Hyperbolic functions

specifications

Hyperbolic Functions

Hyperbolic and inverse hyperbolic functions and their derivatives; applications to integration.

The proofs mentioned below require expressing hyperbolic functions in terms of exponential functions.

To include solution of equations of the form $a\sinh x + b\cosh x = c$.

Use of basic definitions in proving simple identities.

Maximum level of difficulty:

$\sinh(x+y) \equiv \sinh x\cosh y + \cosh x\sinh y$.

The logarithmic forms of the inverse functions, given in the formulae booklet, may be required. Proofs of these results may also be required.

Proofs of the results of differentiation of the hyperbolic functions, given in the formula booklet, are included.

Knowledge, proof and use of:

$$\cosh^2 x - \sinh^2 x = 1$$

$$1 - \tanh^2 x = \text{sech}^2 x$$

$$\coth^2 x - 1 = \text{cosech}^2 x$$

Familiarity with the graphs of

$\sinh x$, $\cosh x$, $\tanh x$, $\sinh^{-1} x$, $\cosh^{-1} x$, $\tanh^{-1} x$.

Introduction

When studying the complex numbers chapter,
we have established that

$$for\ all\ \theta,\ Cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$and \qquad Sin\theta = \frac{e^{i\theta} - e^{i\theta}}{2i}$$

These two functions have an equivalent in the real set of numbers

For all x,

$$\frac{e^{x} + e^{-x}}{2} = Cosh(x) \qquad \text{"}read\ \text{cosh}(x)\text{"}$$

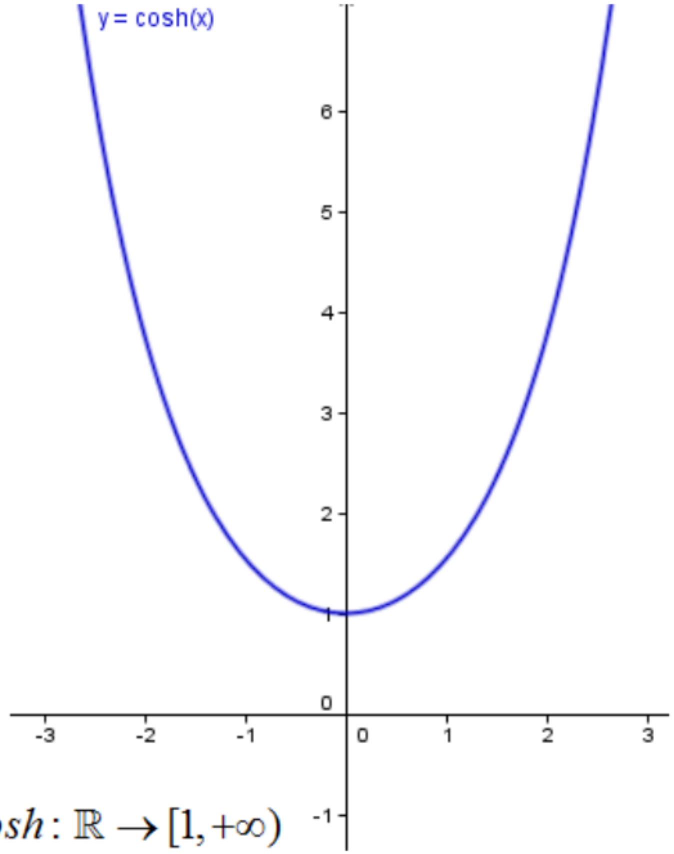
$$\frac{e^{x} - e^{-x}}{2} = Sinh(x) \qquad \text{"}read\ \text{shine(x)"}$$

Property 1:

For all x, $Cosh(x) + Sinh(x) = e^{x}$

Prove it:

Properties and graphs



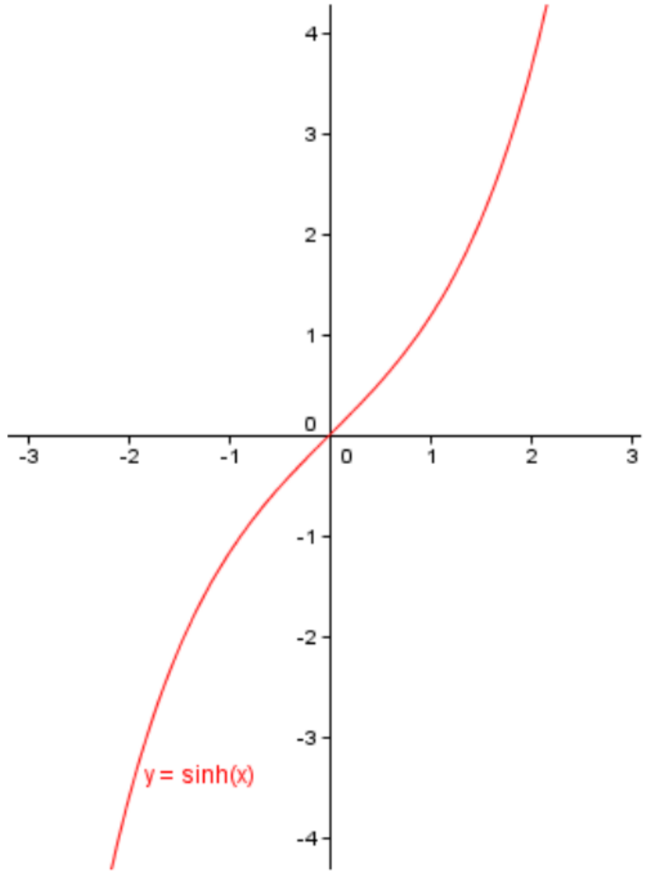
$Cosh : \mathbb{R} \rightarrow [1,+\infty)$

$$x \rightarrow \frac{e^x + e^{-x}}{2}$$

Cosh is an EVEN function

for all x, $Cosh(-x) = Cosh(x)$

Consequence: the graph of *Cosh* is symmetrical around the line $x = 0$



$Sinh : \mathbb{R} \rightarrow \mathbb{R}$

$$x \rightarrow \frac{e^x - e^{-x}}{2}$$

Sinh is an ODD function

for all x, $Sinh(-x) = -Sinh(x)$

Consequence: the graph of *Cosh* is symmetrical around the origin

(Rotational order 2)

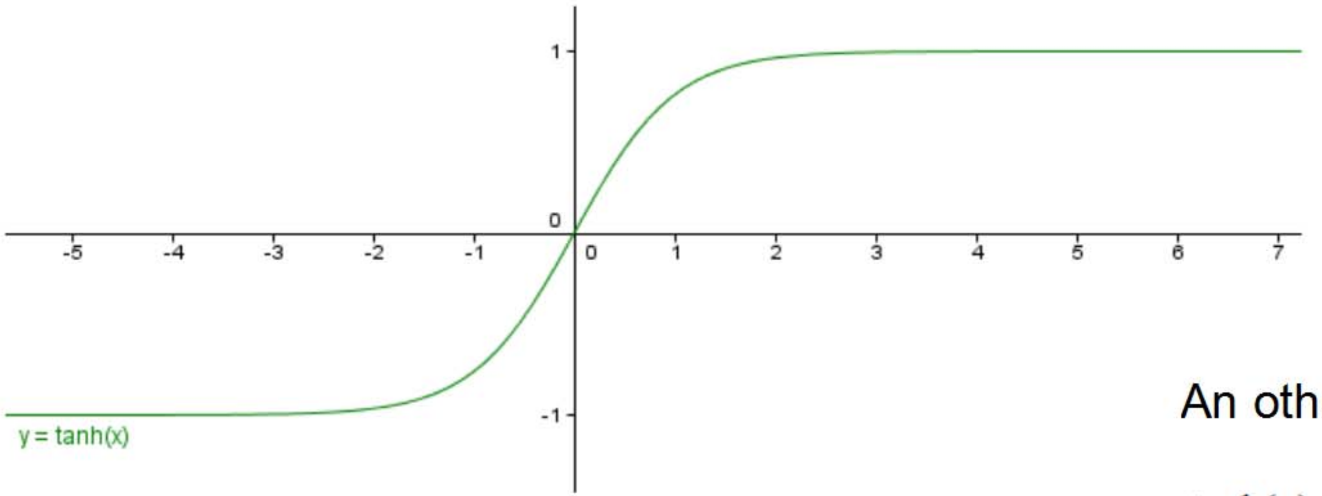
There are four other hyperbolic functions derived from these, just as there are four trigonometric functions. They are:

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \text{‘than } x\text{’}$$

$$\operatorname{cosech} x = \frac{1}{\sinh x} \qquad \text{‘cosheck } x\text{’}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} \qquad \text{‘sheck } x\text{’}$$

$$\coth x = \frac{1}{\tanh x} \qquad \text{‘coth } x\text{’}$$



$$Tanh : \mathbb{R} \to]-1,1[$$

$$x \to \frac{Sinh(x)}{Cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

An other expression of tanh

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \times \frac{e^x}{e^x} = \frac{e^{2x} - 1}{e^{2x} + 1}$$

$$\tanh(x) = \frac{e^{2x} - 1}{e^{2x} + 1}$$

Usual identities

You should know these identities and
you need to be able to prove them using the exponential form of cosh and sinh.

$$Cosh^2(x) - Sinh^2(x) \equiv 1$$

$$Sech^2(x) = 1 - Tanh^2(x)$$

$$\text{Cosech}^2(x) = Coth^2(x) - 1$$

Using the exponential form of these functions, prove the above identities

Other identities

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

$$\sinh 2x = 2\sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$= 2\cosh^2 x - 1$$

$$= 1 + 2\sinh^2 x$$

Using the exponential expression of Cosh and Sinh,
Prove the following identity

$$\sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y$$

Exercises:

Show that $\cosh 3x = 4\cosh^3 x - 3\cosh x$.

Show that $\tanh 2x = \dfrac{2\tanh x}{1+\tanh^2 x}$.

Calculus

Differentiating hyperbolic functions

- $\frac{d}{dx}Cosh(x) = Sinh(x)$
- $\frac{d}{dx}Sinh(x) = Cosh(x)$
- $\frac{d}{dx}Tanh(x) = 1 - Tanh^2 x = Sech^2 x$

- $\int Cosh(x)dx = Sinh(x) + c$
- $\int Sinh(x)dx = Cosh(x) + c$
- $\int Tanh(x)dx = \int \frac{Sinh(x)}{Cosh(x)}dx = \ln(Cosh(x)) + c$

Proof of some of these results:

- $$\frac{d}{dx}Cosh(x) = \frac{d}{dx}\left(e^x + e^{-x}\right) = e^x - e^{-x} = Sinh(x)$$
- $$\frac{d}{dx}Sinh(x) = \frac{d}{dx}\left(e^x - e^{-x}\right) = e^x + e^{-x} = Cosh(x)$$
- $$\frac{d}{dx}Tanh(x) = \frac{d}{dx}\left(\frac{Sinh(x)}{Cosh(x)}\right) = \frac{Cosh(x)\times Cosh(x) - Sinh(x)\times Sinh(x)}{Cosh^2 x}$$

$$\frac{d}{dx}Tanh(x) = \frac{Cosh^2(x) - Sinh^2(x)}{Cosh^2 x} = 1 - Tanh^2(x)$$

Exercises:

Differentiate the following expressions:

(a) $\cosh 3x$, (b) $\cosh^2 3x$, (c) $x^2 \cosh x$,

(d) $\frac{\cosh 2x}{x}$, (e) $x \tanh x$, (f) $\operatorname{sech} x$ [hint: write $\operatorname{sech} x$ as $(\cosh x)^{-1}$],

(g) $\operatorname{cosech} x$.

Differentiate these functions with respect to x.

(a) $\cosh 4x$ (b) $\sinh 5x$ (c) $\sinh x \cosh x$ (d) $\sinh^2 x$

(e) $\sqrt{\cosh x}$ (f) $\ln(\sinh x + \cosh x)$ (g) $\frac{1}{\sinh x + 2\cosh x}$ (h) $\frac{1 + \sinh x}{x + \cosh x}$

(i) $\frac{\cosh x + \sin x}{\cos x + \sinh x}$ (j) $e^x \cosh x$

Find the following indefinite integrals.

(a) $\int \sinh 3x \, dx$ (b) $\int \sinh x \cosh x \, dx$ (c) $\int \cosh^2 x \, dx$

(d) $\int x \sinh x \, dx$ (e) $\int x^2 \cosh 2x \, dx$ (f) $\int e^x \cosh x \, dx$

Evaluate the following definite integrals, correct to 3 significant figures.

(a) $\int_0^1 \cosh 4x \, dx$ (b) $\int_0^{\ln 2} \sinh 4x \, dx$ (c) $\int_{-\ln 3}^{\ln 5} \cosh 2x \, dx$

(d) $\int_0^2 \sinh^2 x \, dx$ (e) $\int_1^2 x \cosh 2x \, dx$ (f) $\int_0^1 \sinh^2 x \cosh x \, dx$

(a) $3\sinh 3x$ (b) $6\sinh 3x \cosh 3x$ (c) $2x\cosh x + x^2 \sinh x$

(d) $\frac{2x\sinh 2x - \cosh 2x}{x^2}$ (e) $\tanh x + x\operatorname{sech}^2 x$ (f) $-\operatorname{sech} x \tanh x$

(g) $-\operatorname{cosech} x \coth x$

(a) $4\sinh 4x$ (b) $5\cosh 5x$

(c) $\cosh 2x$ (d) $2\sinh x \cosh x$

(e) $\frac{\sinh x}{2\sqrt{\cosh x}}$ (f) 1

(g) $-\frac{\cosh x + 2\sinh x}{(\sinh x + 2\cosh x)^2}$

(h) $\frac{x\cosh x - 2\sinh x}{(x + \cosh x)^2}$

(i) $\frac{2\cos x \sinh x}{(\cos x + \sinh x)^2}$

(j) e^{2x}

(a) $\frac{1}{3}\cosh 3x + k$

(b) $\frac{1}{4}\cosh 2x + k$

(c) $\frac{1}{2}(\sinh x \cosh x + x) + k$

(d) $x\cosh x - \sinh x + k$

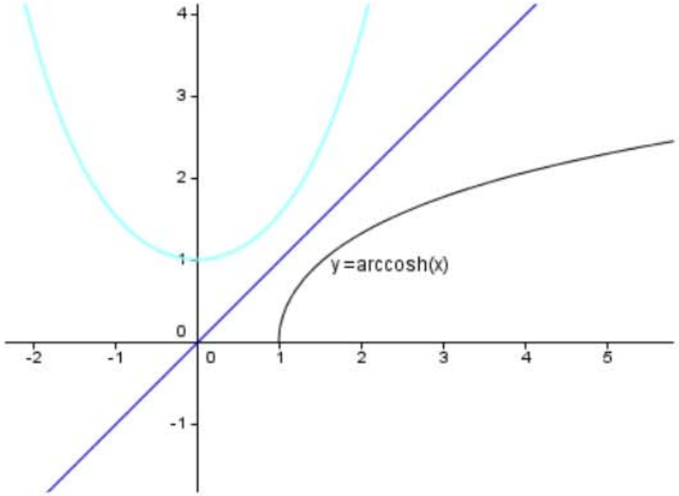
(e) $\frac{1}{2}(x^2 + \frac{1}{2})\sinh 2x - \frac{1}{2}x\cosh 2x + k$

(f) $\frac{1}{4}e^{2x} + \frac{1}{2}x + k$

(a) 6.82 (b) 1.76 (c) 8.46

(d) 5.82 (e) 19.6 (f) 0.541

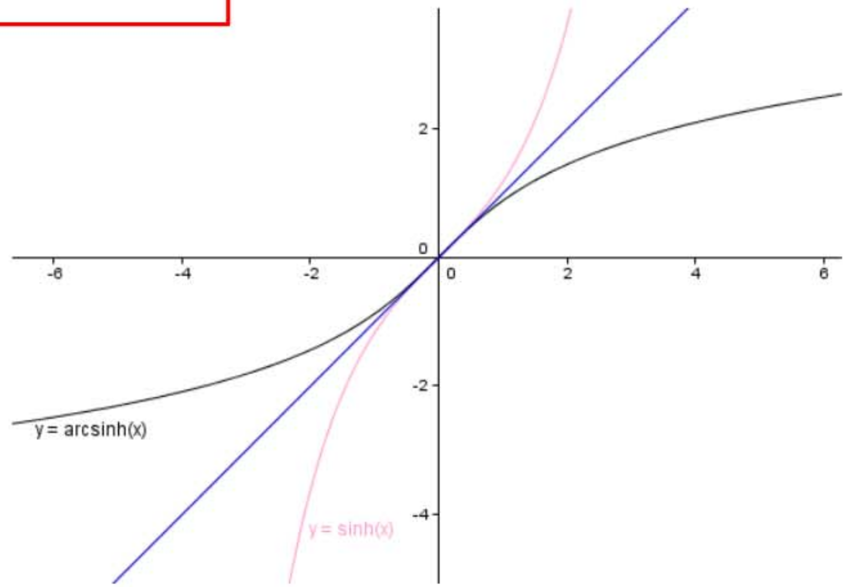
Inverse hyperbolic functions



$$Cosh^{-1}(x) = \text{Arccosh(x)}: \quad [1,+\infty) \to \mathbb{R}^{+}$$

$$x \to y = Cosh^{-1}(x)$$

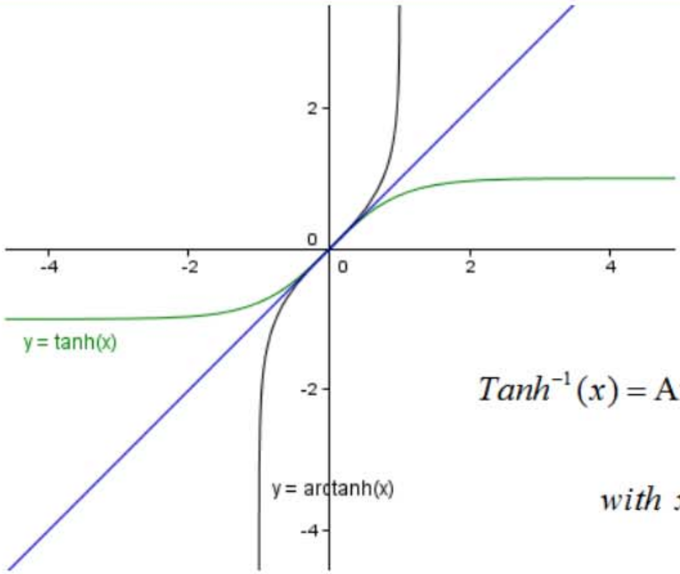
$$with\ x = \cosh(y)$$



$$Sinh^{-1}(x) = \text{Arcsinh}(x): \quad \mathbb{R} \to \mathbb{R}$$

$$x \to y = Sinh^{-1}(x)$$

$$with\ x = Sinh(y)$$



$$Tanh^{-1}(x) = \text{Arctanh}(x): \quad]-1,1[\to \mathbb{R}$$

$$x \to y = Tanh^{-1}(x)$$

$$with\ x = Tanh(y)$$

The logarithm expressions of the inv. hyp. functions

The log expressions are established using the following basic properties:

$$Cosh(y) + Sinh(y) = e^y$$

and

$$Cosh^2(y) - Sinh^2(y) = 1$$

For all x, $\sinh^{-1} x = \ln(x + \sqrt{1 + x^2}\,)$.

For $x \geqslant 1$, $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}\,)$.

For $-1 < x < 1$, $\tanh^{-1} x = \frac{1}{2} \ln \dfrac{1 + x}{1 - x}$.

You need to be able to prove these identities

Proof:

$y = Sinh^{-1}(x)$ *means* $Sinh(y) = x$

We are going to use these to properties

- $Cosh(y) + Sinh(y) = e^y$
- $Cosh^2(y) - Sinh^2(y) = 1$

$$Cosh(y) = \sqrt{1 + Sinh^2(y)}$$

so $Cosh(y) + Sinh(y) = e^y$ *becomes*

$$\sqrt{1 + Sinh^2(y)} + Sinh(y) = e^y$$

$\sqrt{1 + x^2} + x = e^y$ *and finally* by composing *by* ln

$$y = \ln\left(\sqrt{1 + x^2} + x\right)$$

$y = Tanh^{-1}(x)$ *so* $x = Tanh(y)$

We have established that $:Tanh(y) = \dfrac{e^{2y} - 1}{e^{2y} + 1} = x$

By making e^{2y} the subject:

$$e^{2y} - 1 = x(e^{2y} + 1)$$

$$(1 - x)e^{2y} = 1 + x$$

$$e^{2y} = \frac{1 + x}{1 - x}$$

$$2y = \ln\left(\frac{1 + x}{1 - x}\right) \qquad y = \frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right)$$

Exercises:

7 Express as natural logarithms.

a arsinh 2 **b** arcosh 3 **c** artanh $\frac{1}{2}$

8 Express as natural logarithms.

a arsinh $\sqrt{2}$ **b** arcosh $\sqrt{5}$ **c** artanh 0.1

9 Express as natural logarithms.

a arsinh (-3) **b** arcosh $\frac{3}{2}$ **c** artanh $\frac{1}{\sqrt{3}}$

10 Given that artanh x + artanh $y = \ln\sqrt{3}$, show that $y = \dfrac{2x-1}{x-2}$.

Preambule to solving equations

$Cosh(x) = a \quad with \ a \geq 1$

$is \ equivalent \ to:$

$$x = \pm Cosh^{-1}(a)$$

$$x = \pm \ln\left(a + \sqrt{a^2 - 1}\right)$$

$$x = \ln\left(a \pm \sqrt{a^2 - 1}\right)$$

$Sinh(x) = a \quad with \ a \in \mathbb{R}$

$is \ equivalent \ to:$

$$x = Sinh^{-1}(a)$$

$$x = \ln\left(a + \sqrt{1 + a^2}\right)$$

$Tanh(x) = a \quad with \ -1 < a < 1$

$is \ equivalent \ to:$

$$x = Tanh^{-1}(a)$$

$$x = \frac{1}{2}\ln\left(\frac{1+a}{1-a}\right)$$

Solving equations involving hyperbolic functions

- Solve, for real values of x,

  $$6\sinh x - 2\cosh x = 7$$

- Solve $2\cosh^2 x - 5\sinh x = 5$, giving your answers as natural logarithms.

- Solve $\cosh 2x - 5\cosh x + 4 = 0$, giving your answers as natural logarithms where appropriate.

Exercises:

Solve the following equations, giving your answers as natural logarithms.

1 $3\sinh x + 4\cosh x = 4$

2 $7\sinh x - 5\cosh x = 1$

3 $30\cosh x = 15 + 26\sinh x$

4 $13\sinh x - 7\cosh x + 1 = 0$

5 $\cosh 2x - 5\sinh x = 13$

6 $2\tanh^2 x + 5\operatorname{sech} x - 4 = 0$

7 $3\sinh^2 x - 13\cosh x + 7 = 0$

8 $\sinh 2x - 7\sinh x = 0$

9 $4\cosh x + 13e^{-x} = 11$

10 $2\tanh x = \cosh x$

1 $x = \ln\left(\frac{1}{7}\right), x = 0$
2 $x = \ln 3$
3 $x = \ln\left(\frac{7}{2}\right), x = \ln 4$
4 $x = \ln\left(\frac{5}{3}\right)$
5 $x = \ln\left(\frac{-3 + \sqrt{13}}{2}\right)$
$x = \ln(4 + \sqrt{17})$
6 $x = \ln(2 \pm \sqrt{3})$
7 $x = \ln(4 \pm \sqrt{15})$
8 $x = 0, x = \ln\left(\frac{7 \pm 3\sqrt{5}}{2}\right)$
9 $x = \ln\left(\frac{5}{2}\right), x = \ln 3$
10 $x = \ln(1 + \sqrt{2})$

Calculus of inverse hyperbolic functions

- $\dfrac{d}{dx}Cosh^{-1}(x)=\dfrac{1}{\sqrt{x^2-1}}$
- $\dfrac{d}{dx}Sinh^{-1}(x)=\dfrac{1}{\sqrt{1+x^2}}$
- $\dfrac{d}{dx}Tanh^{-1}(x)=\dfrac{1}{1-x^2}$
- $\displaystyle\int \frac{1}{\sqrt{x^2-1}}\,dx=Cosh^{-1}(x)+c$
- $\displaystyle\int \frac{1}{\sqrt{1+x^2}}\,dx=Sinh^{-1}(x)+c$
- $\displaystyle\int \frac{1}{1-x^2}\,dx=Tanh^{-1}(x)+c$

In formulae book

1. Differentiate the following:
   (a) $\tanh^{-1}\frac{x}{3}$, (b) $\sinh^{-1}\frac{x}{3}$, (c) $\cosh^{-1}\frac{x}{4}$,
   (d) $e^x\sinh^{-1}x$, (e) $\frac{1}{x}\cosh^{-1}x^2$.

2. If $y=x\cosh^{-1}x$, find $\frac{dy}{dx}$ when $x=2$, giving your answer in the form $a+\ln b$, where a and b are irrational numbers.

1. (a) $\frac{3}{9-x^2}$ (b) $\frac{1}{\sqrt{9+x^2}}$ (c) $\frac{1}{\sqrt{x^2-16}}$
(d) $e^x\sinh^{-1}x+\frac{e^x}{\sqrt{1+x^2}}$ (e) $\frac{2}{\sqrt{x^4-1}}-\frac{1}{x^2}\cosh^{-1}x^2$
2. $\frac{2\sqrt{3}}{3}+\ln\left(2+\sqrt{3}\right)$

Exercises:

1. Evaluate the following integrals:

(a) $\int \frac{dx}{\sqrt{x^2+9}}$, (b) $\int \frac{dx}{\sqrt{x^2-16}}$, (c) $\int \frac{dx}{\sqrt{4x^2+25}}$,

(d) $\int \frac{dx}{\sqrt{9x^2+49}}$, (e) $\int \frac{dx}{\sqrt{(x+1)^2+4}}$, (f) $\int \frac{dx}{\sqrt{(x-2)^2-16}}$,

(g) $\int \frac{dx}{\sqrt{x^2+4x+5}}$, (h) $\int \frac{dx}{\sqrt{x^2-2x-2}}$.

(a) $\sinh^{-1}\frac{x}{3}+c$ (b) $\cosh^{-1}\frac{x}{4}+c$ (c) $\frac{1}{2}\sinh^{-1}\frac{2x}{5}+c$

(d) $\frac{1}{3}\sinh^{-1}\frac{3x}{7}+c$ (e) $\sinh^{-1}\frac{x+1}{2}+c$ (f) $\cosh^{-1}\frac{x-2}{4}+c$

(g) $\sinh^{-1}(x+2)+c$ (h) $\cosh^{-1}\frac{x-1}{\sqrt{3}}+c$