Solving equations using Numerical methods

Specifications

Numerical Methods

- Finding roots of equations by interval bisection, linear interpolation and the Newton-Raphson method.

Graphical illustration of these methods.

- Solving differential equations of the form $\frac{dy}{dx} = f(x)$

Using a step-by-step method based on the linear approximations $y_{n+1} \approx y_n + h f(x_n)$; $x_{n+1} = x_n + h$, with given values for x_0, y_0 and h.

In the formulae booklet:

Numerical solution of equations

The Newton-Raphson iteration for solving $f(x) = 0$: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

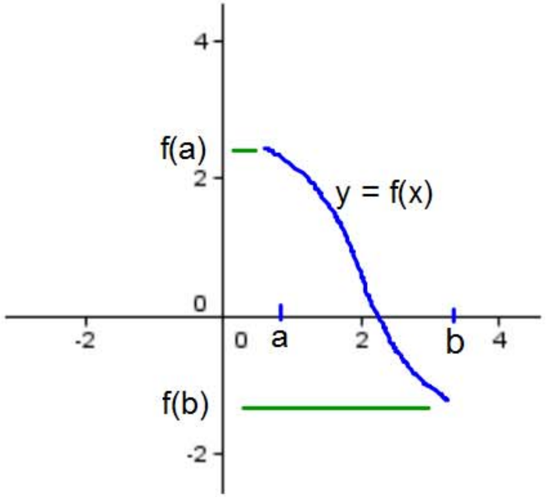
Numerical solution of differential equations

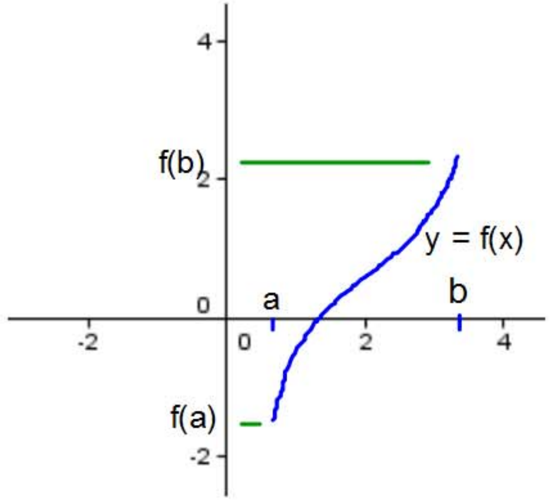
For $\frac{dy}{dx} = f(x)$ and small h, recurrence relations are:

Euler's method: $y_{n+1} = y_n + h f(x_n)$; $x_{n+1} = x_n + h$

A - Change of sign rule

Consider a function f and its graph.
We want to solve f(x) = 0.





> If the graph of $y = f(x)$ is continuous over the interval $a \leq x \leq b$, and $f(a)$ and $f(b)$ have different signs, then a root of the equation $f(x) = 0$ must lie in the interval $a < x < b$.

Application:

Show that the equation $x^3 + x - 120 = 0$ has a solution between 4 and 5.

Exercises:

1 Verify that the equation $x^3 + 7x - 37 = 0$ has a root between 2.6 and 2.7.

2 Show that the equation $x^3 - 2x + 7 = 0$ has a root between -2.3 and -2.2.

3 Prove that the equation $x^5 + 4x = 9$ has a root between 1.30 and 1.31.

4 Verify that the equation $2^x + x = 30$ has a root with value 4.66 to three significant figures.

5 Prove that one root of the equation $x^4 - 6x + 4 = 0$ lies between 0 and 1 and find two consecutive integers between which the other real root lies.

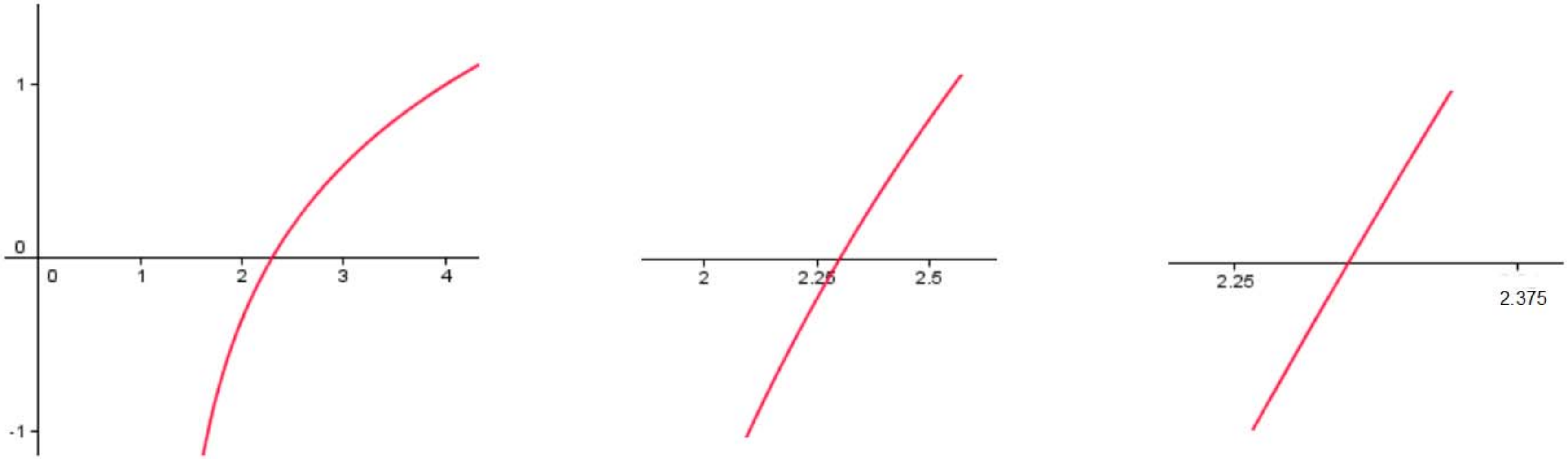
6 Find two consecutive integers between which the real root of the equation $x^3 + 5x + 8 = 0$ lies.

5) $1 < \alpha < 2$

6) $-2 < \alpha < -1$

B - The bisection method

A systematic way to make use of the change of sign technique is the bisection method.
By halving by width of the interval in which the root lies,
we increase the accuracy of this root.



When a root of $f(x) = 0$ is known to lie between $x = a$ and $x = b$, the bisection method requires you to next find the value of $f\left(\frac{a+b}{2}\right)$.

There must then be a change of sign which allows you to bisect the interval in which the root lies.
The procedure is repeated until you have an interval of the desired width containing the root.

Application:

a) Show that the equation $x^3 - 3x + 7 = 0$ has a root between -2.2 and -2.6

b) Use the bisection method to find an interval of width 0.1 in which the root lies

Exercises:

2 **(a)** Show that the equation $x^3 + x - 5 = 0$ has a root between 1.4 and 1.8.

(b) Use the bisection method to find an interval of width 0.1 in which the root lies.

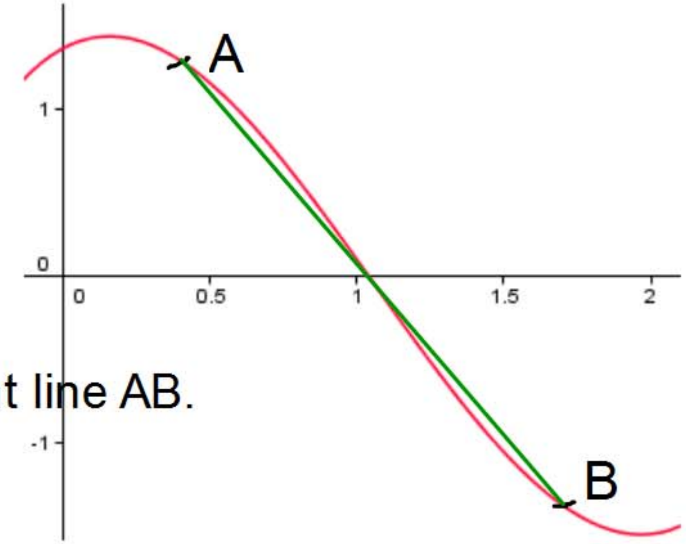
3 **(a)** Show that the equation $x^5 + 2x - 18 = 0$ has a root between 1.6 and 1.8.

(b) Use the bisection method to find an interval of width 0.05 in which the root lies.

4 **(a)** Show that the equation $x^5 + 7x^3 - 2 = 0$ has a root between 0.6 and 0.7.

(b) Use the bisection method to find an interval of width 0.0125 in which the root lies.

C - Linear interpolation



Consider a function f and its graph.
We know that a root of f lies between *a* and *b*.

To find an approximation to the root,
we "approximate" the graph between A and B to the straight line AB.

Let's work out the value of this approximate:

$A(a, f(a))$ *and* $B(b, f(b))$

The gradient of the line AB is $m_{AB} = \dfrac{f(b) - f(a)}{b - a}$

The equation of the line AB is $y - f(a) = m_{AB}(x - a)$

The intersection of this line with the x-axis is given for $y = 0$:

$$-f(a) = m_{AB}(x - a) \quad so \quad x = \frac{-f(a)}{m_{AB}} + a$$

Replacing m_{AB} by its expression, we have $x = \dfrac{-f(a)(b-a)}{f(b) - f(a)} + a$

$$x = \frac{-bf(a) + af(a) + af(b) - af(a)}{f(b) - f(a)} = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

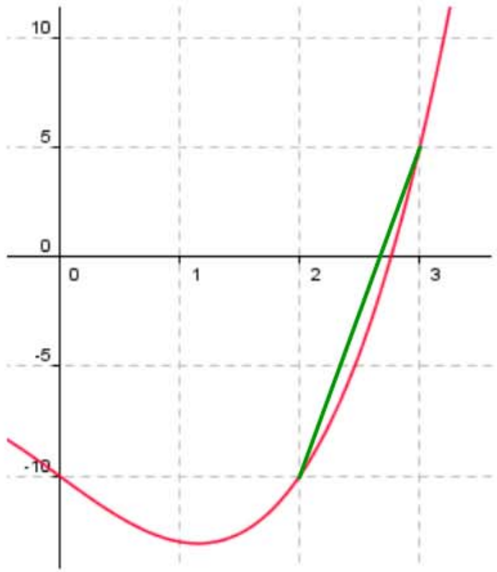
When a root of the equation $f(x) = 0$ is known to lie between $x = a$ and $x = b$, linear interpolation involves replacing the curve by a straight line and gives an approximation to the root as

$$\frac{af(b) - bf(a)}{f(b) - f(a)}.$$

Although you can learn this formula off by heart it is often easier to derive it from first principles using the equation of a straight line or even similar triangles.

Application:

Show that thew equation $x^3 - 4x - 10 = 0$ has a root between 2 and 3. Use linear interpolation to obtain an approximation to this root.



Answer:

$A(2,-10)$ *and* $B(3,5)$

An approximation of the root is $\alpha = \dfrac{af(b) - bf(a)}{f(b) - f(a)}$

$$\alpha = \frac{2\times 5 - 3\times -10}{5 - -10} = \frac{40}{15} = 2.67$$

Exercises:

1 (a) Show that the equation $x^3 + 2x - 1 = 0$ has a root between 0 and 1.

(b) Find an approximation to the root by using linear interpolation between 0 and 1.

2 Show that the equation $x^5 + 4x - 50 = 0$ has a root between 2 and 3.
Use linear interpolation to find an approximation to this root.

3 Show that the equation $2^x - 2x - 5 = 0$ has a root between 3 and 4.
Use linear interpolation to find an approximation to this root.

4 Show that the equation $x^3 + 4x^2 + 1 = 0$ has a root between -4.1 and -3.9.
Use linear interpolation to find an approximation to this root.

5 Show that the equation $\sin x + 3x - 5 = 0$ has a root between 1 and 2.
Use linear interpolation to find an approximation to this root.

6 Show that the equation $\cos x - 2x + 3 = 0$ has a root between 1 and 2.
Use linear interpolation to find an approximation to this root.

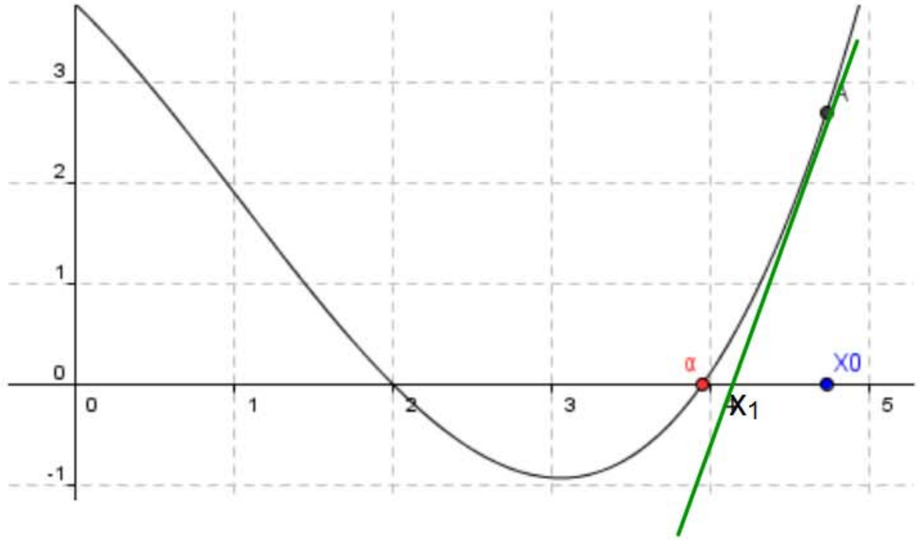
Answers :

$1)b)\frac{1}{3}$ $\quad 2)\frac{440}{215} \approx 2.05$ $\quad 3)3.5$

$4)-4.07$ $\quad 5)1.38$ $\quad 6)1.52$

D - The Newton-Raphson iterative method

x_0 is an approximation to the root α.



To find a better approximation:
Consider $A(x_0, f(x_0))$ and the tangent to the curve at A.
This tangent crosses the x-axis at x_1.
x_1 is a better approximation to the root.

Repeat the process to get even better approximation.

Algebraically:

$A(x_0, f(x_0))$

The equation of the tangent at A is: $y - f(x_0) = f'(x_0)(x - x_0)$

This tangent intersects the x-axis when $y = 0$, this gives

$-f(x_0) = f'(x_0)(x - x_0)$ *and* making x the subject: $x = -\dfrac{f(x_0)}{f'(x_0)} + x_0$

This allowsw us to build a sequence of approximations

the first term is x_0 *and* $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

The Newton–Raphson iterative formula for solving $f(x) = 0$

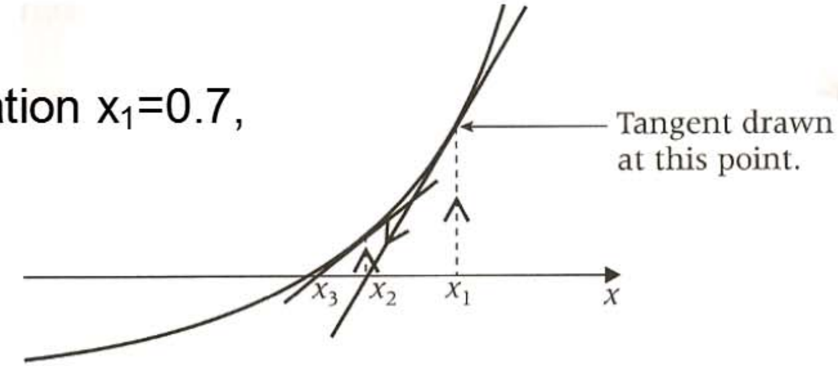
is $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$.

This formula is given in the booklet for use in the examination under the heading *Numerical Solution of Equations.*

Numerical example and calculator use

$$x^3 + x - 1 = 0$$

Use the Newton-Raphson method, with first approximation $x_1=0.7$, to find further approximations, x_2, x_3 and x_4.
Give your answers correct to 4 decimal places.



$$f(x) = x^3 + x - 1 \qquad f'(x) = 3x^2 + 1$$

$$x_1 = 0.7$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.7 - \frac{0.7^3 + 0.7 - 1}{3 \times 0.7^2 + 1} = 0.68259109$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.68232786$$

$x_4 = 0.6823$ correct to 4 dec. places

Calculator:

Type the first approx then "=" (the value is stored in the memory "Ans")
Type: "Ans"-(f("Ans"))/(f '("Ans")) then
press "=" for x_1
pres "=" again for x_2, etc.

Exercises

1 Use the Newton–Raphson method with the value of x_1 given in order to find the next approximation, x_2, to a root of the given equations, giving your answers to three significant figures:

(a) $x^3 - 5x + 11 = 0, \quad x_1 = -3,$

(b) $x^4 - 5x^3 + 6 = 0, \quad x_1 = 1,$

(c) $x^5 - 2x^3 - 17 = 0, \quad x_1 = 2.$

2 Two students are attempting to use Newton–Raphson's method to solve the equation $x^3 - 3x + 4 = 0$. Student A decides to use $x_1 = -1$ and student B uses $x_1 = -3$ as a first approximation. Explain why one of the students will be successful in finding the value of the root and the other will not.

3 The equation $4x^3 - 5x^2 + 2 = 0$ has a single root α. Use the Newton–Raphson iterative method with a first approximation $x_1 = -0.5$ to find the next two successive approximations for α, giving your answers to five decimal places.

4 **(a)** Prove that the equation $x^3 - 5x + 7 = 0$ has a root α between -3 and -2.

(b) The Newton–Raphson method is to be used to find an approximation for α. Use $x_1 = -3$ as a first approximation to find the value of x_2 giving your answer correct to three decimal places. [A]

5 A curve has equation $y = x^2 + 5 + \dfrac{2}{x}$ and it crosses the x-axis at the single point $(\alpha, 0)$.

(a) Show that α lies between -0.4 and -0.3.

(b) Use linear interpolation once to find a further approximation, giving your answer to two decimal places.

(c) Use Newton–Raphson's iterative method with first approximation $x_1 = -0.4$ to find the values of x_2 and x_3, giving your answers to five decimal places.

Answers

1 **(a)** -2.95; **(b)** 1.18; **(c)** 2.02.

2 Student A will not be successful since $f'(-1) = 0$. Student B obtains a next approximation of $-2.417\ldots$ and after several iterations -2.196 (to 3 d.p.).

3 $x_2 = -0.531\,25$, $x_3 = -0.530\,00$.

4 **(b)** $x_2 = -2.773$.

5 **(b)** -0.39; **(c)** $-0.380\,72$.

E - Euler's step-by-step method

Introduction:

A curve with equation $y = f(x)$ is such that

$$\frac{dy}{dx} = 3x^2 + 2x \text{ and goes through the point } (1, 2)$$

Work out the equation of this curve.

Answer:

$$\frac{dy}{dx} = 3x^2 + 2x \quad so \quad y = x^3 + x^2 + c$$

Since $A(1,2)$ belongs to the curve, $2 = 1^3 + 1^2 + c$

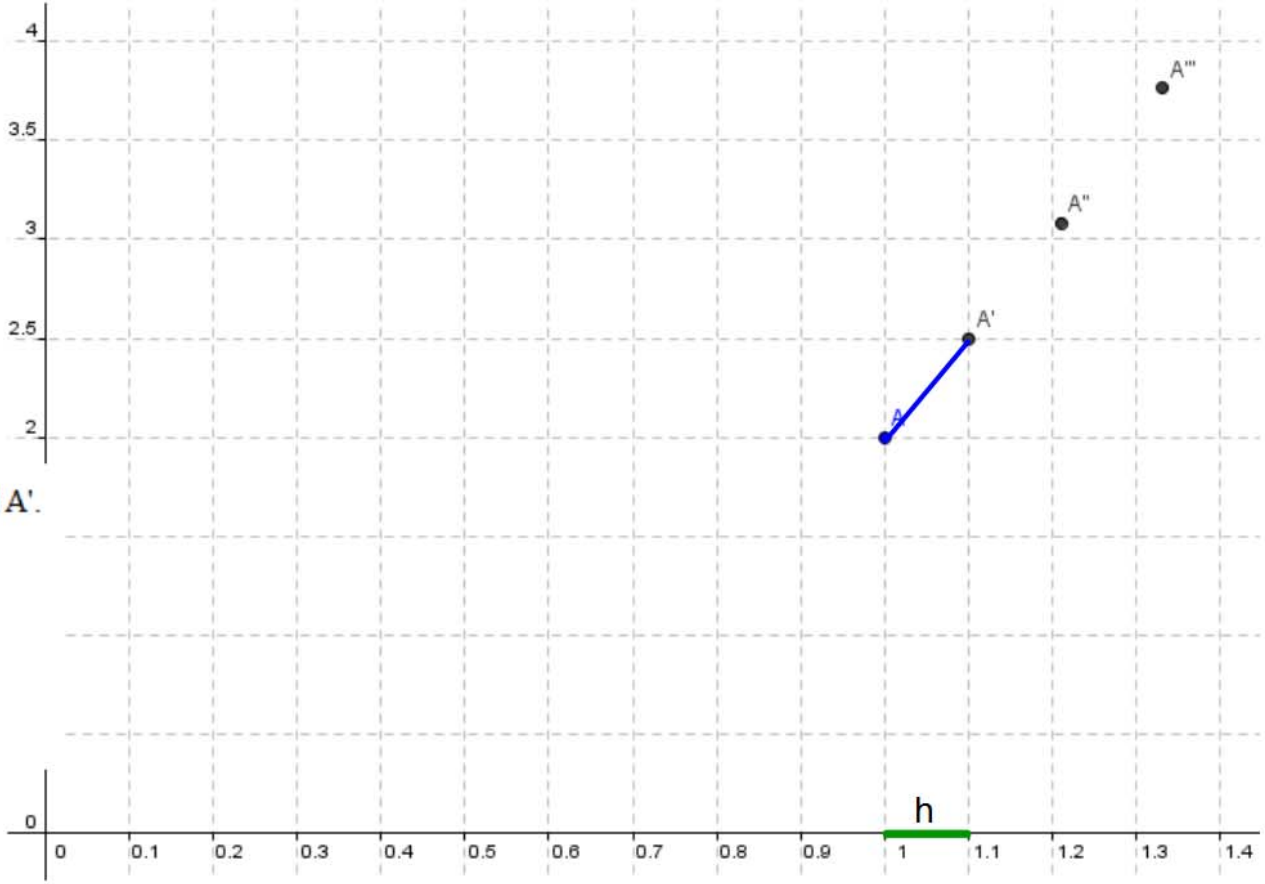
$$c = 0$$

The equation of the curve is $y = x^3 + x^2$

What do you do when you cannot integrate the function?

Euler's method allow you to build the curve point by point in regular intervals.

The principal:



We consider a small interval h, between two point A and A'.
Between these two points,
we approximate the curve by the tangent at A.

Once the coordinates of A' worked out,
we repeated this principle between A' and A''. Etc..

Using Euler method: A(1,2) and the gradient of the curve at A is $\frac{dy}{dx}(x=1) = 3\times1^2 + 2\times1 = 5$

The equation of the tangent at A is : $y - 2 = 5(x-1)$

$$y = 5x - 3$$

By choosing $h = 0.1$

A'$(1+0.1,\ 5(1+0.1)-3)$, A'$(1.1, 2.5)$

Comparing with the actual value:

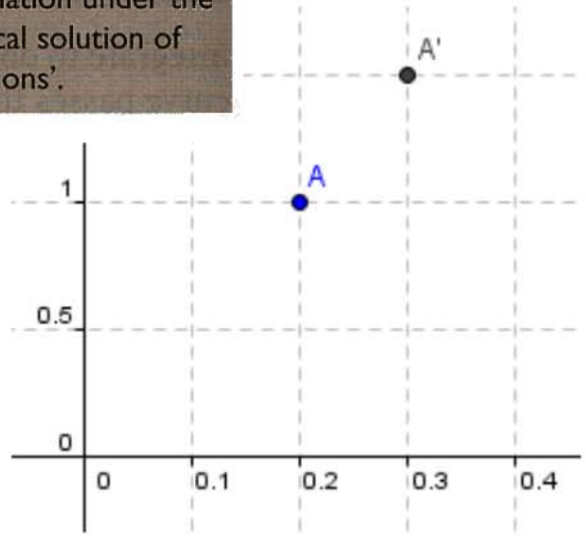
$y = (1.1)^3 + (1.1)^2 = 2.541.$

Not bad!

The formula:

Euler's method is used to find a numerical solution of the differential equation $\frac{dy}{dx} = f(x)$. The formula to find y is given by $y_{n+1} = y_n + hf(x_n)$, where $x_{n+1} = x_n + h$.

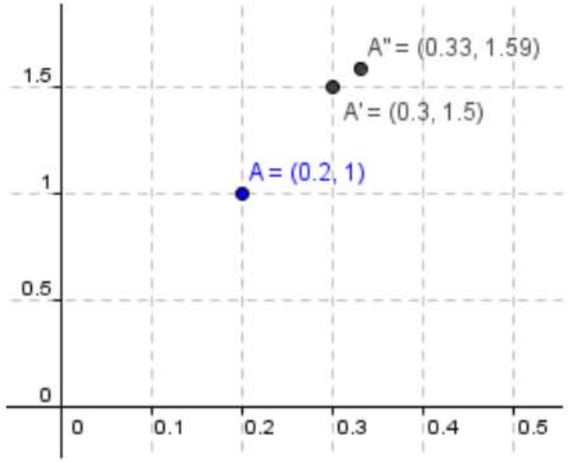
This formula is in the booklet for use in the examination under the heading 'Numerical solution of differential equations'.



Example:

$$\frac{dy}{dx} = 3x^2 + 2x \textit{ and going through } (0.2,1)$$

Using steps of 0.1, estimate the value of y for x = 0.4



Possible layout for answers:

A curve satisfies the differential equation $\frac{dy}{dx} = \frac{4}{x+1}$.

Starting at the point (3,5) on the curve,

use a step-by-step method with a step length 0.2

to estimate the value of y at $x = 3.4$

Solution

From the information given, $h = 0.2$, $x_0 = 3$, $y_0 = 5$ and $f(x) = \frac{4}{x+1}$.

Using $x_{n+1} = x_n + h$, then $x_1 = 3.2$ and $x_2 = 3.4$.

Since $y_{n+1} = y_n + hf(x_n)$, you can find

$$y_1 = y_0 + 0.2 \times f(x_0) = 5 + 0.2 \times \frac{4}{x_0 + 1} = 5 + \frac{0.8}{4} = 5.2$$

$$y_2 = y_1 + 0.2 \times f(x_1) = 5.2 + 0.2 \times \frac{4}{x_1 + 1} = 5.2 + \frac{0.8}{4.2} = 5.39 \text{ (2 d.p.)}$$

Suppose you reduce the step size to 0.1, you could set out the working in a table for convenience.

n	x_n	y_n	$f(x_n)$	h	$hf(x_n)$
0	3	5	1	0.1	0.1
1	3.1	5.1	0.9756	0.1	0.097 56
2	3.2	5.197 56	0.9524	0.1	0.095 24
3	3.3	5.292 80	0.9302	0.1	0.093 02
4	3.4	5.385 82			

The estimate for y when $x = 3.4$, using a step size of 0.1 is 5.386 (3 d.p.).

Exercises:

1 A curve satisfies the differential equation $\frac{dy}{dx} = \frac{1}{x^2 + 3}$.

Starting at the point (1, 2) on the curve, use a step-by-step method with a step length of 0.25 to estimate the value of y at $x = 1.5$, giving your answer to two decimal places.

2 A curve satisfies the differential equation $\frac{dy}{dx} = \sqrt{x^3 + 1}$.

Starting at the point (2, 4) on the curve, use a step-by-step method with a step length of 0.1 to estimate the value of y at $x = 2.3$, giving your answer to three decimal places.

3 A curve satisfies the differential equation $\frac{dy}{dx} = \frac{1}{3 - x}$.

Starting at the point (1, 4) on the curve, use a step-by-step method with a step length of 0.2 to estimate the value of y at $x = 1.6$, giving your answer to three decimal places.

4 A curve satisfies the differential equation $\frac{dy}{dx} = \sqrt{x^4 + 9}$.

Starting at the point (2, −1) on the curve, use a step-by-step method with a step length of 0.1 to estimate the value of y at $x = 2.5$, giving your answer to three decimal places.

5 A curve satisfies the differential equation $\frac{dy}{dx} = \frac{x}{x + 2}$.

Starting at the point (3, 0.7) on the curve, use a step-by-step method with a step length of 0.125 to estimate the value of y at $x = 3.5$, giving your answer to three decimal places.

Answers: 1) 2.12 2) 4.962 3)4.336 4)1.721 5)1.007

Key point summary

1 If the graph of $y = f(x)$ is continuous over the interval $a \leqslant x \leqslant b$, and $f(a)$ and $f(b)$ have different signs, then a root of the equation $f(x) = 0$ must lie in the interval $a < x < b$.

2 When a root of $f(x) = 0$ is known to lie between $x = a$ and $x = b$, the bisection method requires you to next find $f\left(\frac{a+b}{2}\right)$. There must then be a change of sign which allows you to bisect the interval in which the root lies.
The procedure is repeated until you have an interval of the desired width containing the root.

3 When a root of the equation $f(x) = 0$ is known to lie between $x = a$ and $x = b$, linear interpolation involves replacing the curve by a straight line and gives an approximation to the root as

$$\frac{af(b) - bf(a)}{f(b) - f(a)}.$$

4 The Newton–Raphson iterative formula for solving $f(x) = 0$ is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

5 Euler's method is used to find a numerical solution of the differential equation $\frac{dy}{dx} = f(x)$. The formula to find y is given by $y_{n+1} = y_n + hf(x_n)$, where $x_{n+1} = x_n + h$.