = \sqrt{8}, \theta = \frac{\pi}{6}$

b $z = 4\mathrm{e}^{\frac{2\pi\mathrm{i}}{9}}, z = 4\mathrm{e}^{\frac{8\pi\mathrm{i}}{9}}, z = 4\mathrm{e}^{-\frac{4\pi\mathrm{i}}{9}}$