Second order linear differential equations. – Exam questions

Question 1: Jan 2009 Q7

(a) Given that $x = e^t$ and that y is a function of x, show that

$$x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$ *(7 marks)*

(b) Hence show that the substitution $x = e^t$ transforms the differential equation

$$x^2 \frac{d^2y}{dx^2} - 4x\frac{dy}{dx} = 10$$

into

$$\frac{d^2y}{dt^2} - 5\frac{dy}{dt} = 10$$ *(2 marks)*

(c) Find the general solution of the differential equation $\frac{d^2y}{dt^2} - 5\frac{dy}{dt} = 10$. *(5 marks)*

(d) Hence solve the differential equation $x^2 \frac{d^2y}{dx^2} - 4x\frac{dy}{dx} = 10$, given that $y = 0$ and $\frac{dy}{dx} = 8$ when $x = 1$. *(5 marks)*

Question 2: Jan 2007 Q5

Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 6 + 5\sin x$$ *(12 marks)*

Question 3: Jan 2008 Q8

(a) Given that $x = e^t$ and that y is a function of x, show that:

(i) $x\frac{dy}{dx} = \frac{dy}{dt}$; *(3 marks)*

(ii) $x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$. *(3 marks)*

(b) Hence find the general solution of the differential equation

$$x^2 \frac{d^2y}{dx^2} - 6x\frac{dy}{dx} + 6y = 0$$ *(5 marks)*

Question 4: Jan 2006 Q1

(a) Find the roots of the equation $m^2 + 2m + 2 = 0$ in the form $a + ib$. *(2 marks)*

(b) (i) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 4x$$ *(6 marks)*

(ii) Hence express y in terms of x, given that $y = 1$ and $\frac{dy}{dx} = 2$ when $x = 0$. *(4 marks)*

Question 5: Jun 2007 Q1

(a) Find the value of the constant k for which kx^2e^{5x} is a particular integral of the differential equation

$$\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 25y = 6e^{5x}$$ *(6 marks)*

(b) Hence find the general solution of this differential equation. *(4 marks)*

Question 6: Jun 2007 Q5

(a) A differential equation is given by

$$(x^2 - 1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} = x^2 + 1$$

Show that the substitution

$$u = \frac{dy}{dx} + x$$

transforms this differential equation into

$$\frac{du}{dx} = \frac{2xu}{x^2 - 1}$$ *(4 marks)*

(b) Find the general solution of

$$\frac{du}{dx} = \frac{2xu}{x^2 - 1}$$

giving your answer in the form $u = f(x)$. *(5 marks)*

(c) Hence find the general solution of the differential equation

$$(x^2 - 1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} = x^2 + 1$$

giving your answer in the form $y = g(x)$. *(3 marks)*

Question 7: June 2009 Q5

It is given that y satisfies the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 8\sin x + 4\cos x$$

(a) Find the value of the constant k for which $y = k\sin x$ is a particular integral of the given differential equation. *(3 marks)*

(b) Solve the differential equation, expressing y in terms of x, given that $y = 1$ and $\frac{dy}{dx} = 4$ when $x = 0$. *(8 marks)*

Second order linear differential equations. – Exam questions MS

Question 1: Jan 2009 Q7

7(a)	$\frac{dx}{dt} = e^t \{= x\}$	B1	
	$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = e^{-t}\frac{dy}{dt}$	M1 A1	
	$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(e^{-t}\frac{dy}{dt}\right) = \frac{dt}{dx}\frac{d}{dt}\left(e^{-t}\frac{dy}{dt}\right)$	M1	
	$= \frac{dt}{dx}\left(-e^{-t}\frac{dy}{dt} + e^{-t}\frac{d^2y}{dt^2}\right)$	M1	
	$\ldots = e^{-t}\left(-e^{-t}\frac{dy}{dt} + e^{-t}\frac{d^2y}{dt^2}\right)$	A1	
	$\ldots = x^{-2}\left(-\frac{dy}{dt} + \frac{d^2y}{dt^2}\right)$		
	$\Rightarrow x^2\frac{d^2y}{dx^2} = \left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$	A1	7
(b)	$x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} = 10$		
	$\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) - 4\left(\frac{dy}{dt}\right) = 10$	M1	
	$\frac{d^2y}{dt^2} - 5\frac{dy}{dt} = 10$	A1	2
(c)	$\frac{d^2y}{dt^2} - 5\frac{dy}{dt} = 10 \quad (*)$		
	Auxl eqn $m^2 - 5m = 0$	M1	
	$m(m-5) = 0$ $m = 0$ and 5	A1	
	CF: $(y_C =) A + Be^{5t}$	M1	
	PI: $(y_P =) -2t$	B1	
	GS of (*) $\{y\} = A + B\,e^{5t} - 2t$	B1ft	5
(d)	$\Rightarrow y = A + Bx^5 - 2\ln x$	M1	
	$y'(x) = 5Bx^4 - 2x^{-1}$	A1ft	
	Using boundary conditions to find A & B	M1	
	$B = 2; A = -2; \quad \{y = -2 + 2x^5 - 2\ln x\}$	A1;A1ft	5
	Total		**19**

Question 2: Jan 2007 Q5

Auxl. eqn $m^2 - 4m + 3 = 0$	M1	
$m = 3$ and 1	A1	
CF is $A\,e^{3x} + B\,e^x$	A1F	
PI Try $y = a + b\sin x + c\cos x$	M1	
$y'(x) = b\cos x - c\sin x$	A1	
$y''(x) = -b\sin x - c\cos x$	A1F	
Substitute into DE gives	M1	
$a = 2$	B1	
$4c + 2b = 5$ and $2c - 4b = 0$	A1	
$b = 0.5$,	A1F	
$c = 1$	A1F	
GS: $y = A\,e^{3x} + B\,e^x + 2 + 0.5\sin x + \cos x$	B1F	12
Total		**12**

Question 3: Jan 2008 Q8

(a)(i)	$\frac{dx}{dt} = e^t \{= x\}$	B1	
	$x\frac{dy}{dx} = x\frac{dy}{dt}\frac{dt}{dx}$	M1	
	$= x\frac{dy}{dt}\frac{1}{x} = \frac{dy}{dt}$	A1	3
(ii)	$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(x\frac{dy}{dx}\right) =$		
	$= \frac{dx}{dt}\frac{dy}{dx} + x\frac{d}{dt}\left(\frac{dy}{dx}\right)$	M1	
	$\ldots = \frac{dy}{dt} + x\frac{dx}{dt}\frac{d}{dx}\left(\frac{dy}{dx}\right)$	M1	
	$\ldots = \frac{dy}{dt} + x^2\left(\frac{d^2y}{dx^2}\right)$	A1	3
	$\Rightarrow x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$		
(b)	$x^2\frac{d^2y}{dx^2} - 6x\frac{dy}{dx} + 6y = 0$		
	$\Rightarrow \frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 6y = 0$	M1	
	Auxl eqn $m^2 - 7m + 6 = 0$		
	$(m-6)(m-1) = 0$	m1	
	$m = 1$ and 6	A1	
	$y = Ae^{6t} + Be^t$	M1	
	$y = Ax^6 + Bx$	A1√	5
	Total		**11**

Question 4: Jan 2006 Q1

1(a)	$(m+1)^2 = -1$	M1	
	$m = -1 \pm i$	A1	2
b)(i)	CF is $e^{-x}(A\cos x + B\sin x)$	M1	
	{or $e^{-x}A\cos(x + B)$ **but not** $Ae^{(-1+i)x} + Be^{(-1-i)x}$ }	A1√	
	{P.Int.} try $y = px + q$	M1	
	$2p + 2(px + q) = 4x$	A1	
	$p = 2, \quad q = -2$	A1√	
	GS $y = e^{-x}(A\cos x + B\sin x) + 2x - 2$	B1√	6
(ii)	$x=0, \ y=1 \Rightarrow A = 3$	B1√	
	$y'(x) = -e^{-x}(A\cos x + B\sin x) +$	M1	
	$+ e^{-x}(-A\sin x + B\cos x) + 2$	A1√	
	$y'(0) = 2 \Rightarrow 2 = -A + B + 2 \Rightarrow B = 3$	A1√	
	$y = 3e^{-x}(\cos x + \sin x) + 2x - 2$		4
	Total		**12**

Question 5: Jun 2007 Q1

1(a)	$y_{PI} = kx^2e^{5x} \Rightarrow y' = 2kxe^{5x} + 5kx^2e^{5x}$	M1 A1	
	$\Rightarrow y'' = 2ke^{5x} + 10kxe^{5x} + 10kxe^{5x} + 25kx^2e^{5x}$	A1ft	
	$\Rightarrow 2ke^{5x} + 20kxe^{5x} + 25kx^2e^{5x}$ $-10\left(2kxe^{5x} + 5kx^2e^{5x}\right) + 25kx^2e^{5x} = 6e^{5x}$	M1 A1	
	$2k = 6 \Rightarrow k = 3$	A1ft	6
(b)	Aux. eqn. $m^2 - 10m + 25 = 0 \Rightarrow m = 5$	B1	
	CF is $(A + Bx)e^{5x}$	M1	
	GS $y = (A + Bx)e^{5x} + 3x^2e^{5x}$	M1 A1ft	4
	Total		**10**

Question 6: Jun 2007 Q5

5(a)	$u = \frac{dy}{dx} + x \Rightarrow \frac{du}{dx} = \frac{d^2y}{dx^2} + 1$	M1A1	
	$(x^2-1)\left(\frac{du}{dx} - 1\right) - 2x(u - x) = x^2 + 1$	M1	
	DE $\Rightarrow (x^2-1)\frac{du}{dx} - 2xu = 0$		
	$\Rightarrow \frac{du}{dx} = \frac{2xu}{x^2-1}$	A1	4
(b)	$\int \frac{1}{u}\,du = \int \frac{2x}{x^2-1}\,dx$	M1 A1	
	$\ln u = \ln\lvert x^2 - 1\rvert + \ln A$	A1A1	
	$u = A(x^2 - 1)$	A1	5
(c)	$\frac{dy}{dx} + x = A(x^2 - 1)$	M1	
	$\frac{dy}{dx} = A(x^2 - 1) - x$		
	$y = A\left(\frac{x^3}{3} - x\right) - \frac{x^2}{2} + B$	M1	
		A1ft	3
	Total		**12**

Question 7: June 2009 Q5

5(a)	$-k\sin x + 2k\cos x + 5k\sin x = 8\sin x + 4\cos x$	M1 A1	
	$k = 2$	A1	3
(b)	Auxl eqn $m^2 + 2m + 5 = 0$		
	$m = \frac{-2 \pm \sqrt{4-20}}{2}$	M1	
	$m = -1 \pm 2\,i$	A1	
	CF: $\{y_C\} = e^{-x}(A\sin 2x + B\cos 2x)$	A1F	
	GS $\{y\} = e^{-x}(A\sin 2x + B\cos 2x) + k\sin x$	B1F	
	When $x = 0, y = 1 \Rightarrow B = 1$	B1F	
	$\frac{dy}{dx} = -e^{-x}(A\sin 2x + B\cos 2x)$ $+ e^{-x}(2A\cos 2x - 2B\sin 2x) + k\cos x$	M1	
	When $x = 0$, $\frac{dy}{dx} = 4 \Rightarrow 4 = -B + 2A + k$	A1	
	$\Rightarrow A = \frac{3}{2}$		
	$y = e^{-x}\left(\frac{3}{2}\sin 2x + \cos 2x\right) + 2\sin x$	A1	8
	Total		**11**