Second order linear differential equations

Definitions

$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x) \qquad with\ a,b,c\in\mathbb{R}$

• The REDUCED equation is $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$.

The general solution of the reduced equation is called

The COMPLEMENTARY FUNCTION

• A PARTICULAR INTEGRAL satisfies the equation $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$

•The general solution of $a\frac{dy}{dx}+by=f(x)$ is

the sum of the complementary function and the particular integral

$y_G=y_P+y_C$

Solving second order linear differential equations

$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$ is a differential equation where a,b and c are real numbers

The reduced equation is $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$

•The AUXILIARY equation associated with this equation is $a\lambda^2+b\lambda+c=0$

The auxiliary equation is a quadratic equation, three cases are possible:

$Case\,1$: $a\lambda^2+b\lambda+c=0$ has two distinct solutions λ_1 $and\,\lambda_2$

The complementary function is $y=C_1e^{\lambda_1x}+C_2e^{\lambda_2x} \qquad C_1,C_2\in\mathbb{R}$

$Case\,2$: $a\lambda^2+b\lambda+c=0$ has equal/repeated root λ_0

The complementary function is $y=(C_1x+C_2)e^{\lambda_0x} \qquad C_1,C_2\in\mathbb{R}$

$Case\,3$: $a\lambda^2+b\lambda+c=0$ has two conjugate complex solutions $\lambda_1=p+iq$ and $\lambda_2=p-iq$

The complementary function is $y=e^{px}\left(C_1Cos(qx)+C_2Sin(qx)\right) \qquad C_1,C_2\in\mathbb{R}$

•Finding the particular integral:

⊗if $f(x)$ is a polynomial then y_P is also a polynomial of the same degree

⊗if $f(x)=ACos(kx)+BSin(kx)$ then $y_P=aCos(kx)+bSin(kx)$

a and b to be $worked$ out.

⊗if $f(x)=Ae^{kx}$ then $y_P=ae^{kx}$ $if\ \ k\neq\lambda$

$or\quad y_P=axe^{kx}$ $if\ \ k=\lambda_1$ or $\lambda_2 \qquad$ $where\ a\ is\ to\ be\ worked\ out$

$or\quad y_p=ax^2e^{kx}$ $if\ k=\lambda_0$ $(the\ repeated\ root.)$

• The general solution is $y_G=y_P+y_C$

Substitution

$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=R(x)$ is a differential equation

where P,Q and R are functions of x.

Note: this equation is written in its standard form.

These equations are solved using substitution.

The substitution to use will be given in the question.