Proof by induction

Specifications:

Proof by Induction

Applications to sequences and series, and other problems.

E.g. proving that $7^n + 4^n + 1$ is divisible by 6, or

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ where n is a positive integer.

The dominoes analogy

For this domino cascade to work, you need two conditions:

- You must be able to push the first domino
- If a domino falls, the one following it falls as well.

Presentation of the proof by induction

1) Define the statement/ proposition
2) Basis case : show that proposition is true for n = 1
3) Hypothesis / assumption : Assume that the proposition is true for n = k
   Write down what you want to prove
4) Induction : show that the proposition is true for n = k+1 (assuming that it is true for n=k)
5) Conclusion: "If the proposition is true for n=k, then it is true for n=k+1.
   Because it is true for n=1, we can conclude that it is true for all n $\geq$ 1"

Application

1) We have to prove by induction the proposition P_n : for all n $\geq$ 1, 1+3+5+...+(2n-1) = n^2

Basis case : $n=1$

$LHS : 1 \qquad RHS : 1^2 = 1$

The proposition is true for n=1

Hypothesis :

Suppose that the proposition is true for $n = k$

$$i.e: \ 1+3+5+...(2k-1)=k^2$$

Let's show that the proposition is then true for $n=k+1$

$$i.e: \text{Let's show that } 1+3+5+...+(2k-1)+(2k+1)=(k+1)^2$$

Induction:

$$1+3+5+...+(2k-1)+(2k+1)=k^2+(2k+1)$$

$$=k^2+2k+1=(k+1)^2 \quad Q.E.D$$

Conclusion:

If the propostion is true for $n=k$, then it is true for $n=k+1$.

Because the proposition is true for $n=1$, we can conclude that

it is true for all n ≥ 1.

Exercise 1

A sequence u_n is defined by $u_1 = 2$ and $u_{n+1} = u_n + 5$

1) Work out u_2, u_3, u_4
2) Prove by induction that for all $n \geq 1$, $u_n = 5n - 3$

Exercises: "Series"

1 Use the method of mathematical induction to prove that, for all positive integers n,

$$2+4+6+\cdots+(2n)=n(n+1).$$

2 Use the method of mathematical induction to prove that, for $n \in \mathbb{N}$,

$$1^3+2^3+3^3+\cdots+n^3=\tfrac{1}{4}n^2(n+1)^2.$$

3 Use the method of mathematical induction to prove that, for all positive integers n,

$$1\times 4+2\times 5+3\times 6+\cdots+n(n+3)=\tfrac{1}{3}n(n+1)(n+5).$$

Prove by the method of mathematical induction, the following statements for $n \in \mathbb{Z}^+$.

$$\sum_{r=1}^{n} r(r!) = (n+1)! - 1$$

$$\sum_{r=1}^{n} 4^{r-1} = \frac{4^n - 1}{3}$$

"Multiples of ..." problems

Given that $u_n = 5^n + 9^n + 2$

a express $u_{n+1} - 5u_n$, simplifying your answer as far as possible

b hence prove by induction that $5^n + 9^n + 2$ is exactly divisible by 4 for all $n \geqslant 1$.

Given that $u_n = 4 \times 2^n + 3 \times 9^n$

a show that $u_{n+1} - 2u_n = 21 \times 9^n$ and hence prove by induction that u_n is a multiple of 7 for all $n \geqslant 1$

b prove by induction that for all $n \geqslant 1$, u_n is *not* exactly divisible by 3.

Prove, by induction that $n^3 - 7n + 9$ is divisible by 3 for all integers for $n \in \mathbb{Z}^+$.

$f(n) = 13^n - 6^n, n \in \mathbb{Z}^+$.

a Express for $k \in \mathbb{Z}^+$, $f(k + 1) - 6f(k)$ in terms of k, simplifying your answer.

b Use the method of mathematical induction to prove that $f(n)$ is divisible by 7 for all $n \in \mathbb{Z}^+$.

$g(n) = 5^{2n} - 6n + 8, n \in \mathbb{Z}^+$.

a Express for $k \in \mathbb{Z}^+$, $g(k + 1) - 25g(k)$ in terms of k, simplifying your answer.

b Use the method of mathematical induction to prove that $g(n)$ is divisible by 9 for all $n \in \mathbb{Z}^+$.

"Sequences" problems

- Given that $u_{n+1} = 3u_n + 4$, $u_1 = 1$, prove by induction that $u_n = 3^n - 2$.
- Given that $u_{n+1} = 3u_n + 1$, $u_1 = 1$, prove by induction that $u_n = \frac{3^n - 1}{2}$.
- Given that $u_{n+2} = 5u_{n+1} - 6u_n$, $u_1 = 1$, $u_2 = 5$ prove by induction that $u_n = 3^n - 2^n$.
- Given that $u_{n+2} = 6u_{n+1} - 9u_n$, $u_1 = -1$, $u_2 = 0$, prove by induction that $u_n = (n - 2)3^{n-1}$.
- Given that $u_{n+2} = 7u_{n+1} - 10u_n$, $u_1 = 1$, $u_2 = 8$, prove by induction that $u_n = 2(5^{n-1}) - 2^{n-1}$.
- Given that $u_{n+2} = 6u_{n+1} - 9u_n$, $u_1 = 3$, $u_2 = 36$, prove by induction that $u_n = (3n - 2)3^n$.

Miscellaneous exercises:

2 Use induction to prove the following for all natural numbers n.

a $1 + 3 + 3^2 + 3^3 + \cdots + 3^n = \frac{1}{2}(3^{n+1} - 1)$

c $\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

3 Use induction to prove that each statement in question **1** is true for all $n \geqslant 1$.

4 Prove by induction that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$ for all $n \in \mathbb{N}$.

5 A sequence is defined by the iterative formula

$$u_1 = 2$$
$$u_{n+1} = 3u_n + 2 \quad \text{for } n \geqslant 1$$

Prove by induction that $u_n = 3^n - 1$ for all $n \geqslant 1$

6 $\mathbf{A} = \begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix}$

a Prove by induction that $\mathbf{A}^n = \begin{pmatrix} 2n+1 & 4n \\ -n & 1-2n \end{pmatrix}$ for all $n \in \mathbb{N}$.

b Use the result of part **a** to show that $\det(\mathbf{A}^n) = 1$ for all positive integers n.

7 Use induction to prove that $6^n + 4$ is exactly divisible by 10 for all $n \in \mathbb{N}$.

8 Given that $x_n = 3^{2n} - 1$ and $y_n = 3^{2n-1} + 1$

a use induction to prove that, for all $n \in \mathbb{N}$,

i x_n is a multiple of 8 **ii** y_n is exactly divisible by 4

b by simplifying the expression $x_n + 2y_n$, or otherwise, prove that $5 \times 3^{2n-1} + 1$ is a multiple of 8 for all $n \geqslant 1$.

The function f is defined by $f(n) = 5^{2n-1} + 1$, where $n \in \mathbb{Z}^+$.

a Show that $f(n+1) - f(n) = \mu(5^{2n-1})$, where μ is an integer to be determined.

b Hence prove by induction that $f(n)$ is divisible by 6.

Use the method of mathematical induction to prove that $7^n + 4^n + 1$ is divisible by 6 for all $n \in \mathbb{Z}^+$.

A sequence $u_1, u_2, u_3, u_4, \ldots$ is defined by $u_{n+1} = \frac{3u_n - 1}{4}$, $u_1 = 2$.

a Find the first five terms of the sequence.

b Prove, by induction for $n \in \mathbb{Z}^+$, that $u_n = 4\left(\frac{3}{4}\right)^n - 1$.

Summary of key points

1 Mathematical induction is used to prove whether or not general statements are true, usually for positive integers, n.

2 When performing a proof by mathematical induction you need to apply the following four steps:

- **basis:** Show the general statement if true for $n = 1$.
- **assumption:** Assume that the general statement is true for $n = k$.
- **induction:** Show the general statement is true for $n = k + 1$.
- **conclusion:** Then state that the general statement is then true for all positive integers, n.

3 Proof by induction is of no use for deriving formulae from first principles. Proof by induction is used, however, to check whether or not a general statement is true.