# Determinants – exam questions

### Question 1: Jun 2010 – Q5

Factorise fully the determinant $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix}$. *(8 marks)*

### Question 2: Jan 2006 – Q6

(a) Show that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ca & ab \end{vmatrix} = (a-b)(b-c)(c-a)$$ *(5 marks)*

### Question 3: June 2006 – Q3

Express the determinant $\begin{vmatrix} 1 & a^2 & bc \\ 1 & b^2 & ca \\ 1 & c^2 & ab \end{vmatrix}$ as the product of four linear factors. *(6 marks)*

### Question 4: Jan 2007 – Q2

(a) Show that $(a-b)$ is a factor of the determinant

$$\Delta = \begin{vmatrix} a & b & c \\ b+c & c+a & a+b \\ bc & ca & ab \end{vmatrix}$$ *(2 marks)*

(b) Factorise $\Delta$ completely into linear factors. *(5 marks)*

### Question 5: Jan 2011 – Q1

Let $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ x & y & z \\ y+z & z+x & x+y \end{vmatrix}$.

**(a)** Use a row operation to show that $(x+y+z)$ is a factor of $\Delta$. *(2 marks)*

**(b)** Hence, or otherwise, express $\Delta$ as a product of linear factors. *(2 marks)*

### Question 6: June 2007 – Q2

Factorise completely the determinant $\begin{vmatrix} y & x & x+y-1 \\ x & y & 1 \\ y+1 & x+1 & 2 \end{vmatrix}$. *(6 marks)*

### Question 7: Jan 2010 – Q7

(a) It is given that $\Delta = \begin{vmatrix} 16-q & 5 & 7 \\ -12 & -1-q & -7 \\ 6 & 6 & 10-q \end{vmatrix}$.

(i) By using row operations on the first two rows of $\Delta$, show that $(4-q)$ is a factor of $\Delta$. *(2 marks)*

(ii) Express $\Delta$ as the product of three linear factors. *(4 marks)*

### Question 8: June 2008 – Q8

By considering the determinant

$$\begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix}$$

show that $(x+y+z)$ is a factor of $x^3+y^3+z^3-kxyz$ for some value of the constant $k$ to be determined. *(3 marks)*

### Question 9: Jun 2012 – Q3

Let $\Delta = \begin{vmatrix} yz & xz & xy \\ x & y & z \\ x^2 & -y^2 & z^2 \end{vmatrix}$.

**(a)** Show that $(y+z)$ is a factor of $\Delta$. *(2 marks)*

**(b)** Factorise $\Delta$ as completely as possible. *(4 marks)*

### Question 10: Jan 2009 – Q5

(a) Expand the determinant

$$D = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ z & x & y \end{vmatrix}$$ *(2 marks)*

(b) Show that $(x+y+z)$ is a factor of the determinant

$$\Delta = \begin{vmatrix} x & y & z \\ y-z & z-x & x-y \\ x+z & y+x & z+y \end{vmatrix}$$ *(2 marks)*

(c) Show that $\Delta = k(x+y+z)D$ for some integer $k$. *(3 marks)*

### Question 11: June 2009 – Q8

(a) Matrix $\mathbf{M} = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$. Without attempting to factorise, expand fully $\det \mathbf{M}$.

*(2 marks)*

(b) Matrix $\mathbf{N} = \begin{bmatrix} d & e & f \\ f & d & e \\ e & f & d \end{bmatrix}$. Find the product matrix $\mathbf{MN}$. *(3 marks)*

(c) Prove that the product

$$(a^3+b^3+c^3-3abc)(d^3+e^3+f^3-3def)$$

can be written in the form $x^3+y^3+z^3-3xyz$, stating clearly each of $x$, $y$ and $z$ in terms of $a$, $b$, $c$, $d$, $e$ and $f$. *(2 marks)*

# Determinants – exam questions - MS

## Question 1: Jun 2010 – Q5

| | | |
|---|---|---|
| $\Delta = \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix}$ | | |
| $= \begin{vmatrix} x & y-x & z-x \\ x^2 & y^2-x^2 & z^2-x^2 \\ yz & z(x-y) & y(x-z) \end{vmatrix}$ | M1 M1 | |
| $= (y-x)(z-x)\begin{vmatrix} x & 1 & 1 \\ x^2 & y+x & z+x \\ yz & -z & -y \end{vmatrix}$ | A1 A1 | |
| $= (y-x)(z-x)\begin{vmatrix} x & 1 & 0 \\ x^2 & y+x & z-y \\ yz & -z & z-y \end{vmatrix}$ | M1 | |
| $= (y-x)(z-x)(z-y)\begin{vmatrix} x & 1 & 0 \\ x^2 & y+x & 1 \\ yz & -z & 1 \end{vmatrix}$ | A1 | |
| $= (x-y)(y-z)(z-x)(xy+yz+zx)$ | M1 | |
| | A1 | 8 |
| Alternatives using *Cyclic Symmetry* and the *Factor Theorem* are fine | | |
| **Total** | | **8** |

## Question 2: Jan 2006 – Q6

| | | |
|---|---|---|
| $\Delta = \begin{vmatrix} 0 & 0 & 1 \\ a-c & b-c & c \\ b(c-a) & a(c-b) & ab \end{vmatrix}$ | M1 | |
| $= (a-c)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ -b & -a & ab \end{vmatrix}$ | A1 A1 | |
| $= (a-c)(b-c)\ (b-a)$ | M1 | |
| $= (a-b)(b-c)(c-a)$ | A1 | 5 |

## Question 3: June 2006 – Q3

| | | |
|---|---|---|
| $\Delta = \begin{vmatrix} 1 & a^2 & bc \\ 0 & b^2-a^2 & c(a-b) \\ 0 & c^2-a^2 & b(a-c) \end{vmatrix}$ | M1 A1 | |
| $= (b-a)(c-a)\begin{vmatrix} 1 & a^2 & bc \\ 0 & a+b & -c \\ 0 & c+a & -b \end{vmatrix}$ | M1 A1 | |
| $= (b-a)(c-a)(c-b)\begin{vmatrix} 1 & a^2 & bc \\ 0 & a+b & -c \\ 0 & 1 & 1 \end{vmatrix}$ | M1 | |
| $= (a-b)(b-c)(c-a)(a+b+c)$ | A1 | 6 |
| Or by use of factor theorem and cyclic symmetry | | |
| **Total** | | **6** |

## Question 4: Jan 2007 – Q2

| | | |
|---|---|---|
| $\Delta = \begin{vmatrix} a-b & b & c \\ b-a & c+a & a+b \\ c(b-a) & ca & ab \end{vmatrix}$ | M1 | |
| $= (a-b)\begin{vmatrix} 1 & b & c \\ -1 & c+a & a+b \\ -c & ca & ab \end{vmatrix}$ | A1 | 2 |
| **Or** Setting $b = a \Rightarrow C_1 = C_2 \Rightarrow \Delta = 0$ | (M1) | |
| $\Rightarrow (a-b)$ a factor of $\Delta$ | (A1) | (2) |
| **Or** $\Delta = (a-b)(c^3 + a^2b + ab^2 - abc - b^2c - a^2c)$ | (M1) (A1) | (2) |
| $= (a-b)\begin{vmatrix} 1 & b-c & c \\ -1 & c-b & a+b \\ -c & a(c-b) & ab \end{vmatrix}$ | M1 | |
| $= (a-b)(b-c)\begin{vmatrix} 1 & 1 & c \\ -1 & -1 & a+b \\ -c & -a & ab \end{vmatrix}$ | A1 | |
| e.g. $\Delta = (a-b)(b-c)\begin{vmatrix} 0 & 0 & a+b+c \\ -1 & -1 & a+b \\ -c & -a & ab \end{vmatrix}$ | M1 | |
| **and** then expanding final det. | A1 | |
| $\Delta = -(a+b+c)(a-b)(b-c)(c-a)$ | A1 | 5 |
| **Or** By cyclic symmetry, | (M1) | |
| $(b-c)$ and $(c-a)$ are also factors | (A1) (A1) | |
| Final linear factor & checking sign of a coefficient. | (M1) (A1) | (5) |
| **Or** Expanding the determinant fully | (M1) | |
| $\Delta =$ | (A1) | |
| Multiplying out | | |
| $(a-b)(b-c)(c-a)(a+b+c)$ | (M1) | |
| $=$ | (A1) | |
| Fully correct working to show the two things are identically equal & checking for sign | (A1) | (5) |
| **Total** | | **7** |

## Question 5: Jan 2011 – Q1

| | | |
|---|---|---|
| (a) $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ x & y & z \\ x+y+z & y+z+x & z+x+y \end{vmatrix}$ | M1 | |
| $= (x+y+z)\begin{vmatrix} 1 & 2 & 3 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$ | A1 | 2 |
| (b) Expanding remaining det. | M1 | |
| $\Delta = (x+y+z)(x-2y+z)$ | A1 | 2 |
| **Total** | | **4** |

## Question 6: June 2007 – Q2

| | | | |
|---|---|---|---|
| | $\Delta = \begin{vmatrix} y-x & x & x+y-1 \\ x-y & y & 1 \\ y-x & x+1 & 2 \end{vmatrix}$ | M1 | |
| | $= (y-x)\begin{vmatrix} 1 & x & x+y-1 \\ -1 & y & 1 \\ 1 & x+1 & 2 \end{vmatrix}$ | A1 | |
| | $\Delta = (y-x)\begin{vmatrix} 0 & x+y & x+y \\ -1 & y & 1 \\ 1 & x+1 & 2 \end{vmatrix}$ | M1 | |
| | $= (y-x)(y+x)\begin{vmatrix} 0 & 1 & 1 \\ -1 & y & 1 \\ 1 & x+1 & 2 \end{vmatrix}$ | A1 | |
| | Full expansion | M1 | |
| | $\Delta = (y-x)(y+x)(2-x-y)$ | A1 | 6 |
| | **Or** | | |
| | Setting $y = x \Rightarrow C_1 = C_2 \Rightarrow \Delta = 0$ | (M1) | |
| | $\Rightarrow (y-x)$ a factor of $\Delta$ | (A1) | |
| | Setting $y = -x \Rightarrow R_1 = R_2$ | (M1) | |
| | So that $R_1' = R_1 + R_2 \Rightarrow R_1' = 0$ | | |
| | $\Rightarrow \Delta = 0$ and $(y+x)$ a factor of $\Delta$ | (A1) | |
| | Genuine attempt at 3rd factor | (M1) | |
| | Completely correct solution | (A1) | (6) |

## Question 7: Jan 2010 – Q7

| | | | |
|---|---|---|---|
| (i) | $R_1' = R_1 + R_2$ or $R_2' = R_2 + R_1$ | M1 | |
| | leading to $R_{1/2} = (4-q \quad 4-q \quad 0)$ | A1 | 2 |
| ii) | $\Delta = (4-q)\begin{vmatrix} 1 & 1 & 0 \\ -12 & -1-q & -7 \\ 6 & 6 & 10-q \end{vmatrix}$ | | |
| | $= (4-q)\begin{vmatrix} 0 & 1 & 0 \\ q-11 & -1-q & -7 \\ 0 & 6 & 10-q \end{vmatrix}$ | M1 | |
| | $= (4-q)(q-11)\begin{vmatrix} 0 & 1 & 0 \\ 1 & -1-q & -7 \\ 0 & 6 & 10-q \end{vmatrix}$ | A1 | |
| | $= (4-q)(q-11) \times \ldots$ | M1 | |
| | $= (4-q)(q-11)(q-10)$ | A1 | 4 |

## Question 8: June 2008 – Q8

| | | |
|---|---|---|
| Expanding fully: $\Delta = x^3 + y^3 + z^3 - 3xyz$ | B1 | |
| Using row/column operations: | | |
| eg $R_1' = R_1 + (R_2 + R_3)$ | M1 | |
| $\Rightarrow \Delta = (x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ z & x & y \\ y & z & x \end{vmatrix}$ | A1 | 3 |
| NB Any line of argument that leads correctly from $(x+y+z)\,f(x, y, z)$ to $x^3 + y^3 + z^3 - 3xyz$ scores full marks | | |
| **Total** | | **3** |

## Question 9: Jun 2012 – Q3

| | | | |
|---|---|---|---|
| a) | eg $\begin{vmatrix} yz & x(y+z) & xy \\ x & y+z & z \\ x^2 & z^2-y^2 & z^2 \end{vmatrix}$ | M1 | |
| | $= (y+z)\begin{vmatrix} yz & x & xy \\ x & 1 & z \\ x^2 & z-y & z^2 \end{vmatrix}$ | A1 | 2 |
| b) | eg $(y+z)\begin{vmatrix} y(z-x) & x & xy \\ x-z & 1 & z \\ x^2-z^2 & z-y & z^2 \end{vmatrix}$ $C_1' = C_1 - C_3$ | M1 | |
| | $= (x-z)(y+z)\begin{vmatrix} -y & x & xy \\ 1 & 1 & z \\ x+z & z-y & z^2 \end{vmatrix}$ | A1 | |
| | $= (x-z)(y+z)\begin{vmatrix} -(x+y) & x & xy \\ 0 & 1 & z \\ x+y & z-y & z^2 \end{vmatrix}$ | M1 | |
| | $= (x-z)(y+z)(x+y)(xz-xy-yz)$ | A1 | 4 |
| | **Total** | | **6** |

## Question 10: Jan 2009 – Q5

| | | | |
|---|---|---|---|
| (a) | $D = x^2 + y^2 + z^2 - xy - yz - zx$ | M1 A1 | 2 |
| (b) | E.g. by $C_1' = C_1 + (C_2 + C_3)$ | M1 | |
| | $\Rightarrow \Delta = \begin{vmatrix} x+y+z & y & z \\ 0 & z-x & x-y \\ 2(x+y+z) & y+x & z+y \end{vmatrix}$ | | |
| | $= (x+y+z)\begin{vmatrix} 1 & y & z \\ 0 & z-x & x-y \\ 2 & y+x & z+y \end{vmatrix}$ | A1 | 2 |
| (c) | Working on (R/C-ops) or expanding remaining determinant | M1 | |
| | 2nd factor $= -(x^2 + y^2 + z^2 - xy - yz - zx)$ | dM1 | |
| | $k = -1$ | A1 | 3 |
| | **Total** | | **7** |

## Question 11: June 2009 – Q8

| | | | |
|---|---|---|---|
| (a) | $\text{Det}(\mathbf{M}) = a^3 + b^3 + c^3 - 3abc$ | M1 A1 | 2 |
| (b) | $\begin{bmatrix} ad+bf+ce & ae+bd+cf & af+be+cd \\ af+be+cd & ad+bf+ce & ae+bd+cf \\ ae+bd+cf & af+be+cd & ad+bf+ce \end{bmatrix}$ | M1 A1 A1 | 3 |
| (c) | Use of $\det(\mathbf{MN}) = \det(\mathbf{M})\det(\mathbf{N})$ | M1 | |
| | $x = ad + bf + ce$, $y = ae + bd + cf$ and $z = af + be + cd$ | A1 | 2 |
| | **Total** | | **7** |