Proof by induction - exam questions

Question 1: June 2007

(a) Show that

$$\left(1 - \frac{1}{(k+1)^2}\right) \times \frac{k+1}{2k} = \frac{k+2}{2(k+1)} \qquad \textit{(3 marks)}$$

(b) Prove by induction that for all integers $n \geqslant 2$

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\ldots\left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n} \qquad \textit{(4 marks)}$$

Question 2: Jan 2006

(a) Prove by induction that

$$2 + (3 \times 2) + (4 \times 2^2) + \ldots + (n+1)\,2^{n-1} = n\,2^n$$

for all integers $n \geqslant 1$. *(6 marks)*

(b) Show that

$$\sum_{r=n+1}^{2n} (r+1)\,2^{r-1} = n\,2^n(2^{n+1} - 1) \qquad \textit{(3 marks)}$$

Question 3: Jan 2008

Prove by induction that for all integers $n \geqslant 1$

$$\sum_{r=1}^{n} (r^2 + 1)(r!) = n(n+1)! \qquad \textit{(7 marks)}$$

Question 4:Jan 2009

Prove by induction that

$$\frac{2 \times 1}{2 \times 3} + \frac{2^2 \times 2}{3 \times 4} + \frac{2^3 \times 3}{4 \times 5} + \ldots + \frac{2^n \times n}{(n+1)(n+2)} = \frac{2^{n+1}}{n+2} - 1$$

for all integers $n \geqslant 1$. *(7 marks)*

Question 5: June 2008

(a) Explain why $n(n+1)$ is a multiple of 2 when n is an integer. *(1 mark)*

(b) (i) Given that

$$\mathrm{f}(n) = n(n^2 + 5)$$

show that $\mathrm{f}(k+1) - \mathrm{f}(k)$, where k is a positive integer, is a multiple of 6. *(4 marks)*

(ii) Prove by induction that $\mathrm{f}(n)$ is a multiple of 6 for all integers $n \geqslant 1$. *(4 marks)*

Question 6: June 2010

(a) Show that $\dfrac{1}{(k+2)!} - \dfrac{k+1}{(k+3)!} = \dfrac{2}{(k+3)!}$. *(2 marks)*

(b) Prove by induction that, for all positive integers n,

$$\sum_{r=1}^{n} \frac{r \times 2^r}{(r+2)!} = 1 - \frac{2^{n+1}}{(n+2)!}$$ *(6 marks)*

Question 7:Jjan 2010

The sequence u_1, u_2, u_3,... is defined by

$$u_1 = 2, \quad u_{k+1} = 2u_k + 1$$

(a) Prove by induction that, for all $n \geqslant 1$,

$$u_n = 3 \times 2^{n-1} - 1$$ *(5 marks)*

(b) Show that

$$\sum_{r=1}^{n} u_r = u_{n+1} - (n+2)$$ *(3 marks)*

Question 8: June 2006

(a) The function f is given by

$$f(n) = 15^n - 8^{n-2}$$

Express

$$f(n+1) - 8f(n)$$

in the form $k \times 15^n$. *(4 marks)*

(b) Prove by induction that $15^n - 8^{n-2}$ is a multiple of 7 for all integers $n \geqslant 2$. *(4 marks)*

Question 9: Jan 2007

(a) Prove by induction that, if n is a positive integer,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$ *(5 marks)*

(b) Find the value of $\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^6$. *(2 marks)*

(c) Show that

$$(\cos\theta + i\sin\theta)(1 + \cos\theta - i\sin\theta) = 1 + \cos\theta + i\sin\theta$$ *(3 marks)*

(d) Hence show that

$$\left(1 + \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^6 + \left(1 + \cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)^6 = 0$$ *(4 marks)*

Proof by induction - exam questions - MS

Question 1: June 2007

6(a)	$\left(1-\dfrac{1}{(k+1)^2}\right)\times\dfrac{k+1}{2k}=\dfrac{(k+1)^2-1}{(k+1)^2}\times\dfrac{k+1}{2k}$	M1		
	$=\dfrac{k^2+2k}{(k+1)^2}\times\dfrac{k+1}{2k}$	A1		
	$=\dfrac{k+2}{2(k+1)}$	A1	3	AG
(b)	Assume true for $n=k$, then			
	$\left(1-\dfrac{1}{2^2}\right)\left(1-\dfrac{1}{3^2}\right)\ldots\left(1-\dfrac{1}{(k+1)^2}\right)$	M1		
	$=\dfrac{k+2}{2(k+1)}$	A1		
	True for $n=2$ shown $1-\dfrac{1}{2^2}=\dfrac{3}{4}$	B1		
	$P_n \Rightarrow P_{n+1}$ and P_2 true	E1	4	only if the other 3 marks earned
	Total		7	

Question 2: Jan 2006

4(a)	Assume result true for $n=k$			
	$\sum_{r=1}^{k}(r+1)2^{r-1}=k2^k$			
	$\sum_{r=1}^{k+1}(r+1)2^{r-1}=k2^k+(k+2)2^k$	M1A1		
	$=2^k(k+k+2)$	m1		
	$=2^k(2k+2)$			
	$=2^{k+1}(k+1)$	A1		
	$n=1 \quad 2\times2^0=2=1\times2^1$	B1		
	$P_k \Rightarrow P_{k+1}$ and P_1 is true	E1	6	Provided previous 5 marks earned
(b)	$\sum_{r=1}^{2n}(r+1)2^{r-1}-\sum_{r=1}^{n}(r+1)2^{r-1}$	M1		Sensible attempt at the difference between 2 series
	$=2n\ 2^{2n}-n2^n$	A1		
	$=n\left(2^{n+1}-1\right)2^n$	A1	3	AG
	Total		9	

Question 3: Jan 2008

5	Assume result true for $n=k$			
	Then $\sum_{r=1}^{k+1}(r^2+1)r!$			
	$=((k+1)^2+1)(k+1)!+k(k+1)!$	M1A1		
	Taking out $(k+1)!$ as factor	m1		
	$=(k+1)!(k^2+2k+1+1+k)$	A1		
	$=(k+1)(k+2)!$	A1		
	$k=1$ shown $(1^2+1)1!=2$; $1\times 2!=2$	B1		
	$P_k \Rightarrow P_{k+1}$ and P_1 true	E1	7	If all 6 marks earned
	Total		7	

Question 4: Jan 2009

6	Assume result true for $n=k$			
	Then $\sum_{r=1}^{k+1}\frac{2^r\times r}{(r+1)(r+2)}$			SC If no series at all indicated on LHS, deduct 1 and give E0 at end
	$=\frac{2^{k+1}}{k+2}+\frac{2^{k+1}(k+1)}{(k+2)(k+3)}-1$	M1A1		
	$=\frac{2^{k+1}(k+3+k+1)}{(k+2)(k+3)}-1$	M1		Putting over common denominator (not including the -1, unless separated later)
	$=\frac{2^{k+1}2(k+2)}{(k+2)(k+3)}-1$	A1		
	$=\frac{2^{k+2}}{k+3}-1$	A1		
	$k=1$: LHS $=\frac{1}{3}$, RHS $=\frac{2^2}{3}-1$	B1		
	$P_k \Rightarrow P_{k+1}$ and P_1 true	E1	7	Must be completely correct
	Total		7	

Question 5: June 2008

7(a)	Clear reason given	E1	1	Minimum O × E = E
(b)(i)	$(k+1)((k+1)^2+5)-k(k^2+5)$	M1		
	$=3k^2+3k+6$	A1		
	$k^2+k=k(k+1)=M(2)$	E1		Must be shown
	$f(k+1)-f(k)=M(6)$	E1	4	
(ii)	Assume true for $n=k$			
	$f(k+1)-f(k)=M(6)$	M1		Clear method
	$\therefore f(k+1)=M(6)+f(k)$			
	$=M(6)+M(6)$	A1		
	$=M(6)$			
	True for $n=1$	B1		
	$P(n)\to P(n+1)$ and P(1) true	E1	4	Provided all other marks earned in (b)(ii)
	Total		9	

Question 6: June 2010

6(a)	$\frac{1}{(k+2)!} = \frac{k+3}{(k+3)!}$	M1		
	Result	A1	2	
(b)	Assume true for $n = k$			
	For $n = k+1$			
	$\sum_{r=1}^{k+1} \frac{r \times 2^r}{(r+2)!} = 1 - \frac{2^{k+1}}{(k+2)!} + \frac{(k+1)2^{k+1}}{(k+3)!}$	M1A1		If no LHS of equation, M1A0
	$= 1 - 2^{k+1}\left(\frac{1}{(k+2)!} - \frac{k+1}{(k+3)!}\right)$	m1		m1 for a suitable combination clearly shown
	$= 1 - \frac{2^{k+2}}{(k+3)!}$	A1		clearly shown or stated true for $n = k + 1$
	True for $n = 1$	B1		Shown
	Method of induction set out properly	E1	6	Provided previous 5 marks all earned
	Total		**8**	

Question 7:Jjan 2010

7(a)	Assume true for $n = k$			
	$u_{k+1} = 2(3 \times 2^{k-1} - 1) + 1$	M1A1		
	$= 3 \times 2^k - 1$	A1		$2^{(k-1)+1}$ not necessarily seen
	True for $n = 1$ shown	B1		
	Method of induction clearly expressed	E1	5	Provided all 4 previous marks earned
(b)	$\sum_{r=1}^{n} u_r = \sum_{r=1}^{n} 3 \times 2^{r-1} - n$			
	$= 3(2^n - 1) - n$	M1A1		M1 for summation, ie recognition of a GP
	$= u_{n+1} - (n+2)$	A1	3	AG
	Total		**8**	

Question 8: June 2006

6(a)	$f(n+1) - 8f(n) = 15^{n+1} - 8^{n-1}$			
	$-8(15^n - 8^{n-2})$	M1A1		
	$= 15^{n+1} - 8.15^n$			
	$= 15^n(15 - 8)$	M1		For multiples of powers of 15 only
	$= 7.15^n$	A1	4	For valid method ie not using 120^n etc
(b)	Assume $f(n)$ is $M(7)$			
	Then $f(n+1) - 8f(n) = 7 \times 15^n$	M1		Or considering $f(n+1) - f(n)$
	$f(n+1) = M(7) + M(7)$			
	$= M(7)$	A1		
	$n = 2$: $f(n) = 15^2 - 8^0 = 224$			
	$= 7 \times 32$	B1		$n = 1$ B0
	$P(n) \Rightarrow P(n+1)$ and $P(2)$ true	E1	4	Must score previous 3 marks to be awarded E1
	Total		**8**	