Complex numbers

Specifications:

Complex Numbers

The Cartesian and polar co-ordinate forms of a complex number, its modulus, argument and conjugate.

$x + \mathrm{i}y$ and $r(\cos\theta + \mathrm{i}\sin\theta)$.

The sum, difference, product and quotient of two complex numbers.

The parts of this topic also included in module Further Pure 1 will be examined only in the context of the content of this module.

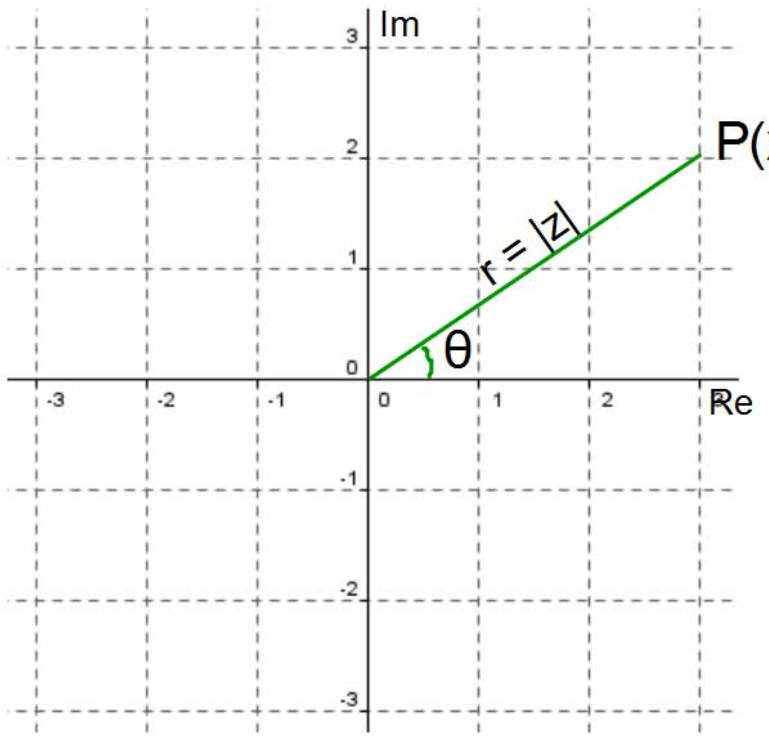
The representation of a complex number by a point on an Argand diagram; geometrical illustrations.

Simple loci in the complex plane.

For example, $|z - 2 - \mathrm{i}| \leqslant 5$, $\arg(z-2) = \frac{\pi}{3}$

Maximum level of difficulty $|z - a| = |z - b|$ where a and b are complex numbers.

Modulus and argument



Consider a complex number
$z = x+iy$

Argand diagram:

There is a natural relationship between any complex $z=x+iy$ and the point P with coordinates (x,y)

If a complex number $z = x + iy$,

We can represent z in an Argand diagram by the point $P(x, y)$.

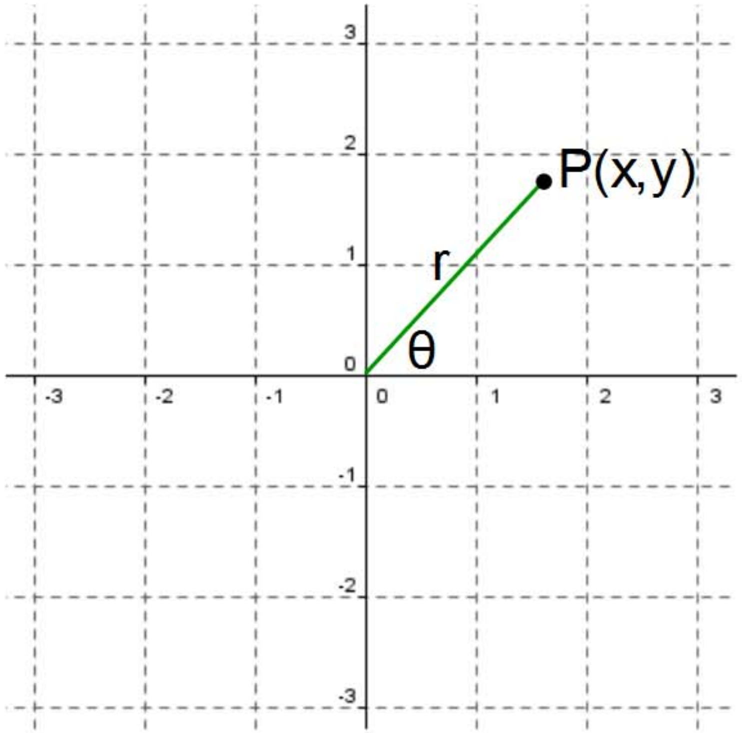
By definition

$$\text{Modulus of } z = |z| = \sqrt{x^2 + y^2} = OP$$

$$\text{Argument of } z = \arg(z) = \theta = angle\left(\widehat{ox, OP}\right)$$

$$(usually, \ -\pi < \theta \le \pi)$$

Polar form of a complex number



z = x+iy is the CARTESIAN form of the complex number

If $z = x + iy$, the polar form of z is

$$z = r(\cos\theta + i\sin\theta) \quad where \;\; r = |z| \; and \; \theta = \arg(z)$$

Converting: Cartesian <-> Polar

Polar to cartesian:

If $z = x + iy = r(\cos\theta + i\sin\theta)$,

then $x = r\cos\theta \quad and \quad y = r\sin\theta$

Cartesian to polar:

If $z = x + iy = r(\cos\theta + i\sin\theta)$,

then $r = |z| = \sqrt{x^2 + y^2}$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \pm \pi$$

It is very useful to know by heart the Sin and Cos of

$\frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}$ *and of course* π *and* $\frac{\pi}{2}$

Exercises:

Question 1:

Complete the table, giving the exact values

Angle θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π
Sin θ						
Cos θ						
Tan θ						

Question 2:

Find, in the form $x + iy$, the complex numbers given in polar coordinate form by:

(a) $z = 2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$, (b) $4\left(\cos -\frac{2\pi}{3} + i\ \sin -\frac{2\pi}{3}\right)$.

Question 3:

Find the arguments of the following complex numbers. Give the answer where appropriate as a rational multiple of π, otherwise give the argument correct to 2 decimal places.

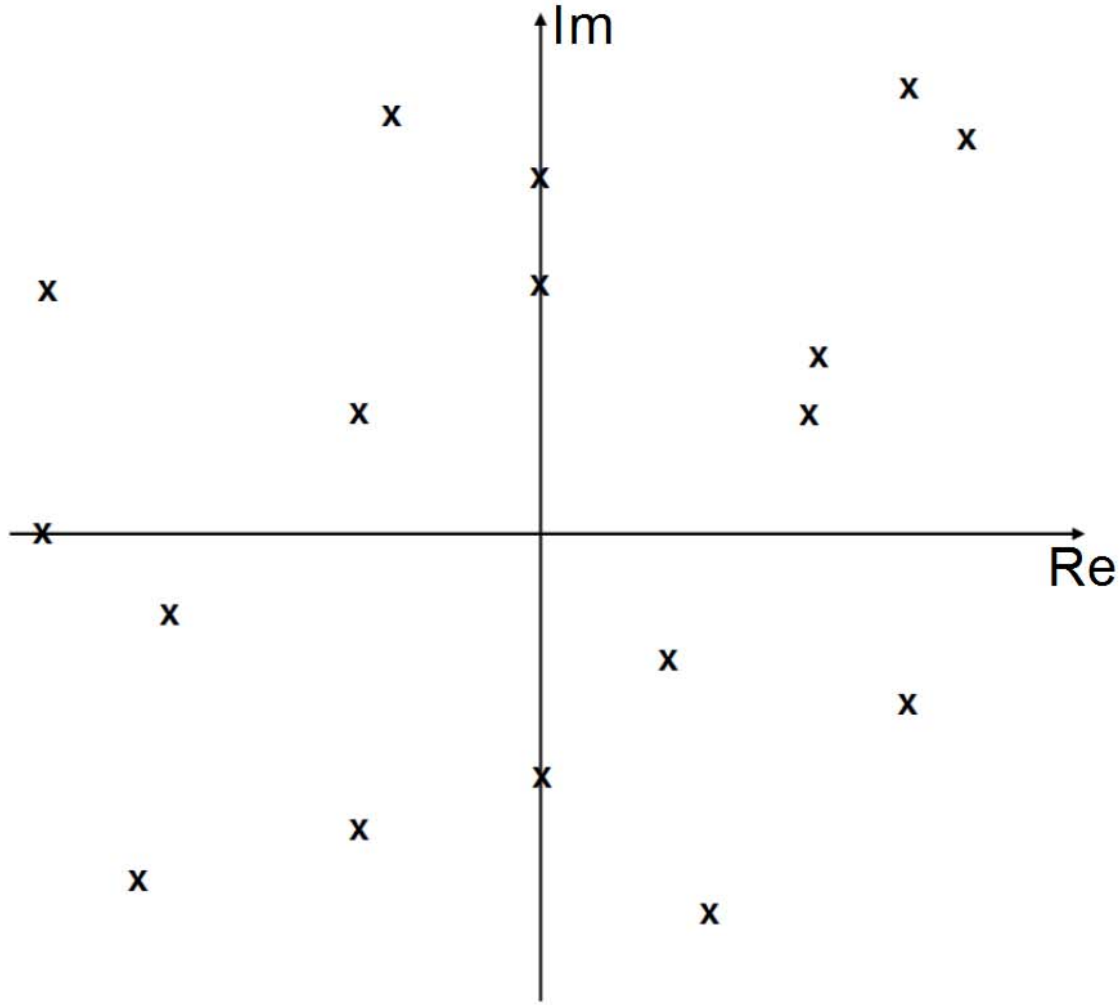
(a) $1 + 2i$ (b) $3 - 4i$ (c) $-5 + 6i$ (d) $-7 - 8i$

(e) 1 (f) $2i$ (g) -3 (h) $-4i$

(i) $\sqrt{2} - \sqrt{2}i$ (j) $-1 + \sqrt{3}i$

Here is a list of 14 complex numbers
Some of them are represented in the Argand diagram
Label these points.

$z_1 = 5 (\cos \pi + i \sin \pi)$	$z_2 = 4 + 5i$
$z_3 = 4 (\cos (-\frac{2}{3}\pi) + i \sin(-\frac{2}{3}\pi)$	$z_4 = - 5 + 3i$
$z_5 = 6 (\cos \frac{1}{4}\pi + i \sin \frac{1}{4}\pi)$	$z_6 = - 4 - i$
$z_7 = 3 (\cos (-\frac{1}{2}\pi) + i \sin (-\frac{1}{2}\pi))$	$z_8 = 3i$
$z_9 = 5 (\cos (-1.1) + i \sin (-1.1))$	$z_{10} = 3 + 2i$
$z_{11} = 3 (\cos \frac{1}{6}\pi + i \sin \frac{1}{6}\pi)$	$z_{12} = 4 - 2i$
$z_{13} = 4.5 (\cos 2 + i \sin 2)$	$z_{14} = 4 (\cos \frac{1}{2}\pi + i \sin \frac{1}{2}\pi)$



Multiplication and division

$z_1 = r_1(\cos\theta + i\sin\theta)$ *and* $z_2 = r_2(\cos\alpha + i\sin\alpha)$

1) a) *Write* $-z_1$

   b) *Express* $(-z_1)$ *in polar form*

   c) Which transformation maps z_1 *to* $-z_1$

2) a) *Write* z_1^*

   b) *Express* z_1^* *in polar form*

   c) Which transformation maps z_1 *to* z_1^*

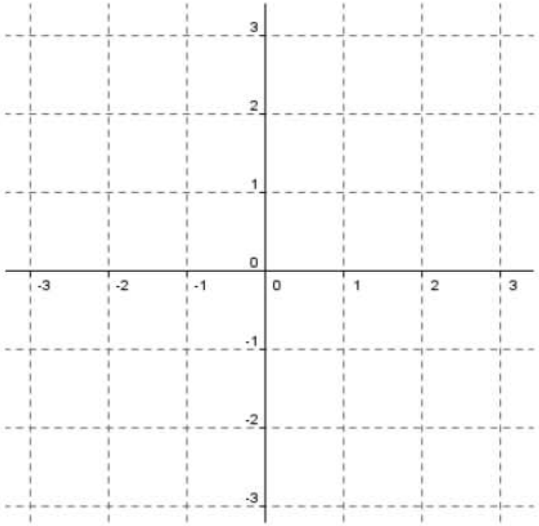
3) a) *Work out* $z_1 \times z_2$

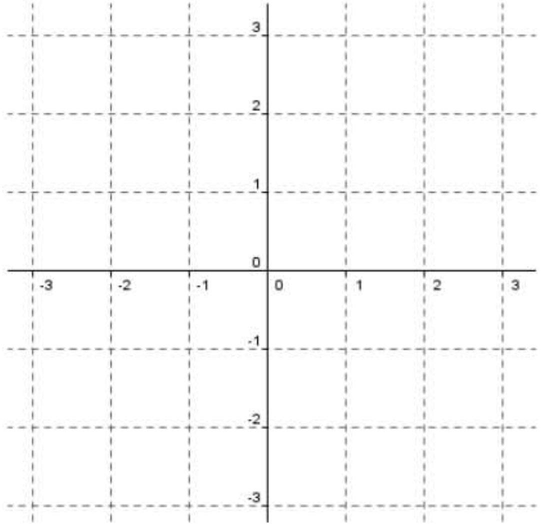
   b) *Express* $z_1 z_2$ *in polar form*

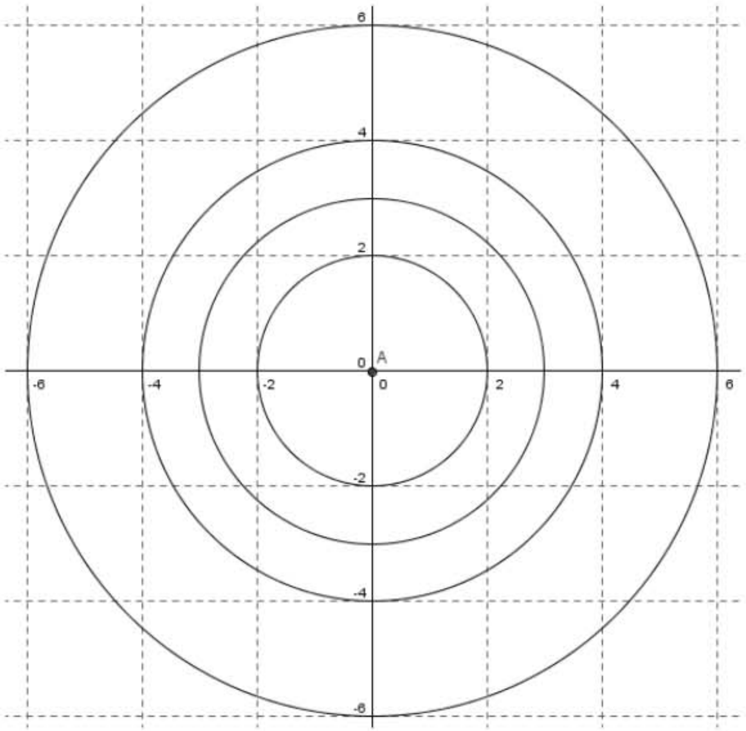
4) a) *Work out* $\frac{z_1}{z_2}$, rationalising your answer

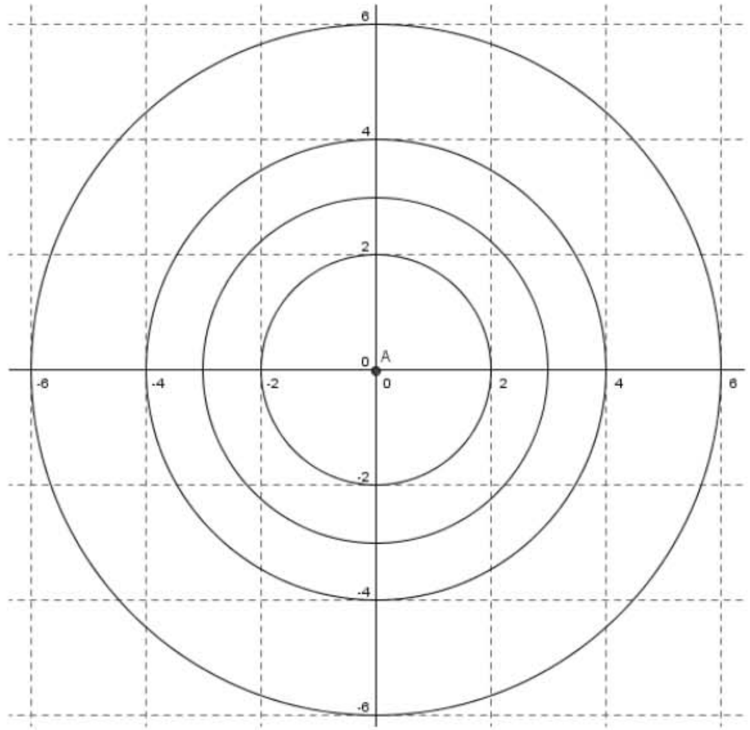
   b) *Express* $\frac{z_1}{z_2}$ *in polar form*

Consequence in the Argand diagram









Summary

If $z_1 = (r_1,\ \theta_1)$ and $z_2 = (r_2,\ \theta_2)$ then $z_1 z_2 = (r_1 r_2,\ \theta_1 + \theta_2)$ – with the proviso that 2π may have to be added to, or subtracted from, $\theta_1 + \theta_2$ if $\theta_1 + \theta_2$ is outside the permitted range for θ

If $z_1 = (r_1,\ \theta_1)$ and $z_2 = (r_2,\ \theta_2)$ then $\dfrac{z_1}{z_2} = \left(\dfrac{r_1}{r_2},\ \theta_1 - \theta_2\right)$ – with the same proviso regarding the size of the angle $\theta_1 - \theta_2$

Exercises:

1. (a) Find $\frac{z_1}{z_2}$ if $z_1 = 2\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)$ and $z_2 = 3\left(\cos\frac{\pi}{6} - \mathrm{i}\sin\frac{\pi}{6}\right)$.

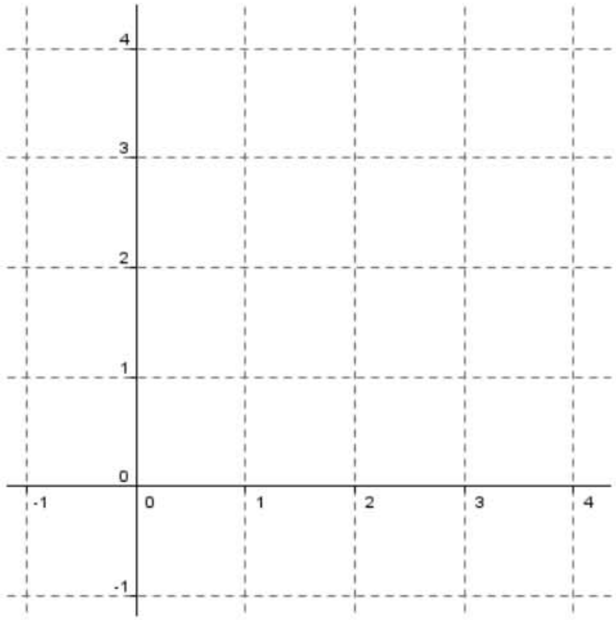
   (b) What can you say about the modulus and argument of $\frac{z_1}{z_2}$?

2 If $z_1 = \left(3,\ \frac{2\pi}{3}\right)$ and $z_2 = \left(2,\ -\frac{\pi}{6}\right)$, find, in polar form, the complex numbers

(a) $z_1 z_2$, (b) $\frac{z_1}{z_2}$, (c) $z_1^{\,2}$, (d) $z_1^{\,3}$, (e) $\frac{z_2}{z_1^{\,2}}$.

Further consideration of $|z_2 - z_1|$ and $\arg(z_2 - z_1)$

Consider the points $A(z_A)$ and $B(z_B)$ with $z_A = 2 + i$ and $z_B = 3 + 2i$



a) Represent z_A and z_B in the Argand diagram

b) Work out $z_C = z_B - z_A$ and represent it in the Argand diag.

c) Work out $|z_C|$

d) Work out the distance AB

e) Work out $\arg(z_C)$

f) Work out the gradient of the line AB hence the angle between the line AB and the x-axis.

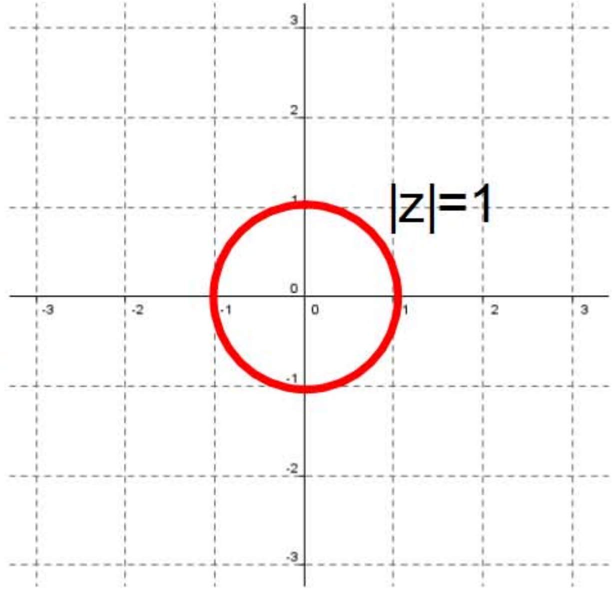
If the complex number z_1 is represented by the point A, and the complex number z_2 is represented by the point B in an Argand diagram, then $|z_2 - z_1| = AB$, and $\arg(z_2 - z_1)$ is the angle between $\overrightarrow{AB}$ and the positive direction of the x-axis

Loci on Argand diagrams

Part 1: Locus of the points M(z) with $| z - z_1 | = r$

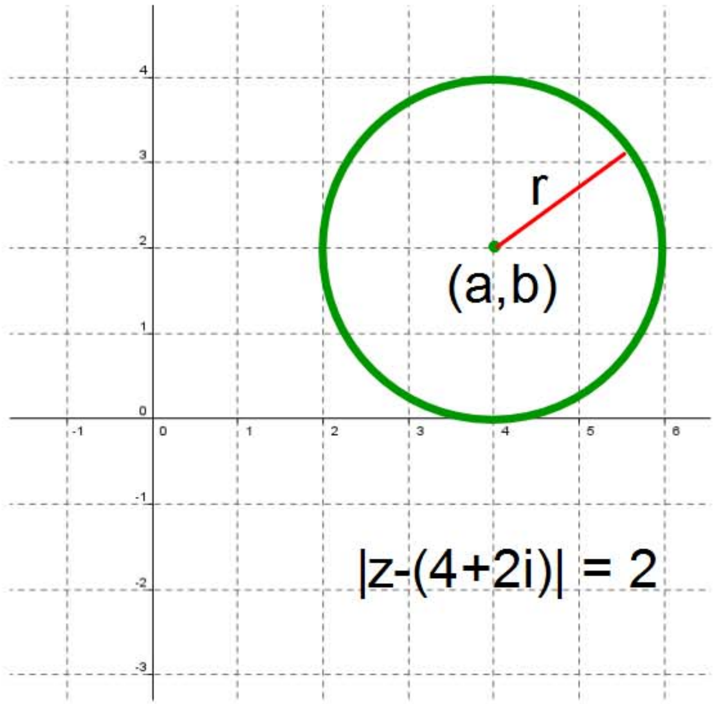
Case 1: $z_1 = 0$ Locus $|z| = r$

The locus is the CIRCLE centre O(0,0) withy radius r.



Case 2: $z_1 \neq 0$, $z_1=a+ib$

The locus is the CIRCLE centre A(a,b) withy radius r.

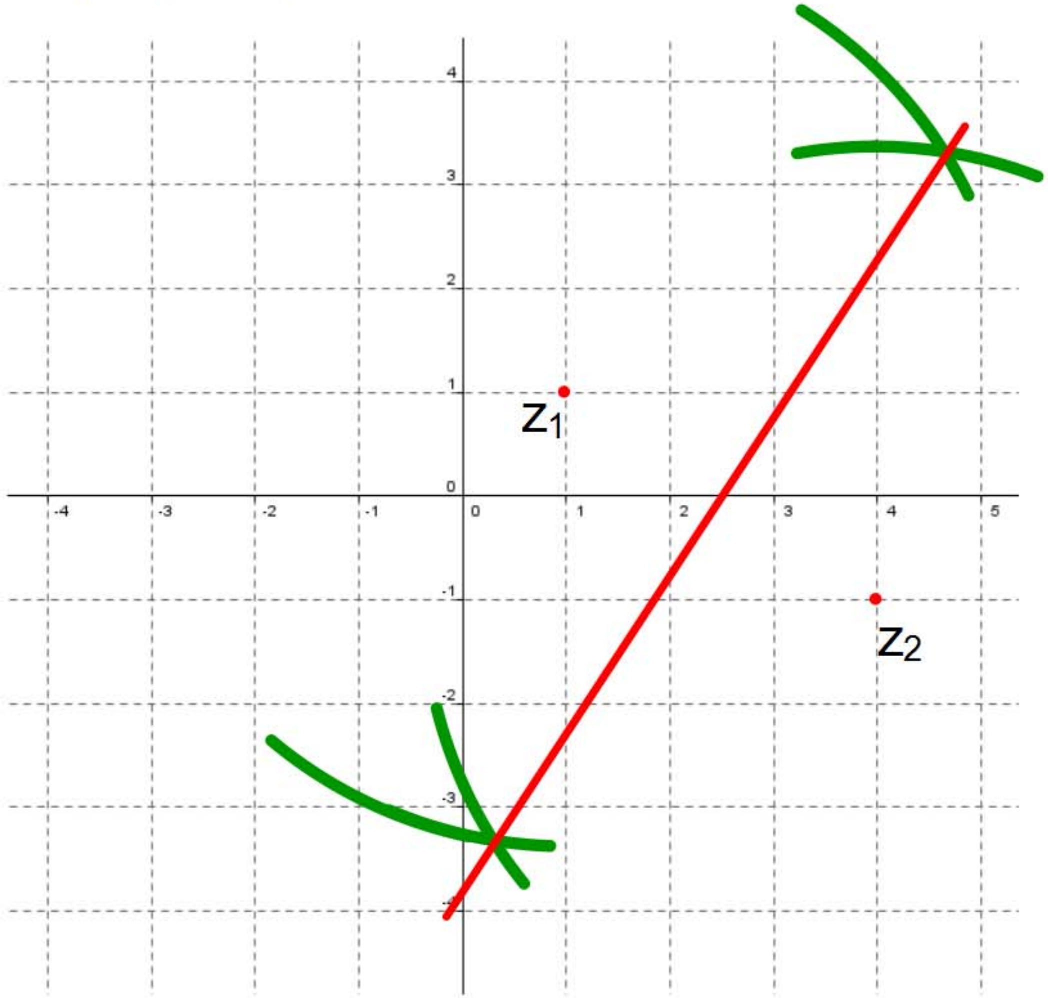


Part 2: Locus of the points $M(z)$ with $|z - z_1| = |z - z_2|$

In an Argand diagram, consider the points $M_1(z_1)$ and $M_2(z_2)$

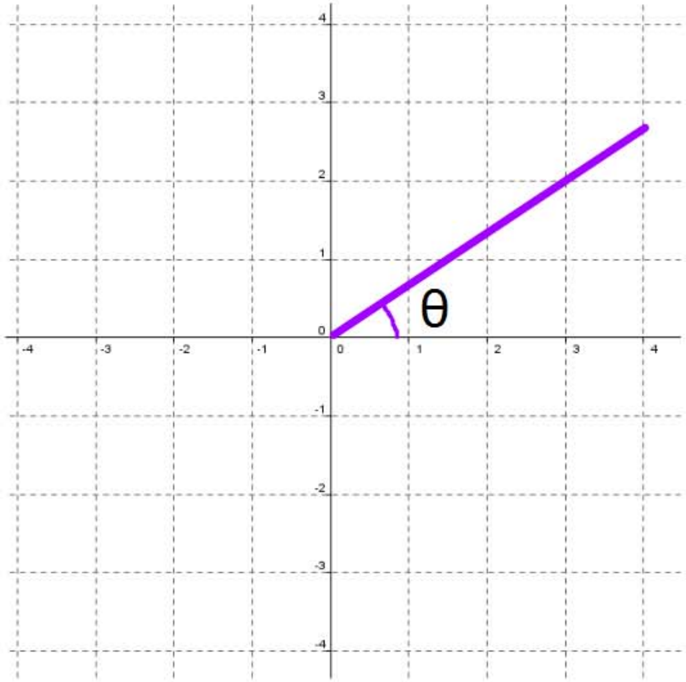
$|z - z_1| = |z - z_2|$ means $MM_1 = MM_2$:

The points belonging to the locus are EQUIDISTANT from M_1 and M_2



The locus is the PERPENDICULAR BISECTOR of the points M_1 and M_2.

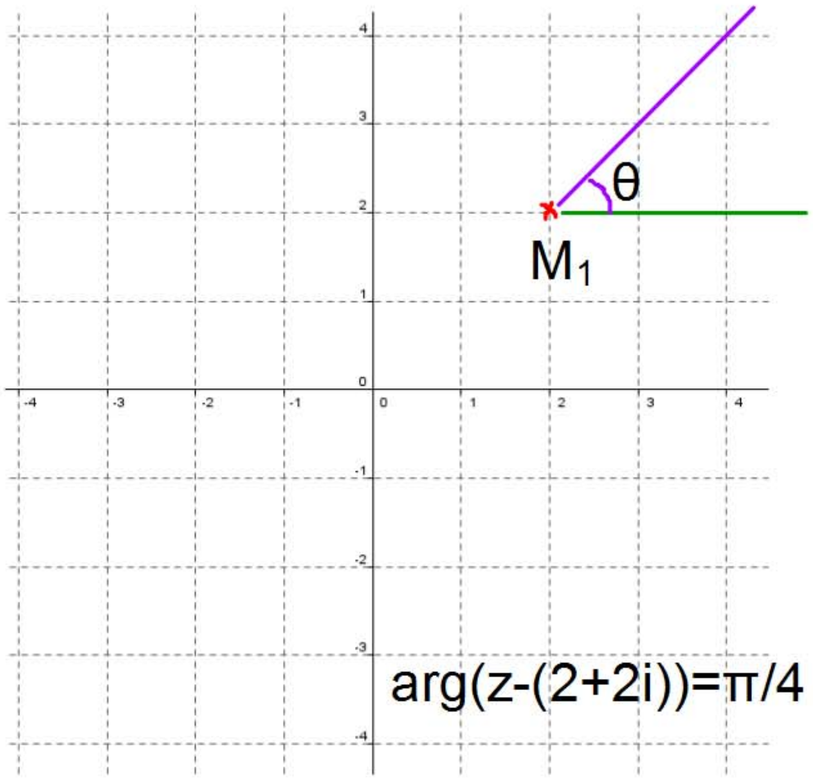
Part 3: Locus of the points M(z) with arg ($z - z_1$) = θ



Case 1: $z_1 = 0$ Locus arg(z) = θ

The locus is the HALF-LINE/RAY with equation $y = \tan(\theta)x$ and $x \geq 0$.

(The ray makes angle θ with the x-axis)



Case 2: $z_1 \neq 0$, $z_1 = a+ib$

The point $M_1(a,b)$

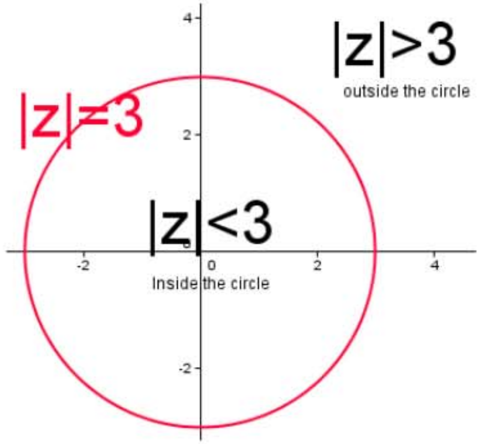
The locus is the HALF-LINE/RAY with equation $(y-b) = \tan(\theta)(x-a)$ and $x \geq a$.

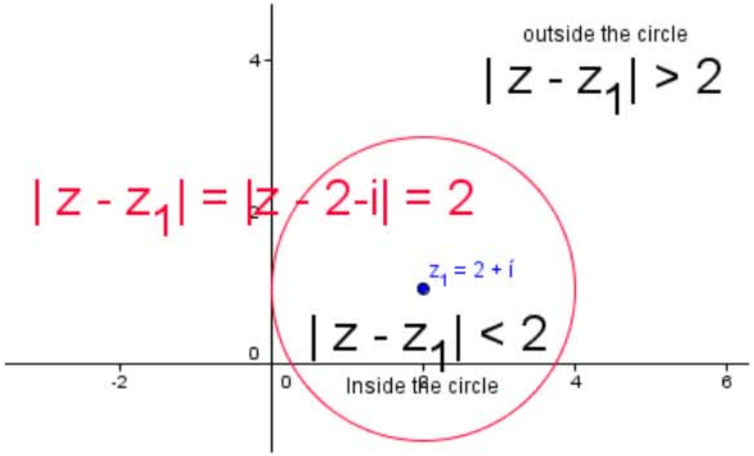
(The ray makes angle θ with the horizontal line going through M_1)

It is a translation of vector $\begin{pmatrix} a \\ b \end{pmatrix}$ of the locus arg(z)= θ

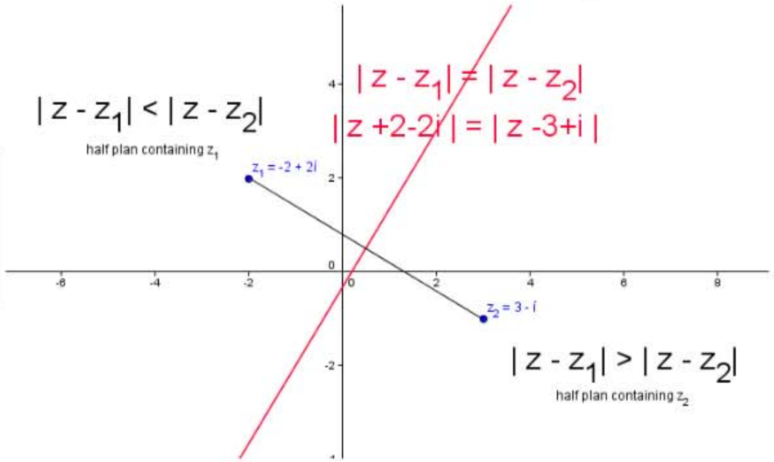
Summary

$|z| = k$ represents a circle with centre O and radius k

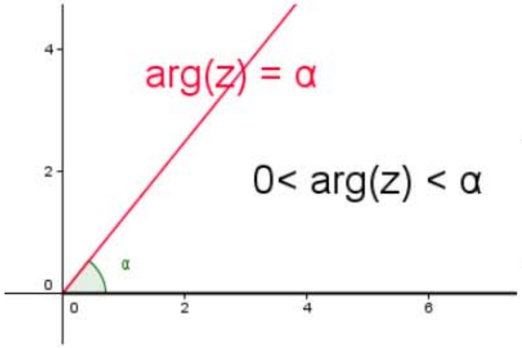




$|z - z_1| = k$ represents a circle with centre z_1 and radius k

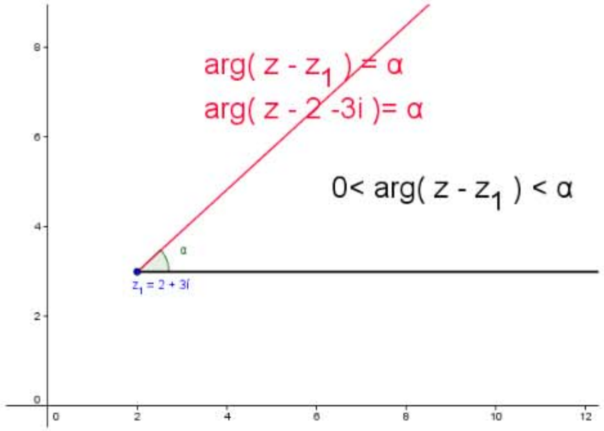


$|z - z_1| = |z - z_2|$ represents a straight line – the perpendicular bisector of the line joining the points z_1 and z_2



$\arg z = \alpha$ represents the *half* line through O inclined at an angle α to the positive direction of Ox

$\arg(z - z_1) = \alpha$ represents the *half* line through *the point* z_1 inclined at an angle α to the positive direction of Ox



Exercises

1. Sketch on Argand diagrams the locus of points satisfying:

   (a) $|z| = 3$, (b) $\arg(z-1) = \frac{\pi}{4}$, (c) $|z-2-\mathrm{i}| = 5$.

2. Sketch on Argand diagrams the regions where:

   (a) $|z-3\mathrm{i}| \le 3$, (b) $\frac{\pi}{2} \le \arg(z-4-2\mathrm{i}) \le \frac{5\pi}{6}$.

3. Sketch on an Argand diagram the region satisfying both $|z-1-\mathrm{i}| \le 3$ and $0 \le \arg z \le \frac{\pi}{4}$.

4. Sketch on an Argand diagram the locus of points satisfying both $|z-\mathrm{i}| = |z+1+2\mathrm{i}|$ and $|z+3\mathrm{i}| \le 4$.