Roots of quadratic equations

Objective: To establish the relationship between
the coefficients of a quadratic expression
and the sum and product of its roots

AQA specifications

Roots and coefficients of a quadratic equation

Manipulating expressions involving $\alpha+\beta$ and $\alpha\beta$.

E.g. $\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$

Forming an equation with roots α^3, β^3 or $\frac{1}{\alpha}, \frac{1}{\beta}$, $\alpha+\frac{2}{\beta}, \beta+\frac{2}{\alpha}$ etc.

Notations:

A quadratic equation will be written $ax^2+bx+c=0$
The coefficient are a, b, c
We note the solutions (the roots) of this equation α and β
The SUM of the roots is $\alpha+\beta$
The PRODUCT of the root is $\alpha\times\beta = \alpha\beta$

The coefficients of the equation and the sum and product of the roots.

Exercise:

Find the solutions α and β of the following equations then work out the SUM $\alpha+\beta$ and the PRODUCT $\alpha\beta$.

Equation	α	β	$\alpha+\beta$	$\alpha\beta$
$x^2 + 5x + 6 = 0$				
$x^2 - x - 12 = 0$				
$x^2 + 3x - 4 = 0$				
$2x^2 - 2x - 4 = 0$				
$3x^2 - 6x - 24 = 0$				

Can you now complete this table without working out α nor β?

Equation	$\alpha+\beta$	$\alpha\beta$
$x^2 + 2x + 6 = 0$		
$x^2 - x - 10 = 0$		
$2x^2 + 8x - 40 = 0$		
$3x^2 - 7x - 8 = 0$		
$ax^2 + bx + c = 0$		

Conclusion:

When the quadratic equation $ax^2 + bx + c = 0$ has roots α and β:

- The SUM of the roots $\alpha+\beta = -\frac{b}{a}$
- The PRODUCT of the roots $\alpha\beta = \frac{c}{a}$

EXERCISE

1 Find the sum and product of the roots for each of the following quadratic equations:

(a) $x^2 + 4x - 9 = 0$

(b) $2x^2 - 3x - 5 = 0$

(c) $2x^2 + 10x - 3 = 0$

(d) $1 + 2x - 3x^2 = 0$

(e) $7x^2 + 12x = 6$

(f) $x(x - 2) = x + 6$

(g) $x(3 - x) = 5x - 2$

(h) $ax^2 - a^2x - 2a^3 = 0$

(i) $ax^2 + 8a = (1 - 2a)x$

(j) $\dfrac{4}{x+5} = \dfrac{x-3}{2}$

	Sum	Product
a)	-4	-9
b)	$\frac{3}{2}$	$-\frac{5}{2}$
c)	-5	$-\frac{3}{2}$
d)	$\frac{2}{3}$	$-\frac{1}{3}$
e)	$-\frac{12}{7}$	$-\frac{6}{7}$
f)	3	-6
g)	-2	-2
h)	a	$-2a^2$
i)	$\frac{1-2a}{a}$	8
j)	-2	-23

Now, the other way round...

Write down a quadratic equation with:

(a) sum of roots $= 5$, product of roots $= 8$,

(b) sum of roots $= -3$, product of roots $= 5$,

(c) sum of roots $= 4$, product of roots $= -7$,

a) $x^2-5x+8 = 0$

b) $x^2+3x+5 = 0$

c) $x^2-4x-7 = 0$

Generalisation:

If α and β are the roots of a quadratic equation,
then they are solution of the equation

$$x^2 - (\text{Sum of roots})x + (\text{Product of the roots}) = 0$$

that is $$x^2 - (\alpha+\beta)x + \alpha\beta = 0$$

Important : do not forget "=0" and There are an infinite number of equivalent equations

Manipulating expressions involving α and β

Exercise:

Considering that $\alpha+\beta = 3$ and $\alpha\beta = 5$,

Work out

a) $(\alpha+3)+(\beta+3)$ and $(\alpha+3)(\beta+3)$

b) $\frac{1}{\alpha}+\frac{1}{\beta}$ *and* $\frac{1}{\alpha}\times\frac{1}{\beta}$ → *Write an equation with roots* $\frac{1}{\alpha}$ *and* $\frac{1}{\beta}$

c) $\frac{1}{\alpha^2\beta}+\frac{1}{\beta^2\alpha}$ *and* $\frac{1}{\alpha^2\beta}\times\frac{1}{\beta^2\alpha}$ → *Write an equation with roots* $\frac{1}{\alpha^2\beta}$ *and* $\frac{1}{\beta^2\alpha}$

Expressions involving squares and cubes

Expand $(\alpha+\beta)^2$

Hence work out and expression of $\alpha^2+\beta^2$ in terms of $\alpha+\beta$ and $\alpha\beta$

Expand $(\alpha+\beta)^3$

Hence work out and expression of $\alpha^3+\beta^3$ in terms of $\alpha+\beta$ and $\alpha\beta$

Remember

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

$$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(a + \beta)$$

EXERCISE

1 Write each of the following expressions in terms of $\alpha + \beta$ and $\alpha\beta$:

(a) $\dfrac{2}{\alpha} + \dfrac{2}{\beta}$ **(b)** $\dfrac{1}{\alpha^2\beta} + \dfrac{1}{\beta^2\alpha}$

(c) $\dfrac{\alpha}{3\beta} + \dfrac{\beta}{3\alpha}$ **(d)** $\alpha^2\beta + \beta^2\alpha$

(e) $(2\alpha - 1)(2\beta - 1)$ **(f)** $\dfrac{\alpha + 5}{\beta} + \dfrac{\beta + 5}{\alpha}$

2 Given that $\alpha + \beta = -3$ and $\alpha\beta = 9$, find the values of:

(a) $\alpha^3\beta + \beta^3\alpha$, **(b)** $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$.

3 Given that $\alpha + \beta = 4$ and $\alpha\beta = 10$, find the values of:

(a) $\alpha^2 + \beta^2$, **(b)** $\alpha^3 + \beta^3$.

4 Given that $\alpha + \beta = 7$ and $\alpha\beta = -2$, find the values of:

(a) $\dfrac{1}{\beta^2} + \dfrac{1}{\alpha^2}$, **(b)** $\dfrac{\alpha^2}{\beta} + \dfrac{\beta^2}{\alpha}$.

5 The roots of the quadratic equation $x^2 - 5x + 3 = 0$ are α and β.

(a) Write down the values of $\alpha + \beta$ and $\alpha\beta$.

(b) Hence find the values of:

(i) $(\alpha - 3)(\beta - 3)$, **(ii)** $\dfrac{\alpha}{\beta^2} + \dfrac{\beta}{\alpha^2}$.

6 The roots of the equation $x^2 - 4x + 3 = 0$ are α and β. Without solving the equation, find the value of:

(a) $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$ **(b)** $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$

(c) $\alpha^2 + \beta^2$ **(d)** $\alpha^2\beta + \alpha\beta^2$

(e) $(\alpha - \beta)^2$ **(f)** $(\alpha + 1)(\beta + 1)$

7 The roots of the quadratic equation $x^2 + 4x + 1 = 0$ are α and β.

(a) Find the values of: **(i)** $\alpha + \beta$, **(ii)** $\alpha\beta$.

(b) Hence find the value of:

(i) $(\alpha^2 - \beta)(\beta^2 - \alpha)$, **(ii)** $\dfrac{\alpha + 2}{\beta + 2} + \dfrac{\beta + 2}{\alpha + 2}$.

1 (a) $\dfrac{2(\alpha + \beta)}{\alpha\beta}$; (b) $\dfrac{\alpha + \beta}{(\alpha\beta)^2}$;
(c) $\dfrac{(\alpha + \beta)^2 - 2\alpha\beta}{3\alpha\beta}$; (d) $\alpha\beta(\alpha + \beta)$;
(e) $4\alpha\beta - 2(\alpha + \beta) + 1$; (f) $\dfrac{(\alpha + \beta)^2 - 2\alpha\beta + 5(\alpha + \beta)}{\alpha\beta}$.
2 (a) -81; (b) -1.
3 (a) -4; (b) -56.
4 (a) $\frac{53}{4}$; (b) $-192\frac{1}{2}$.
5 (a) $\alpha + \beta = 5$, $\alpha\beta = 3$; (b) (i) -3 (ii) $8\frac{8}{9}$.
6 (a) $\frac{4}{3}$; (b) $\frac{10}{3}$; (c) 10; (d) 12; (e) 4; (f) 8.
7 (a) (i) -4, (ii) 1; (b) (i) 54, (ii) -2.

Forming NEW equations with related roots

Let's look at an exam question:

1 The quadratic equation $x^2 - 6x + 18 = 0$ has roots α and β.

(a) Write down the values of $\alpha + \beta$ and $\alpha\beta$. *(2 marks)*

(b) Find a quadratic equation, with integer coefficients, which has roots α^2 and β^2. *(4 marks)*

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Your go...

The roots of the quadratic equation $2x^2 + 5x - 1 = 0$ are α and β

a) Write down i) $\alpha+\beta$ ii) $\alpha\beta$

b) Find a quadratic equation with integer coefficients which has roots $\alpha - 2$ *and* $\beta - 2$.

The roots of the quadratic equation $x^2 - 3x - 7 = 0$ are α and β.

a) Write down the values of

i) $\alpha+\beta$

ii) $\alpha\beta$

b) Find a quadratic eqution with integer coefficients whose roots are $\frac{\alpha}{\beta}$ *and* $\frac{\beta}{\alpha}$

EXERCISE

1 The roots of the equation $x^2 + 6x - 4 = 0$ are α and β. Find a quadratic equation whose roots are α^2 and β^2.

Drag answer here

2 The roots of the equation $x^2 + 3x - 5 = 0$ are α and β. Find a quadratic equation whose roots are $\frac{2}{\alpha}$ and $\frac{2}{\beta}$.

Drag answer here

3 The roots of the equation $x^2 - 9x + 5 = 0$ are α and β. Find a quadratic equation whose roots are $\alpha + 2$ and $\beta + 2$.

Drag answer here

4 Given that the roots of the equation $2x^2 + 5x - 3 = 0$ are α and β, find an equation with integer coefficients whose roots are $\alpha\beta^2$ and $\alpha^2\beta$.

Drag answer here

5 Given that the roots of the equation $3x^2 - 6x + 1 = 0$ are α and β, find a quadratic equation with integer coefficients whose roots are α^3 and β^3.

Drag answer here

6 The roots of the equation $2x^2 + 4x - 1 = 0$ are α and β. Find a quadratic equation with integer coefficients whose roots are $\frac{\beta}{\alpha}$ and $\frac{\alpha}{\beta}$.

Drag answer here

7 The roots of the equation $3x^2 - 6x - 2 = 0$ are α and β.

(a) Find the value of $\alpha^2 + \beta^2$.

(b) Find a quadratic equation whose roots are $\alpha^2 + 1$ and $\beta^2 + 1$.

Drag answer here

8 The roots of the equation $2x^2 + 7x + 3 = 0$ are α and β. Without solving this equation,

(a) find the value of $\alpha^3 + \beta^3$.

(b) Hence, find a quadratic equation with integer coefficients which has roots $\frac{\alpha}{\beta^2}$ and $\frac{\beta}{\alpha^2}$.

Drag answer here

9 Given that α and β are the roots of the equation $5x^2 - 2x + 4 = 0$, find a quadratic equation with integer coefficients which has roots $\frac{1}{\alpha^2}$ and $\frac{1}{\beta^2}$.

Drag answer here

10 The roots of the equation $x^2 + 4x - 6 = 0$ are α and β. Find an equation whose roots are $\alpha^2 + \beta$ and $\beta^2 + \alpha$.

Drag answer here

$x^2 + 10x + 1 = 0$

$16x^2 + 36x + 25 = 0$

$x^2 - 13x + 27 = 0$

$18x^2 + 217x + 12 = 0$

$5x^2 - 6x - 4 = 0$

$x^2 - 24x - 106 = 0$

$27x^2 - 162x + 1 = 0$

$8x^2 - 30x - 27 = 0$

$x^2 - 44x + 16 = 0$

$9x^2 - 66x + 61 = 0$

Extension

5 The roots of the quadratic equation $x^2 - 2x - 5 = 0$ are α and β. Find a quadratic equation which has roots $\alpha^2 + 1$ and $\beta^2 + 1$.

6 The roots of the quadratic equation $x^2 + 5x - 7 = 0$ are α and β.

(a) Without solving the equation, find the values of

(i) $\alpha^2 + \beta^2$, **(ii)** $\alpha^3 + \beta^3$.

(b) Determine an equation with integer coefficients which has roots $\dfrac{\alpha^2}{\beta}$ and $\dfrac{\beta^2}{\alpha}$. [A]

7 The roots of the quadratic equation $3x^2 + 4x - 1 = 0$ are α and β.

(a) Without solving the equation, find the values of

(i) $\alpha^2 + \beta^2$, **(ii)** $\alpha^3\beta + \beta^3\alpha$.

(b) Determine a quadratic equation with integer coefficients which has roots $\alpha^3\beta$ and $\beta^3\alpha$. [A]

8 The roots of the quadratic equation $x^2 + 2x + 3 = 0$ are α and β.

(a) Without solving the equation:

(i) write down the value of $\alpha + \beta$ and the value of $\alpha\beta$,

(ii) show that $\alpha^3 + \beta^3 = 10$,

(iii) find the value of $\dfrac{1}{\alpha^3} + \dfrac{1}{\beta^3}$.

(b) Determine a quadratic equation with integer coefficients which has roots $\dfrac{1}{\alpha^3}$ and $\dfrac{1}{\beta^3}$. [A]

5 $x^2 - 16x + 40 = 0$.

6 (a) (i) 39, (ii) -230; (b) $7x^2 - 230x - 49 = 0$.

7 (a) (i) $\frac{22}{9}$, (ii) $-\frac{22}{27}$; (b) $81x^2 + 66x + 1 = 0$.

8 (a) (i) $\alpha + \beta = -2$, $\alpha\beta = 3$, (iii) $\frac{10}{27}$; (b) $27x^2 - 10x + 1 = 0$.

Key point summary

1 When the quadratic equation $ax^2 + bx + c = 0$ has roots α and β:

- The sum of the roots, $\alpha + \beta = -\frac{b}{a}$;
- and the product of roots, $\alpha\beta = \frac{c}{a}$.

2 A quadratic equation can be expressed as $x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

3 Two useful results are:

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

$$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta).$$

4 The basic method for forming new equations with roots that are related to the roots of a given equation is:

1. Write down the sum of the roots, $\alpha + \beta$, and the product of the roots, $\alpha\beta$, of the given equation.
2. Find the sum and product of the new roots in terms of $\alpha + \beta$ and $\alpha\beta$.
3. Write down the new equation using

$$x^2 - (\text{sum of new roots})x + (\text{product of new roots}) = 0$$