Series

Specifications

Series

Use of formulae for the sum of the squares and the sum of the cubes of the natural numbers.

E.g. to find a polynomial expression for

$$\sum_{r=1}^{n} r^2(r+2) \quad \text{or} \quad \sum_{r=1}^{n}\left(r^2 - r + 1\right).$$

In the formula book

Summations

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1)$$

$$\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1)$$

$$\sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2$$

The sigma sign

Consider the sequence 1 , 4 , 9 , 16 , 25 , 36

The n^{th} term of the sequence is n^2

We want to work out the sum S = 1+4+9+16+25+36

S is called a SERIES (sum of the terms of a sequence)

Notation: $S = 1+4+9+16+25+36 = \sum_{n=1}^{6} n^2$

N.B: The variable used inside the sigma sign could be any letter

Examples: $\sum_{n=2}^{4} 2n^3 =$

$$\sum_{k=1}^{3} 3k+1 =$$

Manipulating the sigma sign

Consider two sequences with n^{th} term f(r) and g(r) and two numbers *a* and *b.*

$$\sum_{r=1}^{n}\left(f(r)+g(r)\right)=\sum_{r=1}^{n}f(r)+\sum_{r=1}^{n}g(r)$$

and

$$\sum_{r=1}^{n}a\times f(r)=a\times\sum_{r=1}^{n}f(r)$$

Sum between *a* and *b*

$$\sum_{r=a}^{b}f(r)=\sum_{r=1}^{b}f(r)-\sum_{r=1}^{a-1}f(r)$$

Examples:

$$\sum_{r=1}^{5}4r+1=$$

$$\sum_{r=10}^{20}2r=$$

$$\sum_{r=1}^{100}2r^2-4r^3=$$

Sum from 1 to n

The purpose of this chapter is to determine/learn
an expression of series, in terms of n, when summing from 1 to n

- $\sum_{k=1}^{n} 1 =$

- Sum of the first "*n*" integers

$$\sum_{r=1}^{n} r =$$

- Sum of the first "*n*" squared numbers

$$\sum_{r=1}^{n} r^2 =$$

- Sum of the first "*n*" cubed numbers

$$\sum_{r=1}^{n} r^3 =$$

In the formula book

Proof:

Sum between two values

$$a)\ \sum_{r=1}^{20} 3r+4=$$

$$b)\ \sum_{r=10}^{50} r^2+2r=$$

EXERCISE

1 Find the value of each of the following:

(a) $\sum_{r=1}^{25} 2$, **(b)** $\sum_{r=1}^{20} r^2$, **(c)** $\sum_{r=1}^{40} r^3$,

(d) $\sum_{r=51}^{100} r^2$, **(e)** $\sum_{r=101}^{130} r^3$, **(f)** $\sum_{r=101}^{140} (r^3-500)$.

2 Evaluate:

(a) $\sum_{k=1}^{10} 6k^2$, **(b)** $\sum_{k=1}^{60} 4k^3$, **(c)** $\sum_{k=16}^{30} 3k^2$,

(d) $\sum_{k=15}^{50} (k^3-100)$, **(e)** $\sum_{k=11}^{30} (k^2+3k)$, **(f)** $\sum_{k=1}^{20} (3k^3-6k^2+7)$.

1 **(a)** 50; **(b)** 2870; **(c)** 672 400;
(d) 295 425; **(e)** 47 002 725; **(f)** 71 894 400.

2 **(a)** 2310; **(b)** 13 395 600; **(c)** 24 645;
(d) 1 611 000; **(e)** 10 300; **(f)** 115 220.

Sum between 1 and n

Find $\sum_{k=1}^{n}(4k^3-12k)$ and factorise your answer.

$$\sum_{k=1}^{n}4k^3-12k=4\sum_{k=1}^{n}k^3-12\sum_{k=1}^{n}k=4\times\frac{1}{4}n^2(n+1)^2-12\times\frac{1}{2}n(n+1)$$

$$=n^2(n+1)^2-6n(n+1)$$

Commons factors are n and $(n+1)$

$$\text{so }\sum_{k=1}^{n}4k^3-12k==n(n+1)\left[n(n+1)-6\right]=n(n+1)(n^2+n-6)=n(n+1)(n+3)(n-2)$$

EXERCISE

1 Prove that $\sum_{r=1}^{n} (6r^2 + 24r) = n(n+1)(2n+13)$.

2 Prove that $\sum_{r=1}^{n} (4r^3 + 6r) = n(n+1)(n^2+n+3)$.

3 Prove that $\sum_{r=1}^{n} 12r^2(r+1) = n(n+1)(3n^2+7n+2)$.

4 Prove that $\sum_{r=1}^{n} (8r^3 + 6r - 3) = n^2(2n^2+4n+5)$.

5 Prove that $\sum_{r=1}^{n} (2r^3 + 6r - 3) = \frac{1}{2}n^2(n^2+2n+7)$.

6 Obtain an expression in terms of n for $\sum_{k=1}^{n} 2k(10k^2+1)$, giving your answer as a product of factors.

7 Find the sums of each of the following, giving your answers in factorised form:

(a) $\sum_{r=1}^{n} (4r^3 + 2r)$, **(b)** $\sum_{r=1}^{n} (6r^2 + 4r)$,

(c) $\sum_{r=1}^{n} (8r^3 - 2r)$, **(d)** $\sum_{r=1}^{n} (8r^3 - 6r^2)$,

(e) $\sum_{r=1}^{n} (12r^3 + 8r)$, **(f)** $\sum_{r=1}^{n} (6r^2 - 6)$.

6 $n(n+1)(5n^2+5n+1)$.

7 **(a)** $n(n+1)(n^2+n+1)$; **(b)** $n(n+1)(2n+3)$;
(c) $n(n+1)(2n^2+2n-1)$; **(d)** $n(n+1)(2n^2-1)$;
(e) $n(n+1)(3n^2+3n+4)$; **(f)** $3n(n-5)$.