Matrix algebra

Specifications:

Matrix Algebra

Matrix algebra of up to 3×3 matrices, including the inverse of a 2×2 or 3×3 matrix.

The identity matrix $\mathbf{I}$ for 2×2 and 3×3 matrices.

Including non-square matrices and use of the results $(\mathbf{AB})^{-1}=\mathbf{B}^{-1}\mathbf{A}^{-1}$ and $(\mathbf{AB})^{\mathrm{T}}=\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$.

Singular and non-singular matrices.

Determinants

Second order and third order determinants, and their manipulation.

Including the use of the result $\det(\mathbf{AB})=\det\mathbf{A}\det\mathbf{B}$, but a general treatment of products is not required.

Vocabulary

Order/dimension of a matrix

- A matrix with n rows and p columns is said to be a matrix of **order $n \times p$**

> When giving the order of a matrix, you should always give the number of rows first, then the number of columns

- If $n=p$, the matrix is called a SQUARE matrix.

Elements

The numbers constituting the matrix is called the ELEMENTS of the matrix.

Identity matrix and zero matrix

- For any order, the zero matrix, written **0**, is the matrix with all its elements=0.

$$\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{is the zero matrix of order } 2 \times 2.$$

$$\mathbf{0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{is the zero matrix of order } 2 \times 3.$$

- An IDENTITY matrix is a square matrix. All the elements are 0 except on the diagonal where they are =1

$$\mathbf{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} etc...$$

Addition , subtraction , multiplication by a scalar

Two matrices can be added, or subtracted, if they have the **same order**. To add, or subtract, two matrices simply add, or subtract, the corresponding elements. For example,

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} + \begin{bmatrix} g & h & i \\ j & k & l \end{bmatrix} = \begin{bmatrix} a+g & b+h & c+i \\ d+j & e+k & f+l \end{bmatrix},$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a-e & b-f \\ c-g & d-h \end{bmatrix}$$

Any matrix can be multiplied by any scalar. Simply multiply each element by the scalar. For example,

$$k\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \\ ke & kf \end{bmatrix}$$

Multiplication of matrices

Order and multiplication

If $\mathbf{A}$ and $\mathbf{B}$ are two matrices,

$\mathbf{A}\times\mathbf{B}$ exists if the number of columns of $\mathbf{A}$ = number of rows of $\mathbf{B}$

Example:

left hand matrix × right hand matrix ⟶ product matrix

4×3 $\qquad$ 3×2 $\qquad$ 4×2

these numbers must be equal

these numbers determine the order of the product matrix

Method to multiply two matrices:

$$\mathbf{A}=\begin{pmatrix}1 & 4\\ -1 & 3\\ 2 & 5\end{pmatrix} \text{ and } \mathbf{B}=\begin{pmatrix}2\\ 3\end{pmatrix}$$

Let's work out $\mathbf{A}\times\mathbf{B}$.

The layout:

$$\begin{pmatrix}2\\ 3\end{pmatrix}$$

$$\begin{pmatrix}1 & 4\\ -1 & 3\\ 2 & 5\end{pmatrix}\begin{pmatrix}\\ \\ \end{pmatrix}$$

Exercises:

Given that $\mathbf{A}=\begin{pmatrix}-1 & 0\\ 2 & 3\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}4 & 1\\ 0 & -2\end{pmatrix}$ find

a $\mathbf{AB}$

b $\mathbf{BA}$.

Matrices are multiplied by multiplying the elements in a row of the first matrix by the elements in a column of the second matrix, and adding the results. For example,

$$\begin{bmatrix}a & b\\ c & d\\ e & f\end{bmatrix}\begin{bmatrix}g & h & i & j\\ k & l & m & n\end{bmatrix}=\begin{bmatrix}ag+bk & ah+bl & ai+bm & aj+bn\\ cg+dk & ch+dl & ci+dm & cj+dn\\ eg+fk & eh+fl & ei+fm & ej+fn\end{bmatrix}$$

The product $\mathbf{AB}$ can be found if the number of columns of matrix $\mathbf{A}$ is equal to the number of rows of matrix $\mathbf{B}$. If the order of matrix $\mathbf{A}$ is $r\times s$ and $\mathbf{B}$ is $s\times t$, then the order of $\mathbf{AB}$ is $r\times t$

Exercises:

1 Let $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix}$. Calculate the matrices

(a) $\mathbf{A} + \mathbf{B}$, (b) $\mathbf{A} - \mathbf{B}$, (c) $3\mathbf{A} + 2\mathbf{B}$, (d) $4\mathbf{A} - 3\mathbf{B}$.

2 Solve for $\mathbf{X}$ the matrix equation $2\mathbf{X} + 3\mathbf{A} = 4\mathbf{X} - 3\mathbf{B}$. What do you need to assume about the sizes of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{X}$?

4 Find the products $\mathbf{AB}$ and $\mathbf{BA}$ where $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix}$.

5 Let $\mathbf{A} = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 3 & 5 \\ -3 & -2 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 & -2 & -3 \\ 5 & 1 & 2 \\ 2 & -4 & 1 \end{pmatrix}$. Calculate $\mathbf{AB}$ and $\mathbf{BA}$.

6 Let $\mathbf{A} = \begin{pmatrix} -3 & 2 & 6 \\ 2 & -1 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 6 & 2 \\ 3 & 2 \\ 2 & -1 \end{pmatrix}$. Calculate $\mathbf{AB}$ and $\mathbf{BA}$.

7 Although in general, it is true that $\mathbf{AB} \neq \mathbf{BA}$, there are matrices $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{BA}$. Find such a pair $\mathbf{A}$ and $\mathbf{B}$ in which none of the elements is 0.

8 Let $\mathbf{A} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -4 & 2 & 3 \\ -2 & 3 & -3 \end{pmatrix}$ and $\mathbf{C} = (2 \quad 3 \quad 1)$. Calculate those of the following matrix products which exist.

(a) $\mathbf{AB}$ (b) $\mathbf{BA}$ (c) $\mathbf{AC}$ (d) $\mathbf{CA}$

(e) $\mathbf{BC}$ (f) $\mathbf{CB}$ (g) $(\mathbf{CA})\,\mathbf{B}$ (h) $\mathbf{C}\,(\mathbf{AB})$

1 (a) $\begin{pmatrix} 3 & 5 \\ 7 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} -1 & -1 \\ -1 & 5 \end{pmatrix}$

(c) $\begin{pmatrix} 7 & 12 \\ 17 & 10 \end{pmatrix}$ (d) $\begin{pmatrix} -2 & -1 \\ 0 & 19 \end{pmatrix}$

2 $\mathbf{X} = \frac{3}{2}(\mathbf{A} + \mathbf{B})$; they are all the same size.

4 $\begin{pmatrix} 10 & 1 \\ 22 & 5 \end{pmatrix}, \begin{pmatrix} 11 & 16 \\ 1 & 4 \end{pmatrix}$

5 $\begin{pmatrix} 21 & 3 & -1 \\ 29 & -19 & 8 \\ -18 & -4 & 7 \end{pmatrix}, \begin{pmatrix} 15 & 12 & -20 \\ 5 & 14 & 4 \\ -3 & -8 & -20 \end{pmatrix}$

6 $\begin{pmatrix} 0 & -8 \\ 13 & 0 \end{pmatrix}, \begin{pmatrix} -14 & 10 & 40 \\ -5 & 4 & 22 \\ -8 & 5 & 10 \end{pmatrix}$

7 For example, $\begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$ and $\begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$.

8 (a) Does not exist. (b) $\begin{pmatrix} -3 \\ -8 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 3 & 1 \\ -2 & -3 & -1 \\ 2 & 3 & 1 \end{pmatrix}$ (d) (0)

(e) Does not exist. (f) Does not exist.

(g) Does not exist. (h) Does not exist.

Laws of matrix arithmetic

Commutativity of addition

For any two matrices which can be added (i.e. which have the same order), $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$

Non-commutativity of multiplication

It cannot be assumed that $\mathbf{AB}=\mathbf{BA}$, even when both products exist

In general AB ≠ BA

Distributive law

For any matrices of appropriate orders,

$$\mathbf{A}(\mathbf{B}+\mathbf{C})=\mathbf{AB}+\mathbf{AC},$$
$$(\mathbf{U}+\mathbf{V})\mathbf{W}=\mathbf{UW}+\mathbf{VW}$$

Associative law

For any matrices of appropriate orders,

$$\mathbf{A}(\mathbf{BC})=(\mathbf{AB})\mathbf{C}$$

For all matrices of suitable orders,

$$\mathbf{A}+\mathbf{0}=\mathbf{A},\ \mathbf{0B}=\mathbf{0}\ \text{ and }\ \mathbf{C0}=\mathbf{0}$$

For all matrices of suitable orders,

$$\mathbf{AI}=\mathbf{A}\ \text{ and }\ \mathbf{IB}=\mathbf{B}$$

Transpose matrices

In some circumstances it can be useful to interchange the rows and columns of a matrix. This process is called **transposing** the matrix.

The transpose of a matrix **M** is obtained by interchanging the rows and columns of **M**. The transpose of **M** is denoted by $\mathbf{M}^T$

Example:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 1 \\ 3 & 4 & -2 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 2 & -1 \\ -1 & 4 \\ 0 & 2 \end{bmatrix}$$

$\mathbf{A}^T =$

$\mathbf{B}^T =$

Work out, if possible,

$\mathbf{AB}$, $(\mathbf{AB})^T$, $\mathbf{A}^T\mathbf{B}^T$, $\mathbf{B}^T\mathbf{A}^T$

What do you notice about **AB** and $\mathbf{B}^T\mathbf{A}^T$?

Note:

For all matrix $\mathbf{A}$, $(\mathbf{A}^T)^T = \mathbf{A}$

for **all** compatible matrices **A** and **B**

$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$$

Learn by heart

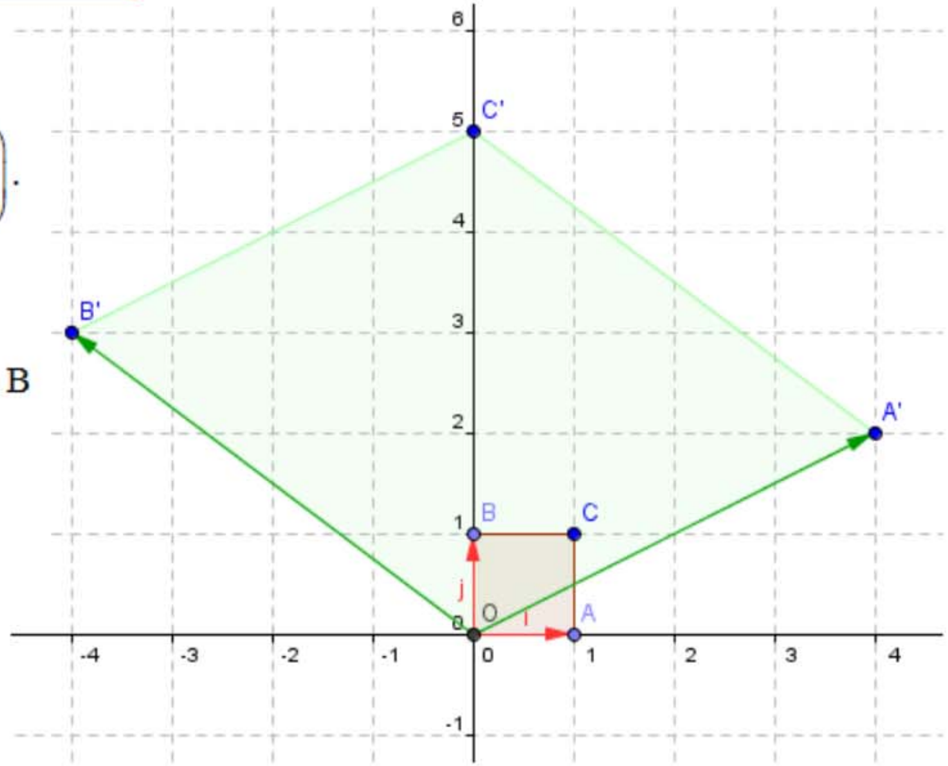
The tranpose of a product is the product of the transpose in the "transpose order"

Determinant of a matrix

Introduction:

Consider the transformation represented by the matrix $\mathbf{M}=\begin{pmatrix} 4 & -4 \\ 2 & 3 \end{pmatrix}$.

$O(0,0)$ $A(1,0)$ $B(0,1)$ $C(1,1)$ *and* the unit square is OACB.



1) Work out the coordinates of O', A', C', B' , the image of O, A, C, B through this transformation.
2) Work out the components of the vector $\overrightarrow{O'A'}$ and $\overrightarrow{O'B'}$.
3) Work out the area of the parallelogram O'A'C'B'.

General case

Consider the transformation represented by the matrix $\mathbf{M}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

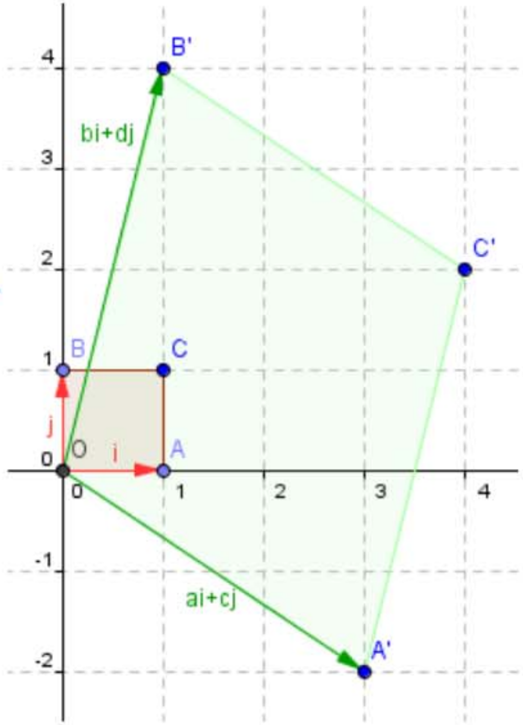
$O(0,0)$ $A(1,0)$ $B(0,1)$ $C(1,1)$ *and* the unit square is OACB.

The points O', A', C', B' are the images of O, A, C, B through this transformation respectively.

Note that $O' = O$

so $\overrightarrow{O'A'} = \overrightarrow{OA'} = a\mathbf{i} + c\mathbf{j}$ *and* $\overrightarrow{O'B'} = \overrightarrow{OB'} = b\mathbf{i} + d\mathbf{j}$

Work out $|\overrightarrow{OA'} \times \overrightarrow{OB'}|$. This number is called the determinant of the matrix **M**.



2x2 matrix determinant

For all matrix $\mathbf{M}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

the determinant of $\mathbf{M}$ is noted det(**M**) or $|\mathbf{M}| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

Find the determinants of the following matrices.

(a) $\begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}$, (b) $\begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$, (c) $\begin{bmatrix} 2 & 5 \\ -2 & 0 \end{bmatrix}$.

3x3 matrix determinant

Transformations can also be represented in 3D with a 3x3 matrix,

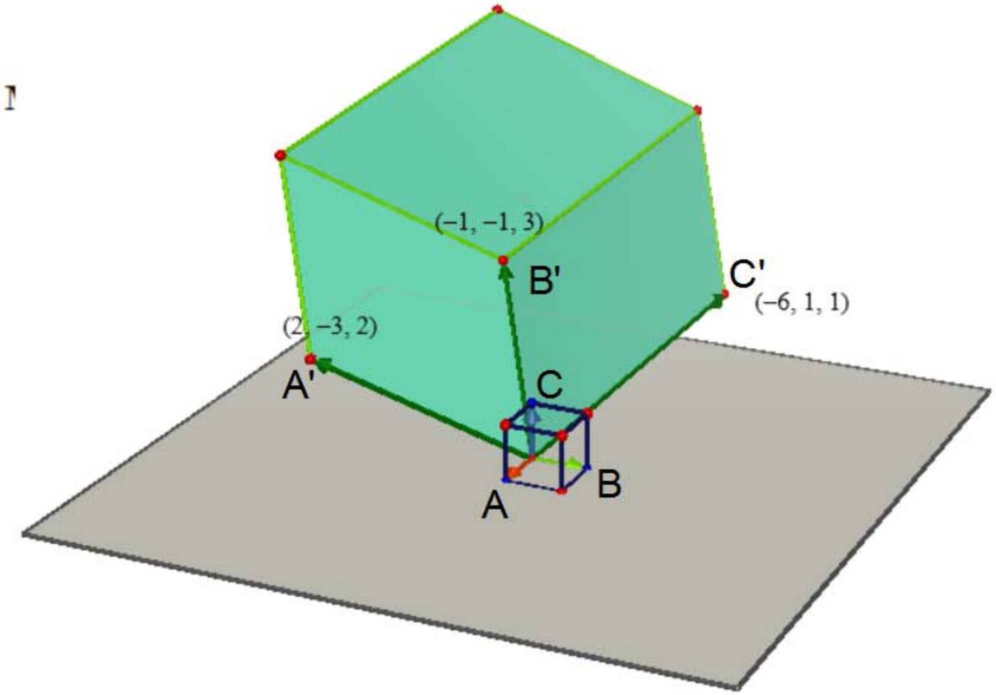
The columns of the matrix are the image of **i**,**j**,**k** respectively.

The unit cube generated by the vector **i**,**j**,**k** is tranformed into

the parallelepiped generated by $\mathbf{i}', \mathbf{j}', \mathbf{k}'$, the images of **i**, **j**, **k**.

The value $\mathbf{i}'.(\mathbf{j}'\times\mathbf{k}')$ is called the determinant of the matrix, **M**

Notice: $|\mathbf{i}'.(\mathbf{j}'\times\mathbf{k}')|$ is the VOLUME of the parallelepiped



Numerical application:

Consider the image of the unit cube through

the 3D transformation represented by the matrix $\mathbf{M}=\begin{pmatrix} 2 & -1 & -6 \\ -3 & -1 & 1 \\ 2 & 3 & 1 \end{pmatrix}$.

$\overrightarrow{OA'} = 2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$, $\overrightarrow{OB'} = -\mathbf{i} - \mathbf{j} + 3\mathbf{k}$, $\overrightarrow{OC'} = -6\mathbf{i} + \mathbf{j} + \mathbf{k}$

1) Using the distributive property of the cross product,

work out $\overrightarrow{OB'} \times \overrightarrow{OC'}$

2) Using the distributive property of the scalar product,

work out $\overrightarrow{OA'}.(\overrightarrow{OB'} \times \overrightarrow{OC'})$

General case:

The determinant of $\mathbf{M} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is $\mathbf{a}.\mathbf{b}\times\mathbf{c}$.

$|\mathbf{M}| = \mathbf{a}.\mathbf{b}\times\mathbf{c}$.

$$|\mathbf{M}| = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

The 'a' column was used to multiply out the determinant. The same process can be used with **any** row or column, the sign of the terms being determined by the pattern

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

Example:

Multiply out $\begin{vmatrix} 2 & -1 & 3 \\ 1 & 4 & -1 \\ 3 & 2 & 0 \end{vmatrix}$

a) using the first column
b) using the second row
c) using the third column

Tip:

To go faster, multiply out/ develop the determinant using the row or the column containing the most number of 0s

Work out $\begin{vmatrix} 1 & 2 & 4 \\ -6 & 1 & 0 \\ -3 & 1 & 0 \end{vmatrix}$

Rules for manipulating determinants

A and B are two square matrices.

- If det(A)= 0 , the matrix is said to be SINGULAR.
- det(A) = det(A^T)
- The determinant of a product is the product of the determinants,

$$|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$$

or det(AB)=det(A)×det(B)

Examples:

- The matrix $\mathbf{A} = \begin{pmatrix} 2 & -1 & 3 \\ 8 & -2 & 7 \\ 4 & 2 & -4 \end{pmatrix}$.

  Show that **A** is singular.

- The matrix $\mathbf{A} = \begin{pmatrix} 3 & k & 0 \\ -2 & 1 & 2 \\ 5 & 0 & k+3 \end{pmatrix}$, where k is a constant.

  a Find det (**A**) in terms of k.

  Given that **A** is singular,

  b find the possible values of k.

Exercises:

1 Find the values of the determinants.

a $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix}$ **b** $\begin{vmatrix} 0 & 4 & 0 \\ 5 & -2 & 3 \\ 2 & 1 & 4 \end{vmatrix}$ **c** $\begin{vmatrix} 1 & 0 & 1 \\ 2 & 4 & 1 \\ 3 & 5 & 2 \end{vmatrix}$ **d** $\begin{vmatrix} 2 & -3 & 4 \\ 2 & 2 & 2 \\ 5 & 5 & 5 \end{vmatrix}$

2 Find the values of the determinants.

a $\begin{vmatrix} 4 & 3 & -1 \\ 2 & -2 & 0 \\ 0 & 4 & -2 \end{vmatrix}$ **b** $\begin{vmatrix} 3 & -2 & 1 \\ 4 & 1 & -3 \\ 7 & 2 & -4 \end{vmatrix}$ **c** $\begin{vmatrix} 5 & -2 & -3 \\ 6 & 4 & 2 \\ -2 & -4 & -3 \end{vmatrix}$

3 The matrix $\mathbf{A} = \begin{pmatrix} 2 & 1 & -4 \\ 2k+1 & 3 & k \\ 1 & 0 & 1 \end{pmatrix}$.

Given that $\mathbf{A}$ is singular, find the value of the constant k.

4 The matrix $\mathbf{A} = \begin{pmatrix} 2 & -1 & 3 \\ k & 2 & 4 \\ -2 & 1 & k+3 \end{pmatrix}$, where k is a constant.

Given that the determinant of $\mathbf{A}$ is 8, find the possible values of k.

5 The matrix $\mathbf{A} = \begin{pmatrix} 2 & 5 & 3 \\ -2 & 0 & 4 \\ 3 & 10 & 8 \end{pmatrix}$ and the matrix $\mathbf{B} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & -2 & -1 \end{pmatrix}$.

a Show that $\mathbf{A}$ is singular.

b Find $\mathbf{AB}$.

c Show that $\mathbf{AB}$ is also singular.

6 The matrix $\mathbf{A} = \begin{pmatrix} 4 & 5 & -2 \\ 2 & -3 & 2 \\ 2 & -4 & 3 \end{pmatrix}$.

a Find $\det(\mathbf{A})$.

b Write down $\mathbf{A}^T$

c Verify that $\det(\mathbf{A}^T) = \det(\mathbf{A})$.

7 **a** Show that, for all values of a, b and c, the matrix $\begin{pmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{pmatrix}$ is singular.

b Show that, for all real values of x, the matrix $\begin{pmatrix} 2 & -2 & 4 \\ 3 & x & -2 \\ -1 & 3 & x \end{pmatrix}$ is non-singular.

8 Find all the values of x for which the matrix $\begin{pmatrix} x-3 & -2 & 0 \\ 1 & x & -2 \\ -2 & -1 & x+1 \end{pmatrix}$ is singular.

1 **a** 6 **b** −56 **c** 1 **d** 0

2 **a** 20 **b** 17 **c** 0

3 17

4 −8, −2

5 **b** $\begin{pmatrix} 7 & 6 & 7 \\ -2 & -10 & -4 \\ 13 & 7 & 12 \end{pmatrix}$

6 **a** −10

b $\begin{pmatrix} 4 & 2 & 2 \\ 5 & -3 & -4 \\ -2 & 2 & 3 \end{pmatrix}$

8 −1, 0, 3

Inverse of a matrix

A is a square matrix 2x2 or 3x3.

If the inverse of the matrix A exists, we call it $\mathbf{A}^{-1}$ and

$$\mathbf{A}\mathbf{A}^{-1}=\mathbf{A}^{-1}\mathbf{A}=\mathbf{I}$$

Numerical examples:

1) $\mathbf{A}=\begin{pmatrix}3 & 4\\ 1 & 3\end{pmatrix}$. *Let's write* $\mathbf{A}^{-1}=\begin{pmatrix}x & y\\ z & t\end{pmatrix}$

Work out the value of x, y, z *and* t.

2) $\mathbf{A}=\begin{pmatrix}3 & 2\\ 6 & 4\end{pmatrix}$. *Let's write* $\mathbf{A}^{-1}=\begin{pmatrix}x & y\\ z & t\end{pmatrix}$

Work out the value of x, y, z *and* t.

Inverse of a 2x2 matrix

- If det(M) = 0, meaning if M is singular, M^{-1} does not exist.
- If M is not singular:

The inverse of $\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, for $ad - bc \neq 0$,

is $\mathbf{M}^{-1} = \dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Proof:

Determinant of an inverse matrix:

For any matrix $\mathbf{M}$ with inverse $\mathbf{M}^{-1}$,

$$\left|\mathbf{M}^{-1}\right| = \frac{1}{|\mathbf{M}|}$$

Proof:

Exercises:

1 Find the inverse of each of the following matrices.

(a) $\begin{bmatrix} 1 & 0 \\ 5 & 7 \end{bmatrix}$, (b) $\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$, (c) $\begin{bmatrix} 4 & -3 \\ -2 & -1 \end{bmatrix}$.

2 Determine which of these matrices are singular and which are non-singular. For those that are non-singular find the inverse matrix.

a $\begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}$ **b** $\begin{pmatrix} 3 & 3 \\ -1 & -1 \end{pmatrix}$ **c** $\begin{pmatrix} 2 & 5 \\ 0 & 0 \end{pmatrix}$

d $\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$ **e** $\begin{pmatrix} 6 & 3 \\ 4 & 2 \end{pmatrix}$ **f** $\begin{pmatrix} 4 & 3 \\ 6 & 2 \end{pmatrix}$

3 Find inverses of these matrices.

a $\begin{pmatrix} a & 1+a \\ 1+a & 2+a \end{pmatrix}$ **b** $\begin{pmatrix} 2a & 3b \\ -a & -b \end{pmatrix}$

4 **a** Given that $\mathbf{ABC} = \mathbf{I}$, prove that $\mathbf{B}^{-1} = \mathbf{CA}$.

b Given that $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & -6 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 2 & 1 \\ -3 & -1 \end{pmatrix}$, find **B**.

5 **a** Given that $\mathbf{AB} = \mathbf{C}$, find an expression for **B**.

b Given further that $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 3 & 6 \\ 1 & 22 \end{pmatrix}$, find **B**.

6 **a** Given that $\mathbf{BAC} = \mathbf{B}$, where **B** is a non-singular matrix, find an expression for **A**.

b When $\mathbf{C} = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$, find **A**.

7 The matrix $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix}$ and $\mathbf{AB} = \begin{pmatrix} 4 & 7 & -8 \\ -8 & -13 & 18 \end{pmatrix}$. Find the matrix **B**.

8 The matrix $\mathbf{B} = \begin{pmatrix} 5 & -4 \\ 2 & 1 \end{pmatrix}$ and $\mathbf{AB} = \begin{pmatrix} 11 & -1 \\ -8 & 9 \\ -2 & -1 \end{pmatrix}$. Find the matrix **A**.

Inverse of a 3x3 matrix

- If det(M) = 0, meaning if M is singular, M^{-1} does not exist.

- If det(M) ≠ 0, follow the algorithm:

1. Find the determinant of **M**. If this is zero then stop.
2. Find the matrix of minor determinants.
3. Alter the signs of the minor determinants in the positions marked with minus signs: ⟶ $\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$
   This new matrix is called the matrix of **cofactors**.
4. Transpose the matrix of cofactors.
5. Divide by $|\mathbf{M}|$.

$$\begin{bmatrix} 1 & 2 & 3 \\ \boxed{4} & \not{0} & \not{5} \\ 6 & 7 & 8 \end{bmatrix}$$

The minor determinant associated with 4 is $\begin{vmatrix} 2 & 3 \\ 7 & 8 \end{vmatrix}$

The cofactor associated with 4 is $-\begin{vmatrix} 2 & 3 \\ 7 & 8 \end{vmatrix}$

Example:

$$\mathbf{M} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 5 \\ 6 & 7 & 8 \end{bmatrix}$$

$$\det(\mathbf{M}) = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 0 & 5 \\ 6 & 7 & 8 \end{vmatrix} = 1\begin{vmatrix} 0 & 5 \\ 7 & 8 \end{vmatrix} - 2\begin{vmatrix} 4 & 5 \\ 6 & 8 \end{vmatrix} + 3\begin{vmatrix} 4 & 0 \\ 6 & 7 \end{vmatrix} = -35 - 4 + 84 = 45$$

$$Cofactor\ matrix = \begin{bmatrix} -35 & -2 & 28 \\ 5 & -10 & 5 \\ 10 & 7 & -8 \end{bmatrix}$$

$$cofactor(M)^T = \begin{bmatrix} -35 & 5 & 10 \\ -2 & -10 & 7 \\ 28 & 5 & -8 \end{bmatrix}$$

$$\mathbf{M}^{-1} = \frac{1}{45}\begin{bmatrix} -35 & 5 & 10 \\ -2 & -10 & 7 \\ 28 & 5 & -8 \end{bmatrix}$$

Have a go:

- Find the inverse of $\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -3 \\ -1 & 0 & 5 \end{bmatrix}$

- Find the inverse of each of the following matrices.

(a) $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}$, (b) $\begin{bmatrix} 2 & 1 & -7 \\ 0 & -3 & 6 \\ -1 & 1 & -1 \end{bmatrix}$, (c) $\begin{bmatrix} 3 & -2 & 4 \\ 2 & 5 & -1 \\ -3 & 4 & 1 \end{bmatrix}$

- The matrix $\mathbf{A} = \begin{pmatrix} -2 & 3 & -3 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{pmatrix}$ and the matrix $\mathbf{B}$ is such that $(\mathbf{AB})^{-1} = \begin{pmatrix} 8 & -17 & 9 \\ -5 & 10 & -6 \\ -3 & 5 & -4 \end{pmatrix}$

a Show that $\mathbf{A}^{-1} = \mathbf{A}$.

b Find $\mathbf{B}^{-1}$.

Exercises:

1 Find the inverses of these matrices.

a $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ **b** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ **c** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & -\frac{4}{5} \\ 0 & \frac{4}{5} & \frac{3}{5} \end{pmatrix}$

2 Find the inverses of these matrices.

a $\begin{pmatrix} 1 & -3 & 2 \\ 0 & -2 & 1 \\ 3 & 0 & 2 \end{pmatrix}$ **b** $\begin{pmatrix} 2 & 3 & 2 \\ 3 & -2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$ **c** $\begin{pmatrix} 3 & 2 & -7 \\ 1 & -3 & 1 \\ 0 & 2 & -2 \end{pmatrix}$

3 The matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$ and the matrix $\mathbf{B} = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}$.

a Find $\mathbf{A}^{-1}$.

b Find $\mathbf{B}^{-1}$.

Given that $(\mathbf{AB})^{-1} = \begin{pmatrix} -\frac{2}{3} & \frac{1}{2} & \frac{1}{2} \\ 1 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{2}{3} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$,

c verify that $\mathbf{B}^{-1}\mathbf{A}^{-1} = (\mathbf{AB})^{-1}$.

4 The matrix $\mathbf{A} = \begin{pmatrix} 2 & 0 & 3 \\ k & 1 & 1 \\ 1 & 1 & 4 \end{pmatrix}$.

a Show that $\det(\mathbf{A}) = 3(k+1)$ **b** Given that $k \neq -1$, find $\mathbf{A}^{-1}$.

5 The matrix $\mathbf{A} = \begin{pmatrix} 5 & a & 4 \\ b & -7 & 8 \\ 2 & -2 & c \end{pmatrix}$.

Given that $\mathbf{A} = \mathbf{A}^{-1}$, find the values of the constants a, b and c.

6 The matrix $\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ 4 & -3 & 0 \\ -3 & 3 & 1 \end{pmatrix}$.

a Show that $\mathbf{A}^3 = \mathbf{I}$. **b** Hence find $\mathbf{A}^{-1}$.

7 The matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 3 & -3 & 1 \\ 0 & 3 & 2 \end{pmatrix}$.

a Show that $\mathbf{A}^3 = 13\mathbf{A} - 15\mathbf{I}$.

b Deduce that $15\mathbf{A}^{-1} = 13\mathbf{I} - \mathbf{A}^2$.

c Hence find $\mathbf{A}^{-1}$.

8 The matrix $\mathbf{A} = \begin{pmatrix} 2 & 0 & 1 \\ 4 & 3 & -2 \\ 0 & 3 & -4 \end{pmatrix}$.

a Show that $\mathbf{A}$ is singular.

The matrix $\mathbf{C}$ is the matrix of the cofactors of $\mathbf{A}$.

b Find $\mathbf{C}$.

c Show that $\mathbf{AC}^{T} = \mathbf{0}$.

1 a $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$ **b** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$ **c** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{4}{5} \\ 0 & -\frac{4}{5} & \frac{3}{5} \end{pmatrix}$

2 a $\begin{pmatrix} 4 & -6 & -1 \\ -3 & 4 & 1 \\ -6 & 9 & 2 \end{pmatrix}$ **b** $\begin{pmatrix} -\frac{3}{5} & -\frac{1}{5} & \frac{7}{5} \\ -\frac{1}{5} & -\frac{2}{5} & \frac{4}{5} \\ \frac{7}{5} & \frac{4}{5} & -\frac{13}{5} \end{pmatrix}$ **c** $\begin{pmatrix} 2 & -5 & -\frac{19}{2} \\ 1 & -3 & -5 \\ 1 & -3 & -\frac{11}{2} \end{pmatrix}$

3 a $\begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix}$ **b** $\begin{pmatrix} \frac{1}{3} & \frac{1}{2} & -\frac{1}{6} \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{pmatrix}$

4 b $\frac{1}{3(k+1)}\begin{pmatrix} 3 & 3 & -3 \\ 1-4k & 5 & 3k-2 \\ k-1 & -2 & 2 \end{pmatrix}$

5 $a = -4, b = 8, c = 3$

6 b $\begin{pmatrix} -3 & 4 & 3 \\ -4 & 5 & 4 \\ 3 & -3 & -2 \end{pmatrix}$

7 c $\frac{1}{15}\begin{pmatrix} 9 & 2 & -1 \\ 6 & -2 & 1 \\ -9 & 3 & 6 \end{pmatrix}$

8 b $\begin{pmatrix} -6 & 16 & 12 \\ 3 & -8 & -6 \\ -3 & 8 & 6 \end{pmatrix}$