Second order linear differential equations

Specifications

Differential Equations – Second Order

Solution of differential equations of the form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$, where a, b and c are integers, by using an auxiliary equation whose roots may be real or complex.

Including repeated roots.

Solution of equations of the form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$ where a, b and c are integers by finding the complementary function and a particular integral

Finding particular integrals will be restricted to cases where $f(x)$ is of the form e^{kx}, $\cos kx$, $\sin kx$ or a polynomial of degree at most 4, or a linear combination of any of the above.

Solution of differential equations of the form:

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$$

where P, Q, and R are functions of x. A substitution will always be given which reduces the differential equation to a form which can be directly solved using the other analytical methods in 18.4 and 18.5 of this specification or by separating variables.

i.e.

- direct integration
- Separating variables
- Integrating factor

Level of difficulty as indicated by:-

(a) Given $x^2\frac{d^2y}{dx^2}-2y=x$ use the substitution $x=e^t$

to show that $\frac{d^2y}{dt^2}-\frac{dy}{dt}-2y=e^t$

Hence find y in terms of t
Hence find y in terms of x

(b) $(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}=0$ use the substitution $u=\frac{dy}{dx}$

to show that $\frac{du}{dx}=\frac{2xu}{1-x^2}$

and hence that $u=\frac{A}{1-x^2}$ where A is an arbitrary constant.

Hence find y in terms of x

Back to the first order linear diff. equations

Consider the diff. equation: $a\frac{dy}{dx}+by=f(x)$ where *a and b are real numbers*

Method 1: To solve this equation, we can re-arrange it in the standard form and use an integrating factor.

Find the general solution of the followiong equation:

$$3\frac{dy}{dx}+5y=x$$

Method 2: This method is an alternative to using an integrating factor. There are two steps in this method

Step 1: Find the general solution of the **REDUCED equation**: $a\frac{dy}{dx}+by=0$
This is called the COMPLEMENTARY function

Step 2: Find a PARTICULAR INTEGRAL which satisfy : $a\frac{dy}{dx}+by=f(x)$

The GENERAL solution of the original equation is the sum of the complementary function and the particular integral.

$$y_G=y_C+y_{PI}$$

Let's establish the method to resolve step 1:

$$a\frac{dy}{dx}+by=f(x)$$ is a differential equation where *a and b are real numbers*

The reduced equation is : $a\frac{dy}{dx}+by=0$

If a function $y=Ae^{\lambda x}$ is a solution of the reduced equation, what equation is satisfied by λ?

$y=Ae^{\lambda x}$ *and* $\frac{dy}{dx}=A\lambda e^{\lambda x}$

The differential equation becomes:

$$a(A\lambda e^{\lambda x})+bAe^{\lambda x}=0$$

$$Ae^{\lambda x}(a\lambda+b)=0$$

This means that $a\lambda+b=0$

This equation is called the AUXILIARY equation.

The complementary function is $y=Ae^{\lambda x}$ where λ is solution of $a\lambda+b=0$

Summary:

$a\frac{dy}{dx}+by=f(x)$ is a differenrial equation where a *and* b are real numbers.

$a\frac{dy}{dx}+by=0$ is called the reduced equation.

- The complementary function is $y=Ae^{\lambda x}$ *with* $A\in\mathbb{R}$ *and where*
  λ is the solution of the AUXILIARY equation $a\lambda+b=0$
- Finding the particular integral:
  - ⊗ if $f(x)=P(x)$, a polynomial, then $y_{PI}=Q(x)$ is a polynomial of the same degree.
  - ⊗ if $f(x)=aCos(kx)+bSin(kx)$, then $y_{PI}=A\cos(kx)+BSin(kx)$
  - ⊗ if $f(x)=ae^{kx}$, then $y_{PI}=Ae^{kx}$ $(or\ y_{PI}=Axe^{kx}$ if k = λ)

Note: A and B are not arbitrary contants, they have to be worked out

Example 1:

Find the general solution of the followiong equation:

$$3\frac{dy}{dx}+5y=x$$

- Complementary function

  The auxiliary equation is $3\lambda+5=0$ so $\lambda=-\frac{5}{3}$

  The complementary function is $y=Ae^{-\frac{5}{3}x}$
- Particular integral

  The "right-hand side" function is a polynomial of order 1

  A particular integral has the form $y=ax+b$

  $y=ax+b \qquad \frac{dy}{dx}=a$

  The equation becomes: $3(a)+5(ax+b)=x$

  $5ax+3a+5b=x$

  This gives $\begin{cases}5a=1\\3a+5b=0\end{cases}$ $\qquad \begin{cases}a=\frac{1}{5}\\b=-\frac{3}{25}\end{cases}$
- General solutions:

  $y=Ae^{-\frac{5}{3}x}+\frac{1}{5}x-\frac{3}{25} \qquad A\in\mathbb{R}$

Example 2: : Find the general solution of the diff.eq. $\frac{dy}{dx} + y = \sin x.$

- Find the CF and a PI of the differential equation $2\frac{dy}{dx} - y = 3e^{\frac{1}{2}x}$. Hence write down the GS.
- Solve the differential equation $\frac{dy}{dx} + 2y = 2x^2 + 3$, given that $y(0) = 5$.

Exercises:

1) Find the general solution of the following differential equations:

(a) $\frac{dy}{dx}+3y=9x^2+1$

(b) $2\frac{dy}{dx}-y=x-3$

(c) $\frac{dy}{dx}+3y=\sin x+2\cos x$

(d) $2\frac{dy}{dx}-y=2\cos x$

(e) $\frac{dy}{dx}+3y=3e^{3x}$

(f) $\frac{dy}{dx}-3y=3e^{3x}$

(g) $2\frac{dy}{dx}+4y=8x+e^{-x}$

2. (a) Find the complementary function and a particular integral of the differential equation

$$\frac{dy}{dx}-3y=6.$$

(b) Hence obtain the solution satisfying the condition $y(1)=0$.

3. (a) Find the complementary function and a particular integral of the differential equation

$$\frac{dy}{dx}-2y=e^{2x}.$$

(b) Hence obtain the solution satisfying the condition $y(0)=2$.

4. Solve the differential equation

$$\frac{dy}{dx}+y=\sin 2x,$$

subject to the condition $y(0)=1$.

1) a) $y=Ae^{-3x}+3x^2-2x+1$ b) $y=Ae^{\frac{1}{2}x}-x+1$

c) $y=Ae^{-3x}+\frac{1}{2}(Sin(x)+Cos(x))$ d) $y=Ae^{\frac{1}{2}x}+\frac{4}{5}Sin(x)-\frac{2}{5}Cos(x)$

e) $y=Ae^{-3x}+\frac{1}{2}e^{3x}$ f) $y=Ae^{3x}+3xe^{3x}$ g) $y=Ae^{-2x}+2x-1+\frac{1}{2}e^{-x}$

2) a) $y_c=Ce^{3x}$; $y_p=-2$ b) $y=2(e^{3x-3}-1)$

3) a) $y_c=Ce^{2x}$; $y_p=xe^{2x}$ b) $y=(2+x)e^{2x}$

4) $y=\frac{1}{5}(7e^{-x}-2Cos(2x)+Sin(2x))$

Second order linear diff. equ.

The method studied with first order linear diff. equations can be extended to second order linear diff. equations:

$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$ is a differential where a, b *and* c *are real numbers.*

The AUXILIARY equation associated with this equation is $a\lambda^2+b\lambda+c=0$

The reduced equation is $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$

Complementary functions: (= General solution of the reduced equation)

Because the auxiliary equation is quadratic, three cases are possible:
(and you need to know them by heart)

Case 1: $a\lambda^2+b\lambda+c=0$ has two distinct solutions λ_1 *and* λ_2.

The complementary function is $y=Ae^{\lambda_1 x}+Be^{\lambda_2 x}$ $\quad A\in\mathbb{R}, B\in\mathbb{R}$

Case 2: $a\lambda^2+b\lambda+c=0$ has equal/repeated solutions, λ_1

The complementary function is $y=(Ax+B)e^{\lambda_1 x}$ $\quad A\in\mathbb{R}, B\in\mathbb{R}$

Case 3: $a\lambda^2+b\lambda+c=0$ has two conjugate solutions $\lambda_1=p+iq$ *and* $\lambda_2=p-iq$

The complementary function is $y=e^{px}(ACos(qx)+BSin(qx))$ $\quad A\in\mathbb{R}, B\in\mathbb{R}$

To know by

Exercises:

1. (a) Write down the general solution of the differential equation

$$\frac{d^2y}{dx^2}+9y=0, \qquad 0\le x\le \frac{\pi}{2}.$$

   (b) Hence find the particular solution satisfying the boundary conditions $y(0)=1$ and $y\left(\frac{\pi}{2}\right)=2.$

2. Find the general solution of each of the following differential equations.

   (a) $\frac{d^2y}{dx^2}-\frac{dy}{dx}-6y=0.$ (b) $9\frac{d^2y}{dx^2}-6\frac{dy}{dx}+y=0.$

   (c) $\frac{d^2y}{dx^2}-6\frac{dy}{dx}+10y=0.$ (d) $4\frac{d^2y}{dx^2}+12\frac{dy}{dx}+9y=0.$

   (e) $2\frac{d^2y}{dx^2}+6\frac{dy}{dx}+5y=0.$ (f) $\frac{d^2y}{dx^2}-6\frac{dy}{dx}=0.$

3. (a) Find the general solution of the differential equation

$$4\frac{d^2y}{dx^2}-y=0.$$

   (b) Hence find the particular solution which is such that $y=1$ and $\frac{dy}{dx}=0$ when $x=0.$

4. Solve the differential equation

$$\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=0$$

subject to the conditions that $y=1$ and $\frac{dy}{dx}=1$ when $x=0.$

1. (a) $y=A\cos 3x+B\sin 3x$ (b) $y=\cos 3x-2\sin 3x$
2. (a) $y=Ae^{3x}+Be^{-2x}$ (b) $y=(Ax+B)e^{\frac{x}{3}}$
   (c) $y=e^{3x}(A\cos x+B\sin x)$ (d) $y=(Ax+B)e^{-\frac{3x}{2}}$
   (e) $y=e^{-\frac{3x}{2}}\left(A\cos\frac{x}{2}+B\sin\frac{x}{2}\right)$ (f) $y=A+Be^{6x}$
3. (a) $y=Ae^{\frac{x}{2}}+Be^{-\frac{x}{2}}$ (b) $y=e^{\frac{x}{2}}+e^{-\frac{x}{2}}$
4. $y=(1-x)e^{2x}$

Particular integrals:

$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$ is a differenrial equation where a, b *and* c are real numbers.

- Finding the particular integral:
  - $\otimes$ if $f(x)=P(x)$, a polynomial, then $y_{PI}=Q(x)$ is a polynomial of the same degree.
  - $\otimes$ if $f(x)=aCos(kx)+bSin(kx)$, then $y_{PI}=A\cos(kx)+BSin(kx)$
  - $\otimes$ if $f(x)=ae^{kx}$, then $y_{PI}=Ae^{kx}$ $(or\ y_{PI}=Axe^{kx}\ or\ y_{PI}=Ax^2e^{kx})$

The general solution of $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$

is the sum of the complementary function and the particular integral:

$$y_G=y_C+y_{PI}$$

Example: Find the general solution of the diff. equation: $\frac{d^2y}{dx^2}-\frac{dy}{dx}-6y=2e^{2x}$

the auxiliary equation: $\lambda^2-\lambda-6=0$

$\lambda=3\ or\ \lambda=-2$

$y_{CF}=Ae^{3x}+Be^{-2x}$

The particular integral: $y=ae^{2x}$

$4ae^{2x}-2ae^{2x}-6ae^{2x}=2e^{2x}$

$-4ae^{2x}=2e^{2x}$ $\qquad$ $so\ a=-\frac{1}{2}$

$y_{PI}=-\frac{1}{2}e^{2x}$

$$y=Ae^{3x}+Be^{-2x}-\frac{1}{2}e^{2x} \qquad A,B\in\mathbb{R}$$

Have a go:

Solve the differential equation

$$\frac{d^2 y}{dx^2} + 2\frac{dy}{dx} + 2y = 5\cos x$$

subject to the conditions that $y = 1$ and $\frac{dy}{dx} = 0$ when $x = 0$.

$$y = -2e^{-x}\sin x + \cos x + 2\sin x$$
$$= 2\left(1 - e^{-x}\right)\sin x + \cos x.$$

Exercises:

1. The complementary function of the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = f(x)$$

is $(A+Bx)e^{2x}$. In each of the following cases find a particular integral, and hence write down the general solution of the differential equation in these cases.

(a) $f(x) = e^{-x}$, (b) $f(x) = 4x^2 + 6$, (c) $f(x) = 25\sin x$.

2. The function $y(x)$ satisfies the differential equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = e^{3x}.$$

(a) Find the complementary function.

(b) Show that there is a particular integral of the form $y = ax^2e^{3x}$, and find the appropriate value of a.

(c) Hence write down the general solution for $y(x)$.

3. Find a particular integral for the differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = x^4.$$

4. Solve the differential equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 4$$

subject to the conditions that $y = 0$ when $x = 0$ and $y \to -2$ as $x \to \infty$.

5. Find the solution of the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 8\sin x$$

satisfying the conditions that $y = 0$ and $\frac{dy}{dx} = 0$ when $x = 0$.

1. (a) PI: $y = \frac{1}{9}e^{-x}$; GS: $y = (A+Bx)e^{2x} + \frac{1}{9}e^{-x}$

(b) PI: $y = x^2 + 2x + 3$; GS: $y = (A+Bx)e^{2x} + x^2 + 2x + 3$

(c) PI: $y = 4\cos x + 3\sin x$; GS: $y = (A+Bx)e^{2x} + 4\cos x + 3\sin x$

2. (a) $y = (A+Bx)e^{3x}$ (b) $a = \frac{1}{2}$ (c) $y = (A+Bx)e^{3x} + \frac{1}{2}x^2e^{3x}$

3. $y = x^4 - 4x^3 + 24x - 24$

4. $y = 2(e^{-x} - 1)$

5. $y = e^{2x}(\sin x - \cos x) + \sin x + \cos x$

Second order linear diff. equ. with variable coefficients

$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=R(x)$ is a differential equation where P, Q and R are three functions of x.

Note: This is the equation written in its standard form.

These equation are solved using SUBSTITUTION.
The substitution to use will be given in the question.

After substitution, the equation obtained will be solved using either of the techniques seen in this chapter or the previous one (integrating factors).

Example:

The function $y(x)$ satisfies the differential equation

$$\frac{d^2y}{dx^2}+\frac{2}{x}\frac{dy}{dx}=0\ ,\ x>0.$$

(a) Show that the substitution $\frac{dy}{dx}=u$ reduces the differential equation to

$$\frac{du}{dx}+\frac{2u}{x}=0.$$

(b) Hence find the general solution for u in terms of x.

(c) Deduce the general solution for $y(x)$.

(d) Find the particular solution for $y(x)$ which is such that $y(1)=0$ and $y(x)\to 1$ as $x\to\infty$.

Exercises:

1) The function $y(x)$ satisfies the differential equation $\frac{d^2y}{dx^2} - \frac{1}{x-2}\frac{dy}{dx} = 0$ where $x > 2$.

$a)$ Show that the substitution $\frac{dy}{dx} = u$ reduces the differential equation to $\frac{du}{dx} - \frac{u}{x-2} = 0$.

$b)$ Find the general solution for $u(x)$.

$c)$ Hence find the general solution for $y(x)$.

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2) The function $y(x)$ satisfies the differential equation $Sin(x)\frac{d^2y}{dx^2} - 2Cos(x)\frac{dy}{dx} = 0$, $0 < x < \pi$.

$a)$ Use the substitution $\frac{dy}{dx} = u$ to transform the diff.eq. to one of first order in u.

Find the general solution for u and show that it can be expressed as

$$u = C(1 - Cos(2x)) \quad where\ C \text{ is an arbitrary constant.}$$

$b)$ Given that $y = \pi$ and $\frac{dy}{dx} = 1$ when $x = \frac{\pi}{2}$, find $y(x)$.

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3) $a)$ Use the substitution $\frac{dy}{dx} = u$ to transform the diff.eq. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = e^{3x}$

into a first order linear differential equation in u.

$b)$ Obtain an integrating factor for the diff.eq in u, and hence show that the general solution is

$$u = e^{3x} + Ae^{2x} \qquad where\ A \text{ is an arbitrary constant.}$$

$c)$ Given that $y = 0$ and $\frac{dy}{dx} = 0$ when $x = 0$, find y in terms of x.

1. (b) $u(x) = A(x-2)$ (c) $y(x) = A(\frac{1}{2}x^2 - 2x) + B$

2. (a) $(\sin x)\frac{du}{dx} - (2\cos x)u = 0$, GS is $u = A\sin^2 x$

   (b) $y(x) = \frac{1}{2}(x - \frac{1}{2}\sin 2x) + \frac{3}{4}\pi$

3. (a) $\frac{du}{dx} - 2u = e^{3x}$ (b)(i) e^{-2x} (c) $y(x) = \frac{1}{3}e^{3x} - \frac{1}{2}e^{2x} + \frac{1}{6}$

If you want more practice: Miscellaneous questions

In questions 1–10 solve each of the differential equations, giving the general solution.

1. $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 5y = 10$
2. $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 12y = 36x$
3. $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 12y = 12e^{2x}$
4. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 15y = 5$
5. $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 8x + 12$
6. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 25\cos 2x$
7. $\frac{d^2y}{dx^2} + 81y = 15e^{3x}$
8. $\frac{d^2y}{dx^2} + 4y = \sin x$
9. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 25x^2 - 7$
10. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 26y = e^x$

In questions 1–5 find the solution subject to the given boundary conditions for each of the following differential equations.

1. $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 12e^x$ $y = 1$ and $\frac{dy}{dx} = 0$ at $x = 0$
2. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 12e^{2x}$ $y = 2$ and $\frac{dy}{dx} = 6$ at $x = 0$
3. $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 42y = 14$ $y = 0$ and $\frac{dy}{dx} = \frac{1}{6}$ at $x = 0$
4. $\frac{d^2y}{dx^2} + 9y = 16\sin x$ $y = 1$ and $\frac{dy}{dx} = 8$ at $x = 0$
5. $4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = \sin x + 4\cos x$ $y = 0$ and $\frac{dy}{dx} = 0$ at $x = 0$

7. Use the substitution $y = \frac{z}{x}$ to transform the differential equation

$$x\frac{d^2y}{dx^2} + (2 - 4x)\frac{dy}{dx} - 4y = 0 \text{ into the equation } \frac{d^2z}{dx^2} - 4\frac{dz}{dx} = 0.$$

Hence solve the equation $x\frac{d^2y}{dx^2} + (2 - 4x)\frac{dy}{dx} - 4y = 0$, giving y in terms of x.

8. Use the substitution $y = \frac{z}{x^2}$ to transform the differential equation

$$x^2\frac{d^2y}{dx^2} + 2x(x + 2)\frac{dy}{dx} + 2(x + 1)^2y = e^{-x} \text{ into the equation } \frac{d^2z}{dx^2} + 2\frac{dz}{dx} + 2z = e^{-x}.$$

Hence solve the equation $x^2\frac{d^2y}{dx^2} + 2x(x + 2)\frac{dy}{dx} + 2(x + 1)^2y = e^{-x}$, giving y in terms of x.

9. Use the substitution $z = \sin x$ to transform the differential equation

$$\cos x\frac{d^2y}{dx^2} + \sin x\frac{dy}{dx} - 2y\cos^3 x = 2\cos^5 x \text{ into the equation } \frac{d^2y}{dz^2} - 2y = 2(1 - z^2).$$

Hence solve the equation $\cos x\frac{d^2y}{dx^2} + \sin x\frac{dy}{dx} - 2y\cos^3 x = 2\cos^5 x$, giving y in terms of x.

1. $y = Ae^{-x} + Be^{-5x} + 2$
2. $y = Ae^{6x} + Be^{2x} + 2 + 3x$
3. $y = Ae^{-4x} + Be^{3x} - 2e^{2x}$
4. $y = Ae^{-5x} + Be^{3x} - \frac{1}{3}$
5. $y = (A + Bx)e^{4x} + 1 + \frac{1}{2}x$
6. $y = (A + Bx)e^{-x} + 4\sin 2x - 3\cos 2x$
7. $y = A\cos 9x + B\sin 9x + \frac{1}{6}e^{3x}$
8. $y = A\cos 2x + B\sin 2x + \frac{1}{3}\sin x$
9. $y = e^{2x}(A\cos x + B\sin x) + 3 + 8x + 5x^2$
10. $y = e^x(A\cos 5x + B\sin 5x) + \frac{1}{25}e^x$

1. $y = e^{-3x} - e^{-2x} + e^x$
2. $y = 2 - \frac{3}{2}e^{-2x} + \frac{3}{2}e^{2x}$
3. $y = \frac{1}{6}e^{-6x} + \frac{1}{6}e^{7x} - \frac{1}{3}$
4. $y = \cos 3x + 2\sin 3x + 2\sin x$
5. $y = \sin x\,(1 - e^{-\frac{1}{2}x})$

7. $y = \frac{A}{x} + \frac{B}{x}e^{4x}$
8. $y = \frac{e^{-x}}{x^2}(A\cos x + B\sin x + 1)$
9. $y = Ae^{\sqrt{2}\sin x} + Be^{-\sqrt{2}\sin x} + \sin^2 x$