Chapter 9: Differential Calculus

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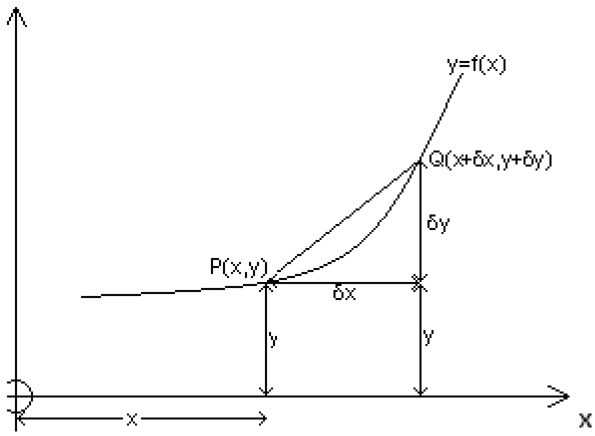
1 Differentiation from First Principles

Consider a point P(x,y) on the curve y=f(x), and let $Q(x + \delta x, y + \delta y)$ be a point on the curve which is very close to P:



The gradient of the chord PQ is:

$$\frac{\delta y}{\delta x} = \frac{(y + \delta y) - y}{(x + \delta x) - x} \qquad (1)$$

$$= \frac{f(x + \delta x) - f(x)}{(x + \delta x) - x} \qquad (2)$$

The ratio $\frac{\delta y}{\delta x}$ approaches a definite limit as δx gets smaller and approaches 0.

This limit is the gradient of the tangent at P, which is the gradient of the curve at P.

It is called the rate of the change of y with respect to x at the point P, and is denoted by $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \left(\frac{\delta y}{\delta x} \right) \qquad (3)$$

$$= \lim_{\delta x \to 0} \left[\frac{f(x + \delta x) - f(x)}{(x + \delta x) - x} \right] \qquad (4)$$

$$= \lim_{\delta x \to 0} \left[\frac{f(x + \delta x) - f(x)}{(\delta x)} \right] \qquad (5)$$

$\frac{dy}{dx}$ is called the differential coefficient or the first derivative of y with respect to x.

If y=f(x), you can use the notation $\frac{dy}{dx} = f'(x)$.

In this case f' is often called the derived function of f or the gradient function (since it gives an expression for the gradient of the curve at any point). The procedure used to find $\frac{dy}{dx}$ from y is called differentiating y with respect to x.

Example

Find $\frac{dy}{dx}$ from first principles if $y = x^2$

Solution

$$\begin{aligned}
\text{Let } f(x) &= x^2 && (6)\\
\frac{dy}{dx} &= \lim_{\delta x \to 0}\left[\frac{f(x+\delta x)-f(x)}{\delta x}\right]\\
&= \lim_{\delta x \to 0}\left[\frac{(x+\delta x)^2-x^2}{\delta x}\right]\\
&= \lim_{\delta x \to 0}\left[\frac{x^2+2x\delta x+(\delta x)^2-x^2}{\delta x}\right]\\
&= \lim_{\delta x \to 0}\left[\frac{2x\delta x+(\delta x)^2}{\delta x}\right]\\
&= \lim_{\delta x \to 0}[2x+\delta x]\\
\therefore \frac{dy}{dx} &= 2x && (7)
\end{aligned}$$

2 General formula for $\frac{dy}{dx}$ when $y = ax^n$

1. If $y = ax^n$ (where a and n are constants), then
   $\frac{dy}{dx} = anx^{n-1}$
   i.e,multiply by the power and then subtract one from the power.
2. If y=ax (where a is a constant), then $\frac{dy}{dx} = a$
3. If y=a (where a is a constant), then $\frac{dy}{dx} = 0$

Example

Find $\frac{dy}{dx}$ for each of the following:

$$\begin{aligned}
(i) &\quad y = 6x^3 && (8)\\
(ii) &\quad y = \frac{3}{8x^2} && (9)\\
(iii) &\quad y = x^{\frac{1}{4}} && (10)\\
(iv) &\quad y = \frac{1}{\sqrt[3]{x}} && (11)
\end{aligned}$$

Solution

$$
\begin{aligned}
(i) \quad y &= 6x^3 && (12)\\
\frac{dy}{dx} &= 6(3x^{3-1}) = 18x^2 && (13)\\
(ii) \quad y &= \frac{3}{8x^2} = \frac{3}{8}x^{-2} && (14)\\
\frac{dy}{dx} &= \frac{3}{8}(-2x^{-2-1}) = -\frac{3}{4}x^{-3} = -\frac{3}{4x^3} && (15)\\
(iii) \quad y &= x^{\frac{1}{4}} && (16)\\
\frac{dy}{dx} &= \frac{1}{4}x^{\frac{1}{4}-1} = \frac{1}{4}x^{-\frac{3}{4}} = \frac{1}{4x^{\frac{3}{4}}} && (17)\\
(iv) \quad y &= \frac{1}{\sqrt[3]{x}} = x^{-\frac{1}{3}} && (18)\\
\frac{dy}{dx} &= -\frac{1}{3}x^{-\frac{1}{3}-1} = -\frac{1}{3}x^{-\frac{4}{3}}\\
&= -\frac{1}{3x^{\frac{4}{3}}}\\
&= -\frac{1}{3(\sqrt[3]{3})^4} && (19)
\end{aligned}
$$

3 Sum or Difference of Two Functions

$$
\begin{aligned}
\text{If } y &= f(x) \pm g(x), \text{ then} && (20)\\
\frac{dy}{dx} &= f'(x) \pm g'(x). && (21)
\end{aligned}
$$

Example 1

Find f'(x) of for each of the following:

$$(i) \quad f(x) = 3x^4 + \sqrt{x} + 2x \qquad (22)$$

$$(ii) \quad f(x) = \frac{6}{\sqrt{x}} - \frac{4}{x^3} + 10 \qquad (23)$$

Solution 1

$$
\begin{aligned}
(i) \qquad f(x) &= 3x^4 + \sqrt{x} + 2x = 3x^4 + x^{\frac{1}{2}} + 2x && (24)\\
f'(x) &= 12x^3 + \frac{1}{2}x^{-\frac{1}{2}} + 2\\
&= 12x^3 + \frac{1}{2x^{\frac{1}{2}}} + 2\\
&= 12x^3 + \frac{1}{2\sqrt{x}} + 2 && (25)\\
(ii) \qquad f(x) &= \frac{6}{\sqrt{x}} - \frac{4}{x^3} + 10 = \frac{6}{x^{\frac{1}{2}}} - \frac{4}{x^3} + 10 && (26)\\
&= 6x^{-\frac{1}{2}} - 4x^{-3} + 10\\
\therefore f'(x) &= -3x^{-\frac{3}{2}} + 12x^{-4}\\
&= -\frac{3}{x^{\frac{3}{2}}} + \frac{12}{x^4}\\
f'(x) &= -\frac{3}{(\sqrt{x})^3} + \frac{12}{x^4} && (27)\\
& && (28)
\end{aligned}
$$

Example 2

Find $\frac{dy}{dx}$ for each of the following:

$$(i) \qquad y = (\sqrt{x} + 3)^2 \qquad (29)$$

$$(ii) \qquad y = \frac{3x^2 + 2}{x} \qquad (30)$$

Solution 2

$$
\begin{aligned}
&(i) \quad \text{Expand brackets} && (31)\\
&y = (\sqrt{x} + 3)^2 = x + 6\sqrt{x} + 9 = x + 6x^{\frac{1}{2}} + 9\\
&\therefore \frac{dy}{dx} = 1 + 3x^{-\frac{1}{2}} = 1 + \frac{3}{x^{\frac{1}{2}}} = 1 + \frac{3}{\sqrt{x}} && (32)\\
&(ii) \quad y = \frac{3x^2 + 2}{x} = \frac{3x^2}{x} + \frac{2}{x} = 3x + 2x^{-1} && (33)\\
&\therefore \frac{dy}{dx} = 3 - 2x^{-2} = 3 - \frac{2}{x^2} = \frac{3x^2 - 2}{x^2} && (34)
\end{aligned}
$$

4 Second Derivative

We can repeat the differentiation process to find the differential coefficient of $\frac{dy}{dx}$ with respect to x, i.e.,

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) \qquad (35)$$

This is called the second derivative of y with respect to x and is written as:

$$\frac{d^2y}{dx^2} \qquad (36)$$

If $y = f(x)$, $\frac{d^2y}{dx^2}$ is written as $f''(x)$.

Example

If $y = 2x + \frac{3}{x}$, find $\frac{d^2y}{dx^2}$

Solution

$$y = 2x + \frac{3}{x} = 2x + 3x^{-1} \quad (37)$$

$$\frac{dy}{dx} = 2 - 3x^{-2} \quad (38)$$

$$\frac{d^2y}{dx^2} = 6x^{-3} = \frac{6}{x^3} \quad (39)$$

5 Gradient of a Curve

The gradient of a curve at point P is equal to the gradient of the tangent at P, which is given by the value of $\frac{dy}{dx}$ at P.

Example 1

Find the gradient of the curve $y = 3x^2 + x + 1$ at the point (1,5).

Solution 1

$$y = 3x^2 + x + 1 \quad (40)$$

$$\frac{dy}{dx} = 6x + 1 \quad (41)$$

$$\text{when } x = 1, \frac{dy}{dx} = 6 \times 1 + 1 = 7 \quad (42)$$

$$\therefore \text{ ANS } = 7 \quad (43)$$

Example 2

Find the coordinates of the point on the curve $y = 2x^2 + 3x + 1$ where the gradient is -1.

Solution 2

$$y = 2x^2 + 3x + 1 \quad (44)$$

$$\frac{dy}{dx} = 4x + 3$$

$$\frac{dy}{dx} = -1 \Rightarrow 4x + 3 = -1$$

$$\therefore 4x = -4$$

$$\therefore x = -1$$

$$\text{when } x = -1, \quad y = 2(-1)^2 + 3(-1) + 1 = 0$$

$$\therefore \text{ ANS } = (-1, 0) \quad (45)$$

6 Equation of a Tangent to a curve

The equation of the line passing through the point (x_1, y_1) of gradient m is given by:

$y - y_1 = m(x - x_1)$.

Therefore, to find the equation of the tangent to a curve:

1. Differentiate to find the gradient, m
2. Use $y - y_1 = m(x - x_1)$.

Example

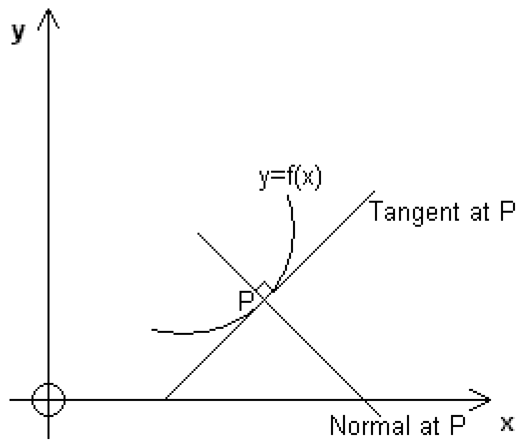
Find the equation of the tangent to the curve $y = 4x^3 - 2x^2 - 5x$ at the point where x=1.

Solution

$$
\begin{aligned}
& y = 4x^3 - 2x^2 - 5x && (46)\\
& \frac{dy}{dx} = 12x^2 - 4x - 5 \\
& \text{When } x = 1, \frac{dy}{dx} = 12(1)^2 - 4(1) - 5 = 3 \\
& \text{When } x = 1, y = 4x^3 - 2x^2 - 5x \\
& = 4(1)^3 - 2(1)^2 - 5(1) = -3 \\
& \therefore m = 3, \quad x_1 = 1 \quad y_1 = -3 \\
& \text{Using } y - y_1 = m(x - x_1) \text{ gives} \\
& y - (-3) = 3(x - 1) \\
& \therefore y + 3 = 3x - 3 \\
& \therefore y = 3x - 3 - 3 \\
& \therefore y = 3x - 6 && (47)
\end{aligned}
$$

7 Equation of a Normal to a Curve

A normal is a line perpendicular to the tangent at the point of contact.



If the gradient of the tangent is m, then the gradient of the normal is $\frac{-1}{m}$, i.e. to find the gradient of the normal turn the gradient of the tangent upside down and change the sign.

Example

Find the equation of the normal to the curve $y = x^2 + 3x - 2$ at the point where $x = 4$.

Solution

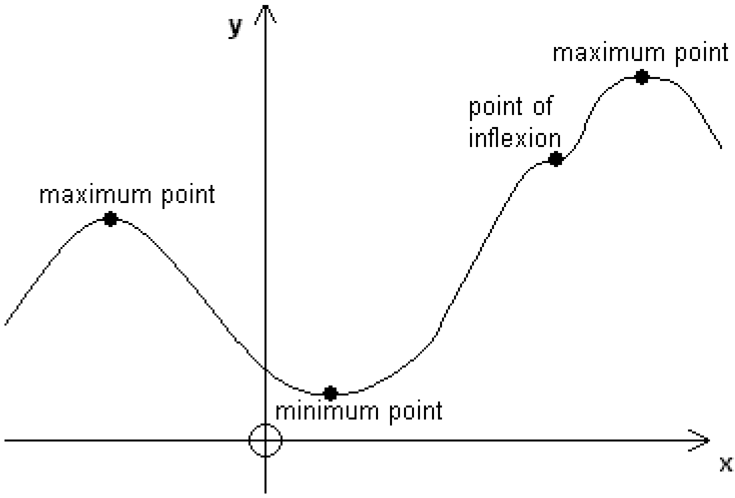
$$\begin{aligned}
&y = x^2 + 3x - 2 &(48)\\
&\frac{dy}{dx} = 2x + 3\\
&\text{When } x = 4, \quad \frac{dy}{dx} = 2(4) + 3 = 11\\
&\therefore m_{tangent} = 11\\
&\therefore m_{normal} = -\frac{1}{11}\\
&\text{When } x = 4, \quad y = x^2 + 3x - 2\\
&= 4^2 + 3(4) - 2 = 26\\
&\therefore m = -\frac{1}{11}, \quad x_1 = 4, \quad y_1 = 26\\
&\text{Using } y - y_1 = m(x - x_1) \text{ gives}\\
&y - 26 = -\frac{1}{11}(x - 4)\\
&y - 26 = -\frac{1}{11}x + \frac{4}{11}\\
&y = -\frac{1}{11}x + 26\frac{4}{11} &(49)
\end{aligned}$$

8 Stationary Points

A stationary point is one where $\frac{dy}{dx} = 0$.

There are three main types of stationary point:

1. A (local) maximum turning point.
2. A (local) minimum turning point.
3. A point of inflexion or a saddle point.



To find and distinguish between the stationary points of a curve:

1. Differentiate y twice to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$
2. Put $\frac{dy}{dx}$=0 and solve for x. Then find the corresponding values for y from the original equation.

   At this stage you have found the stationary points
3. To determine the nature of stationary point P substitute the coordinates into $\frac{d^2y}{dx^2}$

   - If $\frac{d^2y}{dx^2} < 0$, the point is a maximum turning point
   - If $\frac{d^2y}{dx^2} > 0$, the point is a minimum turning point
   - If $\frac{d^2y}{dx^2} = 0$, and $\frac{dy}{dx}$ has the same sign on either side of P, the point is a point of inflexion.
   - **Note**: A point of inflexion is a point where the tangent of the curve actually crosses the curve at that point. Therefore, it is possible to have points of inflexion that are not stationary.

Example 1

Find the coordinates and nature of the stationary points on the curve with equation $y = (x+2)(x-1)^2$. Hence sketch the curve.

Solution 1

$$\begin{aligned} y &= (x+2)(x-1)^2 \quad (50)\\ &= (x+2)(x^2-2x+1)\\ &= x^3-2x^2+x+2x^2-4x+2\\ y &= x^3-3x+2\\ \frac{dy}{dx} &= 3x^2-3 \quad (51) \end{aligned}$$

Stationary points occur when $\frac{dy}{dx} = 0$

$$\begin{aligned} \frac{dy}{dx} &= 0 \Rightarrow 3x^2-3=0 \quad (52)\\ \therefore\ & 3(x+1)(x-1)=0\\ \therefore\ & x=-1 \text{ or } x=1\\ \text{When } x=-1, \quad & y=(x+2)(x-1)^2\\ & = (1)(-2)^2 = 4\\ \text{When } x=1, \quad & y=(x+2)(x-1)^2\\ & (3)(0)^2 = 0 \quad (53) \end{aligned}$$

$\therefore$ The stationary points are (-1,4) and (1,0). (54)

$\frac{d^2y}{dx^2} = 6x$

when $x=-1$, $\frac{d^2y}{dx^2} = 6\times(-1) = -6 < 0$

$\therefore$ $x=-1$ corresponds to a maximum turning point.

When $x=1$, $\frac{d^2y}{dx^2} = 6\times 1 = 6 > 0$

$\therefore$ $x=1$ corresponds to a minimum turning point.

$\therefore$ The stationary points are:

$(-1,4)$ Maximum turning point

$(1,0)$ Minimum turning point

Curve meets y-axis when $x=0$

$\therefore$ $y=(2)(-1)^2=2$

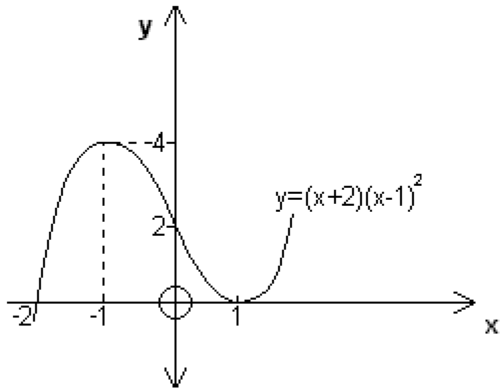
$\therefore$ Curve meets y-axis at $(0,2)$

Curve meets x-axis when $y=0$

$\therefore$ $(x+2)(x-1)^2=0$

$\therefore$ $x=-2$ or $x=1$

$\therefore$ Curve meets x-axis at (-2,0) and (1,0). (55)



Example 2

Find the coordinates of any stationary points on the curve $y = 5x^6 - 12x^5$ and distinguish between them.

Hence sketch the curve.

Solution 2

$$y = 5x^6 - 12x^5 \quad (56)$$

$$\frac{dy}{dx} = 30x^5 - 60x^4 \quad (57)$$

Stationary points occur when $\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = 0 \Rightarrow 30x^5 - 60x^4 = 0 \quad (58)$$

$$\therefore 30x^4(x-2) = 0$$

$$\therefore x = 0 \text{ or } x = 2$$

When $x = 0$, $\quad y = 5x^6 - 12x^5$

$$= 0 - 0 = 0$$

When $x = 2$, $\quad y = 5x^6 - 12x^5$

$$= 5(2^6) - 12(2^5) = -64 \quad (59)$$

$\therefore$ The stationary points are (0,0) and (2,-64). (60)

$\frac{d^2y}{dx^2} = 150x^4 - 240x^3$

When $x = 2$, $\frac{d^2y}{dx^2} = 150(2^4) - 240(2^3) = 480 > 0$

$\therefore$ $x = 2$ corresponds to a minimum turning point.

When $x = 0$, $\frac{d^2y}{dx^2} = 0 - 0 = 0$

$\therefore$ $x = 0$ *may* correspond to a point of inflexion.

When $x = -0.1$, $\frac{dy}{dx} = 30(-0.1)^5 - 60(-0.1)^4$

$= -0.0063$

When $x = 0.1$, $\frac{dy}{dx} = 30(0.1)^5 - 60(0.1)^4$

$= -0.0057$

$\therefore$ $x = 0$ does correspond to a point of inflexion.

$\therefore$ Stationary points are:

$(0, 0)$ Point of inflexion

$(2, -64)$ Minimum turning point

Curve meets y-axis when $x = 0$

$\therefore$ $y = 0 - 0 = 0$

$\therefore$ Curve meets y-axis at (0,0)

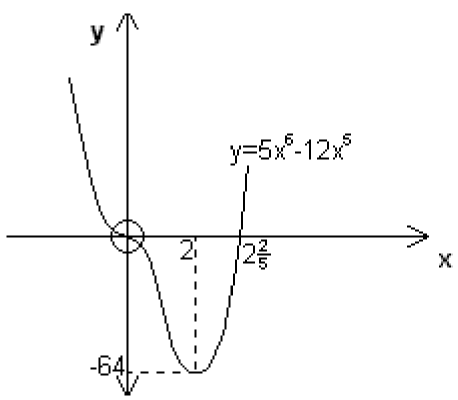
Curve meets x-axis when y=0

$\therefore 5x^6 - 12x^5 = 0$

$\therefore x^5(5x - 12) = 0$

$\therefore x = 0$ or $x = 2\frac{2}{5}$

$\therefore$ Curve meets x-axis at (0,0) and $(2\frac{2}{5}, 0)$. (61)



9 Increasing and Decreasing Functions

A function f which increases as x increases in the interval from x=a to x=b is called an increasing function in the interval (a,b).

For such a function, $f'(x) > 0$ throughout the interval.

A function f which decreases as x increases in the interval from x=c to x=d is called an decreasing function in the interval (c,d).

For such a function, $f'(x) < 0$ throughout the interval.

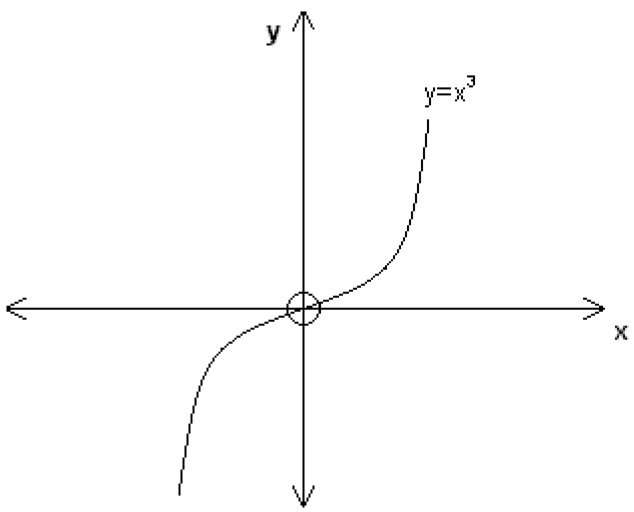
Example

Determine if the following functions are increasing or decreasing:

(i) $f(x) = x^3$
(ii) $f(x) = -(x-1)^2$

Solution

$$(i) \quad f(x) = x^3 \tag{62}$$



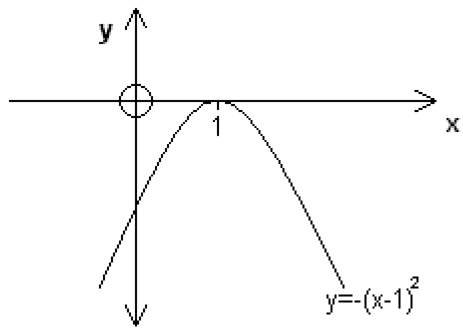
$$y = x^3 \tag{63}$$

$$\frac{dy}{dx} = 3x^2 > 0 \text{ for all real values except x=0} \tag{64}$$

$$\therefore \quad f(x) = x^3 \text{ is increasing for all real values of x except x=0} \tag{65}$$

$$\tag{66}$$

$$(ii) \qquad f(x) = -(x-1)^2 \tag{67}$$



$$y = -(x-1)^2 = -(x^2 - 2x + 1) \tag{68}$$

$$= -x^2 + 2x - 1$$

$$\therefore \quad \frac{dy}{dx} = -2x + 2$$

f(x) is increasing when $\frac{dy}{dx} > 0$

$$\frac{dy}{dx} > 0 \Rightarrow -2x + 2 > 0$$

$$\Rightarrow x < 1$$

f(x) is decreasing when $\frac{dy}{dx} < 0$

$$\frac{dy}{dx} < 0 \Rightarrow -2x + 2 < 0$$

$$\Rightarrow x > 1$$

$\therefore$ The function $f(x) = -(x-1)^2$ is increasing in the interval (69)

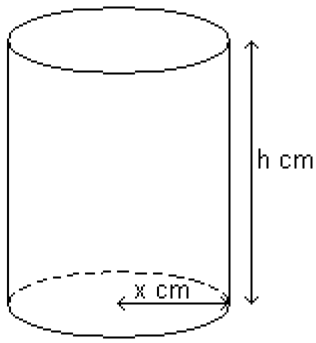
$x < 1$ and is decreasing in the interval $x > 1$ (70)

10 Using Differentiation to Solve Practical Problems

Example

A cylindrical tin, closed at both ends, is made of thin sheet metal. Find the dimensions of a tin like this that holds $1000cm^3$ and has a minimum total surface area.

Solution



Let radius = x cm (71)

and height = h cm

Volume $= \pi x^2 h = 1000$

$\therefore \quad h = \dfrac{1000}{\pi x^2} \qquad (1)$

Curved surface area $= 2\pi x h$

Area of 2 circular ends $= 2\pi x^2$

$\therefore$ Total surface area is:

$$A = 2\pi x h + 2\pi x^2 \qquad (2)$$

Substituting (1) into (2) gives

$A = 2\pi x(\dfrac{1000}{\pi x^2}) + 2\pi x^2$

$\therefore \quad A = \dfrac{2000}{x} + 2\pi x^2$

$\therefore \quad A = 2000x^{-1} + 2\pi x^2$

$\dfrac{dA}{dx} = -2000x^{-2} + 4\pi x = -\dfrac{2000}{x^2} + 4\pi x$

For stationary value, $\dfrac{dA}{dx} = 0$

$\therefore \quad -\dfrac{2000}{x^2} + 4\pi x = 0$

$\therefore \quad \dfrac{2000}{x^2} = 4\pi x$

$\therefore \quad x^3 = \dfrac{2000}{4\pi} = \dfrac{500}{\pi}$

$\therefore \quad x = \sqrt[3]{\dfrac{500}{\pi}} = 5.42$ cm

Substituting x=5.42 into (1) gives

$$h = \frac{1000}{\pi(5.42)^2} = 10.8 \text{ cm}$$

$$\frac{d^2y}{dx^2} = 4000x^{-3} + 4\pi = \frac{4000}{x^3} + 4\pi$$

Clearly $\dfrac{d^2A}{dx^2} > 0$ when $x = 5.42$ cm

$\therefore$ x=5.42 corresponds to the minimum value of A

ANS: Radius $= 5.42$ cm

Height $= 10.8$ cm (72)

11 Rates of Change

Example

At time t seconds the length of edge x cm of an expanding cube is given by x=2t. Express the volume, $V cm^3$, and the surface area, $A cm^2$, in terms of t. Hence find the rate of change of V and the rate of change of A with respect to t at the instant when t=3.

Solution

$$\begin{aligned} x &= 2t \quad (73) \\ V &= x^3 \\ \therefore \quad V &= (2t)^3 = 8t^3 \\ A &= 6x^2 \\ \therefore \quad A &= 6(2t)^2 = 24t^2 \\ \frac{dV}{dt} &= 24t^2 \\ \frac{dA}{dt} &= 48t \quad (74) \end{aligned}$$

$$\begin{aligned} \text{When } t = 3s, \quad \frac{dV}{dt} &= 24 \times 3^2 = 216 \text{ cm}^3/s \quad (75) \\ \frac{dA}{dt} &= 48 \times 3 = 144 \text{ cm}^2/s. \quad (76) \end{aligned}$$

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