Complex numbers - exam questions

Question 1: Jan 2009

(a) Indicate on an Argand diagram the region for which $|z - 4i| \leqslant 2$. *(4 marks)*

(b) The complex number z satisfies $|z - 4i| \leqslant 2$. Find the range of possible values of $\arg z$. *(4 marks)*

Question 2: Jan 2007

(a) Sketch on one diagram:

(i) the locus of points satisfying $|z - 4 + 2i| = 2$; *(3 marks)*

(ii) the locus of points satisfying $|z| = |z - 3 - 2i|$. *(3 marks)*

(b) Shade on your sketch the region in which

both $$|z - 4 + 2i| \leqslant 2$$

and $$|z| \leqslant |z - 3 - 2i|$$ *(2 marks)*

Question 3: Jan 2008

A circle C and a half-line L have equations

$$|z - 2\sqrt{3} - i| = 4$$

and $$\arg(z + i) = \frac{\pi}{6}$$

respectively.

(a) Show that:

(i) the circle C passes through the point where $z = -i$; *(2 marks)*

(ii) the half-line L passes through the centre of C. *(3 marks)*

(b) On one Argand diagram, sketch C and L. *(4 marks)*

(c) Shade on your sketch the set of points satisfying both

$$|z - 2\sqrt{3} - i| \leqslant 4$$

and $$0 \leqslant \arg(z + i) \leqslant \frac{\pi}{6}$$ *(2 marks)*

Question 4: June 2010

Two loci, L_1 and L_2, in an Argand diagram are given by

$$L_1 : |z + 1 + 3\text{i}| = |z - 5 - 7\text{i}|$$

$$L_2 : \arg z = \frac{\pi}{4}$$

(a) Verify that the point represented by the complex number $2 + 2\text{i}$ is a point of intersection of L_1 and L_2. *(2 marks)*

(b) Sketch L_1 and L_2 on one Argand diagram. *(5 marks)*

(c) Shade on your Argand diagram the region satisfying

both $$|z + 1 + 3\text{i}| \leqslant |z - 5 - 7\text{i}|$$

and $$\frac{\pi}{4} \leqslant \arg z \leqslant \frac{\pi}{2}$$ *(2 marks)*

Question 5: Jan 2010

(a) On the same Argand diagram, draw:

(i) the locus of points satisfying $|z - 4 + 2\text{i}| = 4$; *(3 marks)*

(ii) the locus of points satisfying $|z| = |z - 2\text{i}|$. *(3 marks)*

(b) Indicate on your sketch the set of points satisfying both

$$|z - 4 + 2\text{i}| \leqslant 4$$

and $$|z| \geqslant |z - 2\text{i}|$$ *(2 marks)*

Question 6: Jan 2006

The complex numbers z_1 and z_2 are given by

$$z_1 = \frac{1 + \text{i}}{1 - \text{i}} \quad \text{and} \quad z_2 = \frac{1}{2} + \frac{\sqrt{3}}{2}\text{i}$$

(a) Show that $z_1 = \text{i}$. *(2 marks)*

(b) Show that $|z_1| = |z_2|$. *(2 marks)*

(c) Express both z_1 and z_2 in the polar form (r , θ), with r>0 and $-\pi < \theta \leqslant \pi$. *(3 marks)*

(d) Draw an Argand diagram to show the points representing z_1, z_2 and $z_1 + z_2$. *(2 marks)*

(e) Use your Argand diagram to show that

$$\tan \frac{5}{12}\pi = 2 + \sqrt{3}$$ *(3 marks)*

Complex numbers - exam questions - answers

Question 1: Jan 2009

$a) |z-4i| \le 2$ is the region inside the circle centre $A(0,4)$ and radius $r = 2$.

$b)$ Draw the two tangents to the circle from the origin O. We call the points of contact $P_1(z_1)$ and $P_2(z_2)$.

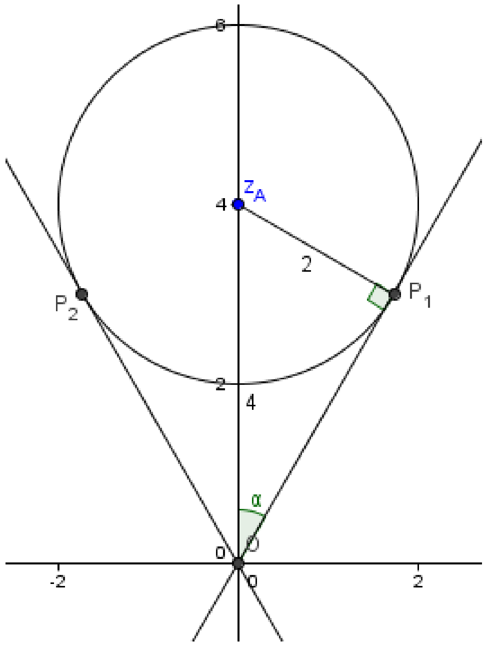
Use trig.properties to work out the argument of z_1 and z_2 :

In the right-angles triangle OAP_1, $\sin\alpha = \dfrac{opp}{hyp} = \dfrac{2}{4} = \dfrac{1}{2}$

$$so\ \alpha = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$\arg(z_1) = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3} \quad and \quad \arg(z_2) = \arg(z_1) + 2\alpha = \frac{2\pi}{3}$$

$$\frac{\pi}{3} \le \arg(z) \le \frac{2\pi}{3}$$



Question 2: Jan 2007

$a) i)$ Let $z_A = 4-2i$ and $A(4,-2)$

The point M represents z in the Argand diagram.

$|z-4+2i| = 2$

$|z-z_A| = 2$ is equivalent to $AM = 2$

The locus of M is the circle centre $A(4,-2)$ radius $r = 2$

$ii)$ Let $z_B = 3+2i$ and $B(3,2)$

$|z| = |z-3-2i|$

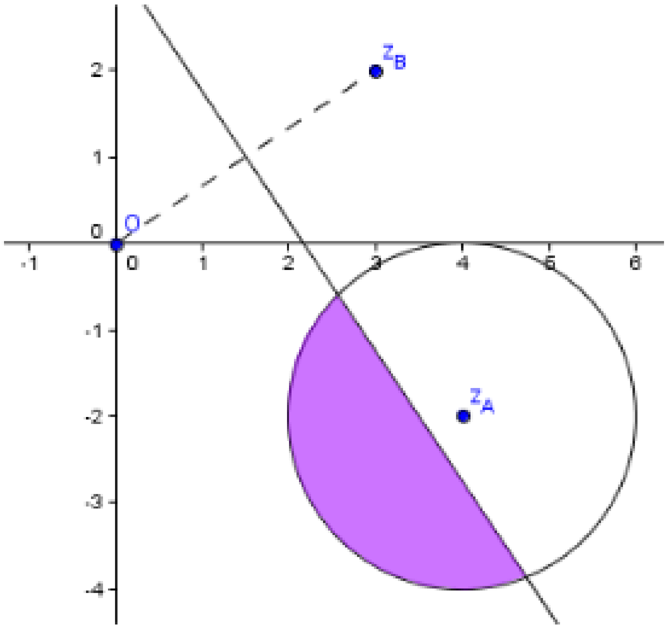
$|z-z_o| = |z-z_B|$ is equivalent to

OM= BM

The locus of M is the prependicular bisector of OB.

b) $|z-4+2i| \le 2$ is "inside" the circle

$|z| \le |z-3-2i|$ is the "half-plane" containing O.



Question 3: Jan 2008

$$a) i) \left|-i-2\sqrt{3}-i\right| = \left|-2\sqrt{3}-2i\right| = \sqrt{(-2\sqrt{3})^2+(-2)^2} = \sqrt{12+4} = \sqrt{16} = 4$$

The circle C passes through the point where $z = -i$

$ii)$ The centre of C is the point where $z = 2\sqrt{3}+i$

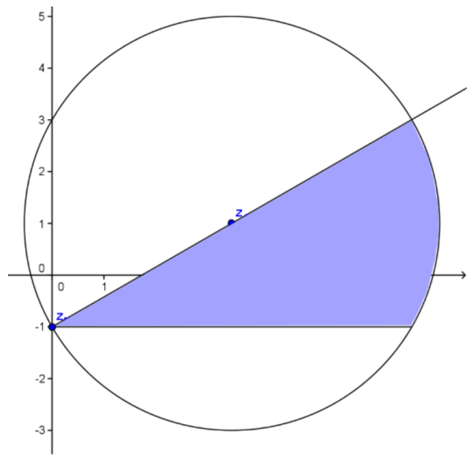
$\arg(z+i) = \arg(2\sqrt{3}+i+i) = \arg(2\sqrt{3}+2i)$

$$Tan^{-1}\left(\frac{2}{2\sqrt{3}}\right) = \frac{\pi}{6}.$$

The half-line L passes through the centre of C.

$b)$

$c)$



Question 4: June 2010

$z = 2 + 2i \text{ and } M(z)$

Does M belong to L_1 ?

$|z+1+3i| = |2+2i+1+3i| = |3+5i| = \sqrt{9+25} = \sqrt{34}$

$|z-5-7i| = |2+2i-5-7i| = |-3-5i| = \sqrt{9+25} = \sqrt{34}$

$M(z = 2+2i)$ belongs to L_1

Does M belong to L_2 ?

$\arg(z) = \arg(2+2i) = \tan^{-1}\left(\frac{2}{2}\right) = \frac{\pi}{4}$

$M(z = 2+2i)$ belongs to L_2

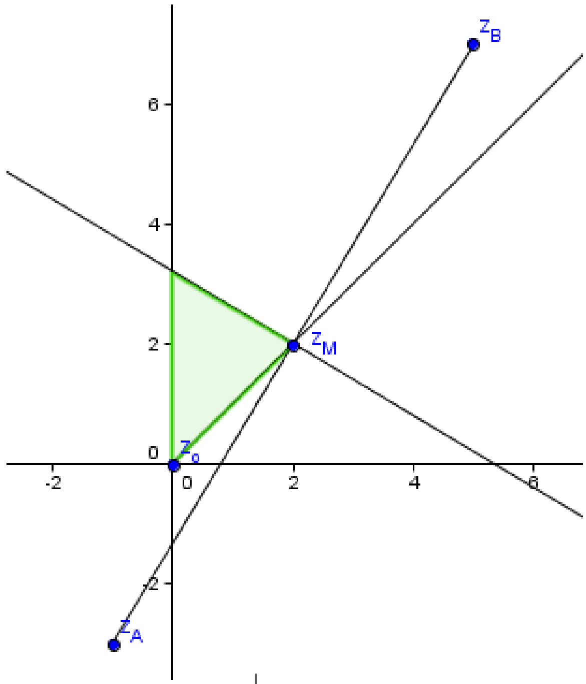
$M(z)$ is a point of the intersection between L_1 *and* L_2

b) L_1 is the perpendicular bisector of the line AB

with $A(z_A = -1-3i)$ and $B(z_B = 5+7i)$

L_2 is the half line from O with gradient $\tan\frac{\pi}{4} = 1$.

c)



Question 5: Jan 2010

a) i) $|z-4+2i| = 4$

this is the circel centre $A(z_A)$ *with* $z_A = 4-2i$

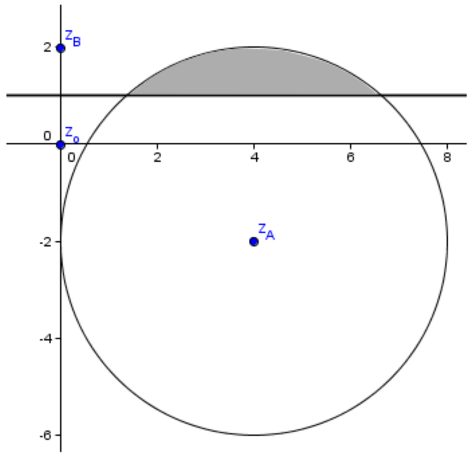
and radius $r = 4$

ii) $|z| = |z-2i|$

This is the perpendicular bisector of the line OB

with $z_B = 2i$ and $z_O = 0$

b) The region is the intersection of the inside of the circle and the half-plane containing B.



Question 6: Jan 2006

$z_1 = \frac{1+i}{1-i}$ and $z_2 = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

a) $z_1 = \frac{1+i}{1-i} \times \frac{1+i}{1+i} = \frac{1+2i+i^2}{1-i^2} = \frac{2i}{2} = i$

b) $|z_1| = |i| = |0+1i| = \sqrt{0^2+1^2} = 1$

$|z_2| = \left|\frac{1}{2} + \frac{\sqrt{3}}{2}i\right| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4}+\frac{3}{4}} = \sqrt{1} = 1$

$|z_1| = |z_2|$

c) $z_1 = \left(1, \frac{\pi}{2}\right)$ and $z_2 = \left(1, \frac{\pi}{3}\right)$

e) $\frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$

$\frac{1}{2} \text{ of } \frac{\pi}{6} = \frac{\pi}{12}$

The argument of z_3 is $\arg(z_2) + \frac{\arg(z_1) - \arg(z_2)}{2}$

$\arg(z_3) = \frac{\pi}{3} + \frac{\pi}{12} = \frac{5\pi}{12}$.

and $z_3 = z_1 + z_2 = i + \frac{1}{2} + \frac{\sqrt{3}}{2}i = \frac{1}{2} + i(1 + \frac{\sqrt{3}}{2})$

So $\tan\frac{5\pi}{12} = \frac{1+\frac{\sqrt{3}}{2}}{\frac{1}{2}} = 2+\sqrt{3}$