The Normal Distribution (Part 1)

Specification

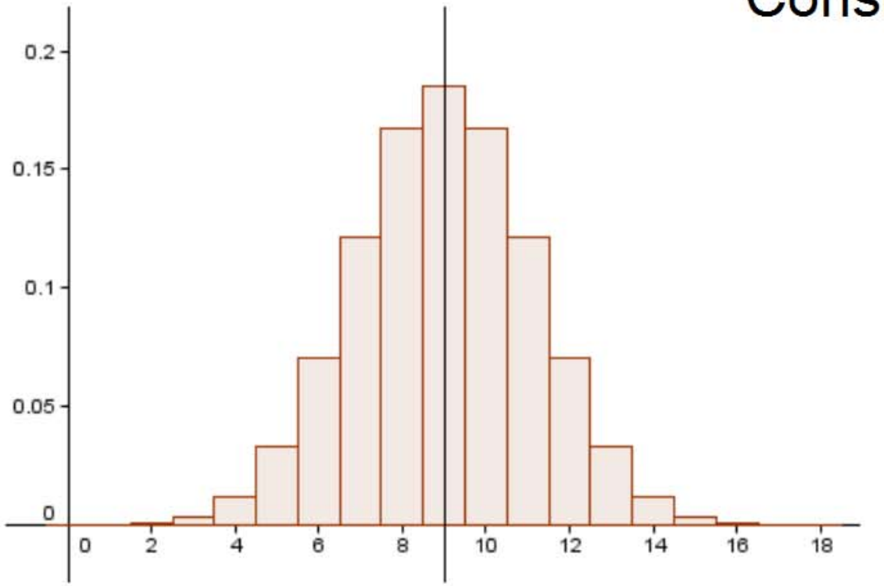
Normal Distribution

Continuous random variables.	Only an understanding of the concepts; not examined beyond normal distributions.
Properties of normal distributions.	Shape, symmetry and area properties. Knowledge that approximately $\frac{2}{3}$ of observations lie within $\mu \pm \sigma$, and equivalent results.
Calculation of probabilities.	Transformation to the standardised normal distribution and use of the supplied tables. Interpolation will not be essential; rounding z −values to two decimal places will be accepted.
Mean, variance and standard deviation of a normal distribution.	To include finding unknown mean and/or standard deviation by making use of the table of percentage points. (Candidates may be required to solve two simultaneous equations.)

In the formula book

Distribution of X	Probability density function	Mean	Variance
Normal $N(\mu, \sigma^2)$	$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$	μ	σ^2

Discrete to continuous
Binomial to Normal distribution



Consider a random variable

$X \sim B(18, 0.5)$

0	0
1	0.0001
2	0.0006
3	0.0031
4	0.0117
5	0.0327
6	0.0708
7	0.1214
8	0.1669
9	0.1855
10	0.1669
11	0.1214
12	0.0708
13	0.0327
14	0.0117
15	0.0031
16	0.0006
17	0.0001
18	0

Consider the probability : $P(X \leq 8)$

we can work it out using the table of values: $P(X \leq 8) =$

This would be working out the area of the bar X=8 and all the bar before it

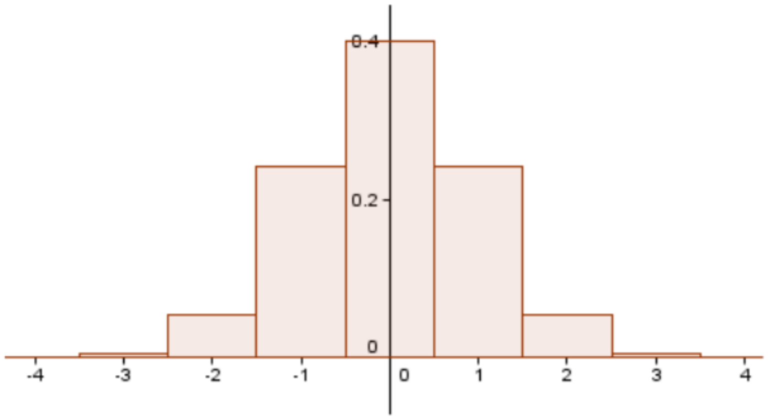
The experiment:

We measure the temperature of items to find out if they are still frozen or not.

(The random variable, Z , is the temperature in °)

Step 1: We group our results interval of width 1°

here is the histogram (or distribution) which represents this grouped frequency table



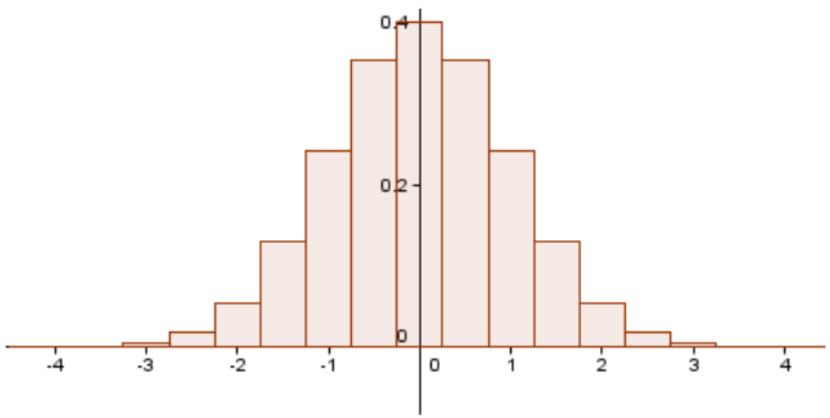
Z	Probability	Prob density
-4.5 , -3.5	0	
-3.5 , -2.5	0	
-2.5 , -1.5	0.05	
-1.5 , -0.5	0.24	
-0.5 , 0.5	0.4	
0.5 , 1.5	0.24	
1.5 , 2.5	0.05	
2.5 , 3.5	0	
3.5 , 4.5	0	

$$\text{Probability density} = \frac{\text{Probability}}{\text{Width of the interval}}$$

This histogram is "chunky", so to gain in details,

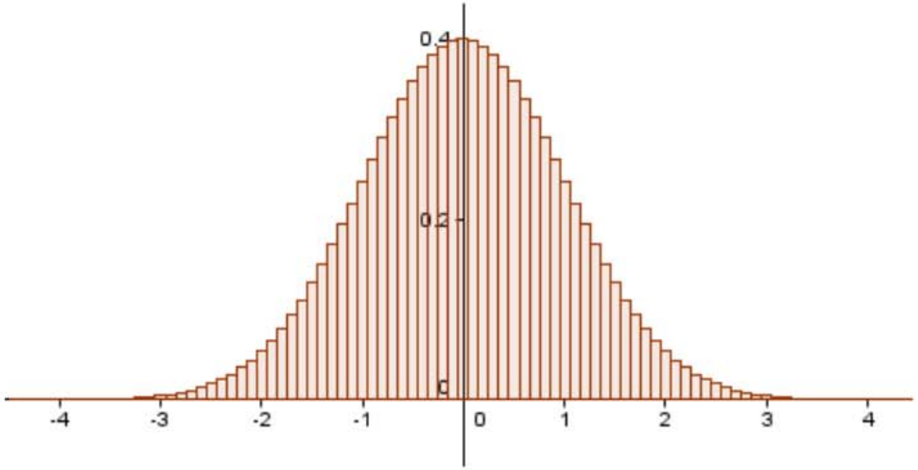
we decide to reduce the width of the interval

Step 2: interval of width 0.5°

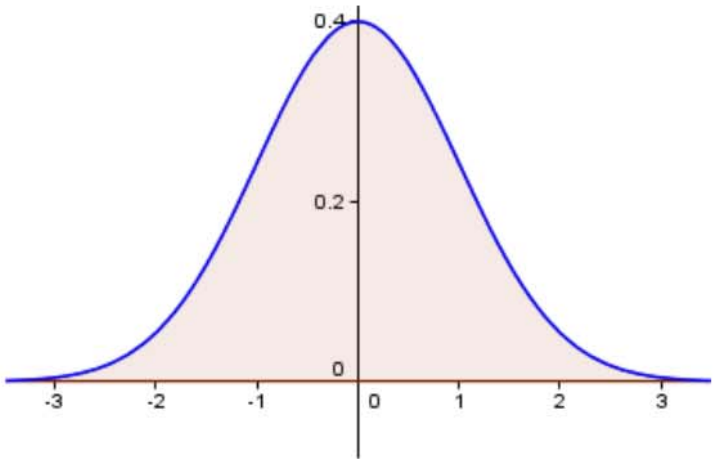


Z	Prob	Prob density
-3.75 , -3.25	0	0
-3.25 , 2.75	0	0
-2.75 , -2.25	0.01	0.02
-2.25 , -1.75	0.025	0.05
-1.75 , -1.25	0.065	0.13
-1.25 , -0.75	0.12	0.24
-0.75 , -0.25	0.175	0.35
-0.25 , 0.25	0.2	0.4
0.25 , 0.75	0.175	0.35
0.75 , 1.25	0.12	0.24
1.25 , 1.75	0.065	0.13
1.75 , 2.25	0.025	0.05
2.25 , 2.75	0.01	0.02
2.75 , 3.25	0	0
3.25 , 3.75	0	0

Step 3: Interval with width 0.1



We can see that eventually, by reducing further the width of the bar, the contour of the graph becomes a smooth curve:

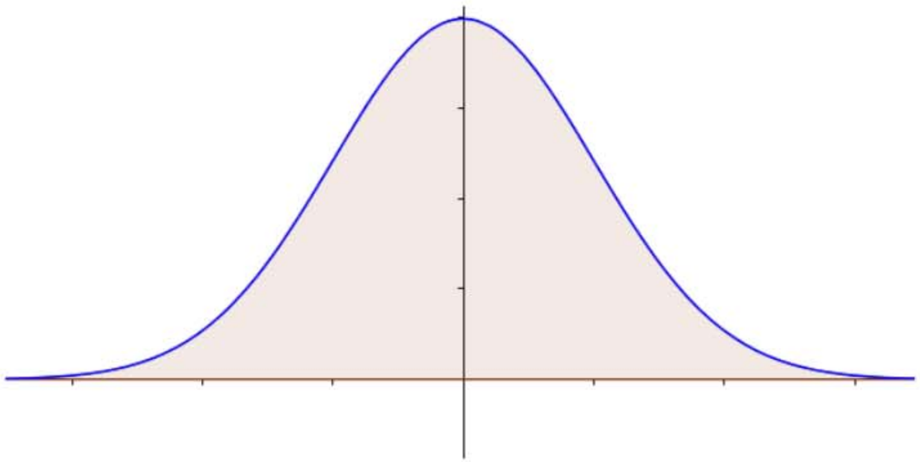


This curve is the graph of what is called the probability density function (p.d.f)

The normal distributions

The main features of normal distribution are that it is:

- bell shaped
- symmetrical (about the mean)
- the total area under the curve is 1 (as with all probability density functions).



This normal distribution is called the STANDARD normal distribution

The mean $\mu = 0$

and

the standard deviance $\sigma = 1$

The equation of the probability density function (p.d.f.) is

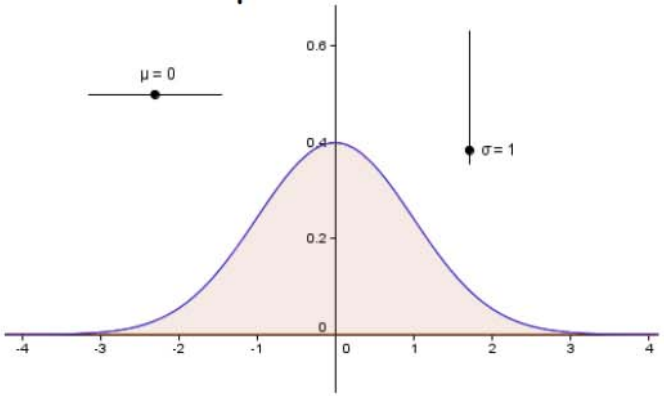
$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \quad \text{for } -\infty < z < \infty.$$

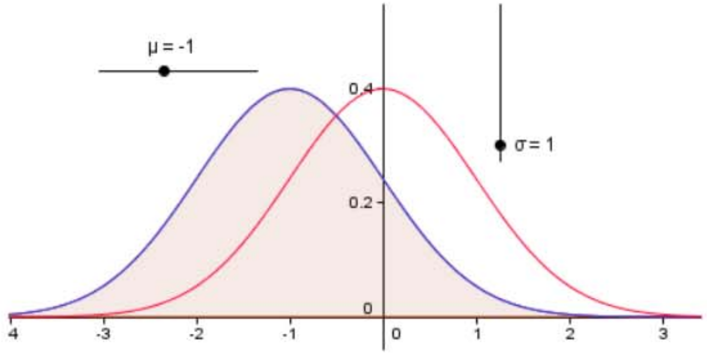
Z is, by convention, used to denote the standard normal variable.

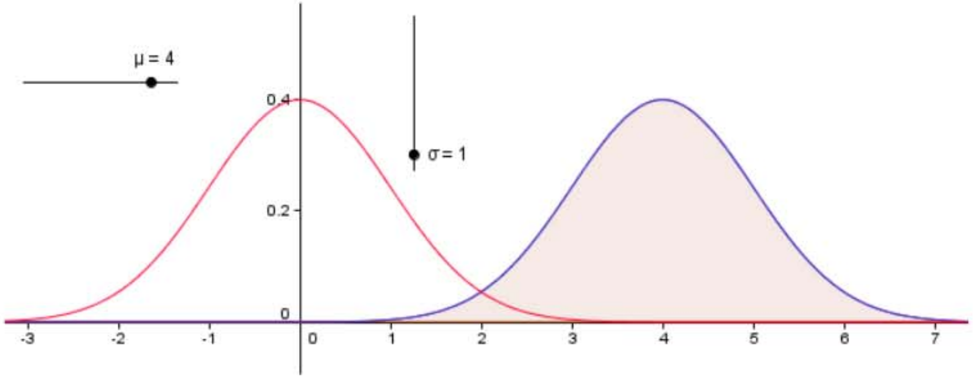
Comparing normal distributions

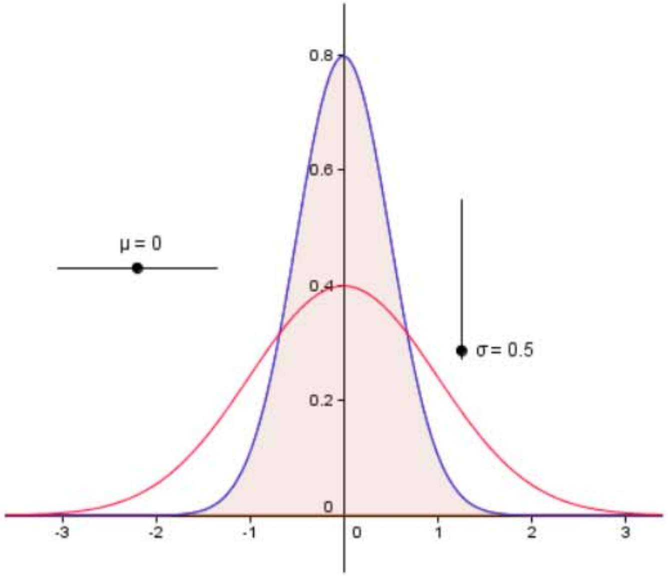
The STANDARD Normal distribution

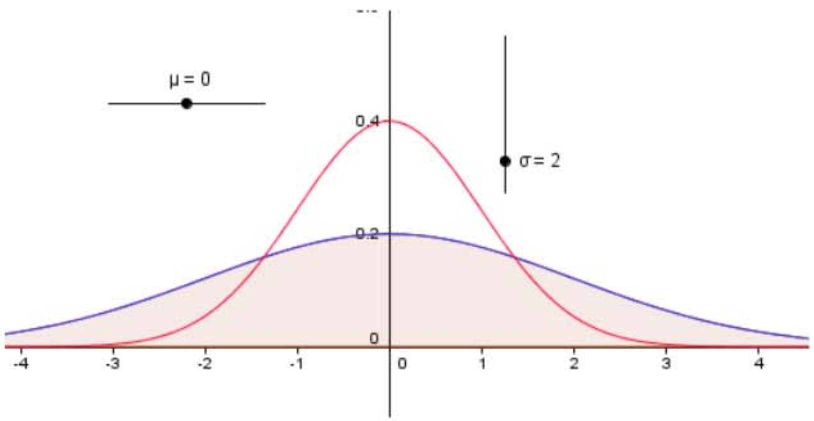
μ=0 and σ=1









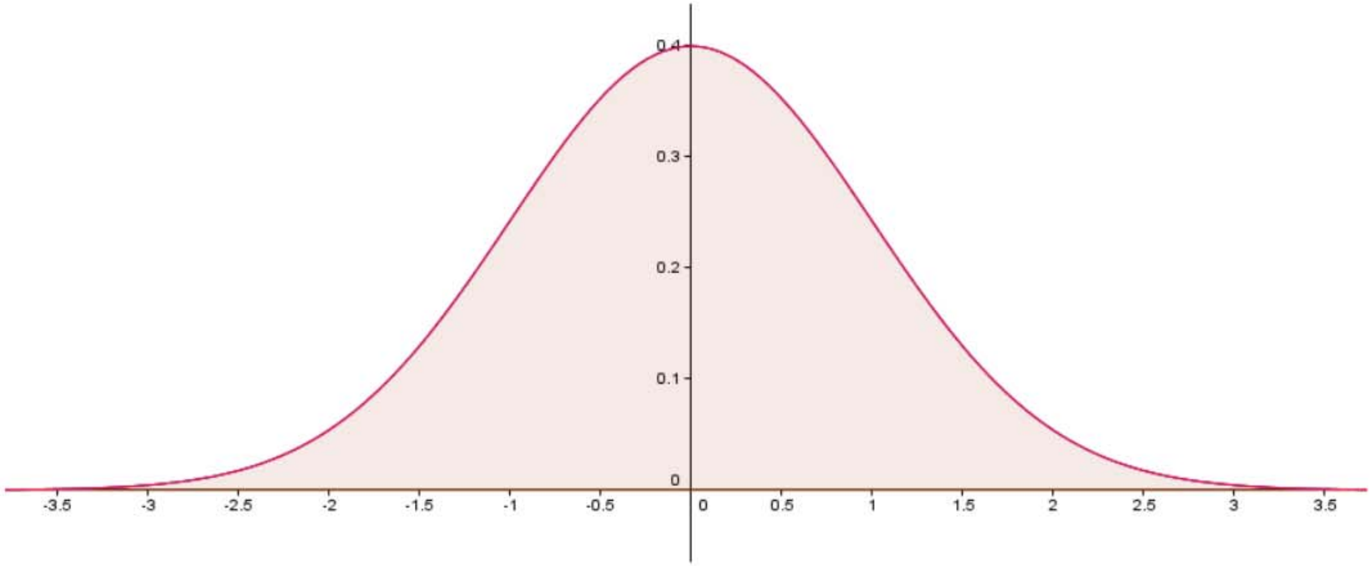


Notation:

a variable Z distributed normally will be noted $Z \sim N(\mu,\sigma^2)$

Working out probabilities

Like an histogram, a probability (or frequency) is determined by working out the area of a certain part of the representation



- $P(Z = z)$: Since $Z=z$ is represented by a line, its area is 0 so $P(Z = z) = 0$ for all z
- $P(Z \leq z)$ is the area of the shaded region, underneath the curve, from $-\infty$ to z

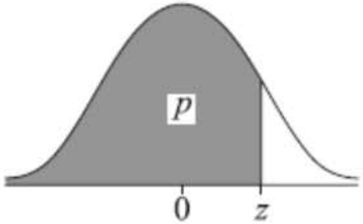
  Note: $P(Z\leq z) = P(Z<z)$

But Sir!, how do we work out the area underneath the curve?

There is no other way than the table of values given in the formula book!

TABLE 3 NORMAL DISTRIBUTION FUNCTION

The table gives the probability, p, that a normally distributed random variable Z, with mean = 0 and variance = 1, is less than or equal to z.



Extract of the table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	z
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586	0.0
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535	0.1
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409	0.2
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173	0.3
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793	0.4
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240	0.5
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490	0.6
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524	0.7
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327	0.8
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891	0.9
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214	1.0
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298	1.1

Let's work out the following probabilities:

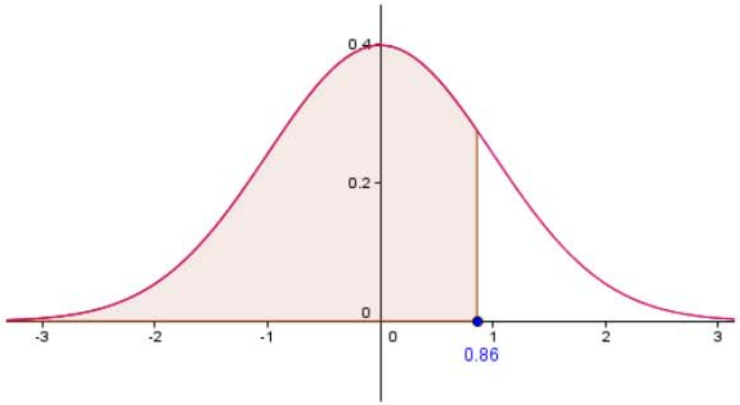
$P(Z \leq 0.86) =$

$P(Z \leq 1.19) =$

$P(Z \leq 0.4) =$

$P(Z \leq 0.07) =$

$P(Z \leq 1.1486) =$



EXERCISE 5A

Find the probability that an observation from a standard normal distribution will be less than:

(a) 1.23, **(b)** 0.97, **(c)** 1.85, **(d)** 0.42, **(e)** 0.09,

(f) 1.57, **(g)** 1.94, **(h)** 0.603, **(i)** 2.358, **(j)** 1.053 79.

EXERCISE 5A

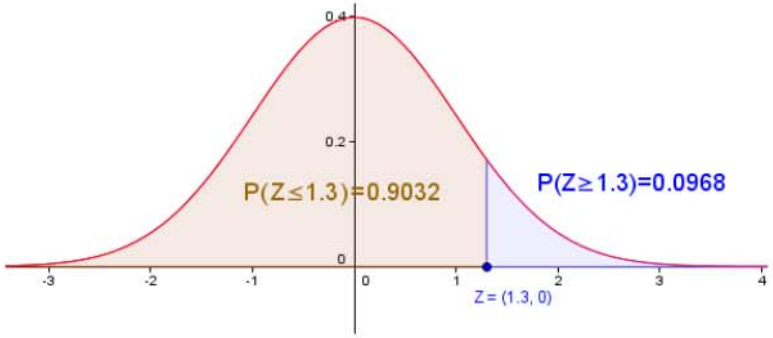
1 **(a)** 0.891; **(b)** 0.834; **(c)** 0.968;
(d) 0.663; **(e)** 0.536; **(f)** 0.942;
(g) 0.974; **(h)** 0.726; **(i)** 0.991;
(j) 0.853.

Probability $P(Z \geq z)$

Remember: the total area underneath the curve is =1

So $P(Z \geq z) = 1 - P(Z < z)$

also $P(Z \geq z) = 1 - P(Z \leq z)$



EXERCISE 5B

1 Find the probability that an observation from a standard normal distribution will be greater than:

(a) 1.36, **(b)** 0.58, **(c)** 1.23, **(d)** 0.86,

(e) 0.32, **(f)** 1.94, **(g)** 2.37, **(h)** 0.652,

(i) 0.087, **(j)** 1.3486.

EXERCISE 5B

1 **(a)** 0.0869; **(b)** 0.281; **(c)** 0.109;
(d) 0.195; **(e)** 0.374; **(f)** 0.0262;
(g) 0.008 89; **(h)** 0.258; **(i)** 0.464;
(j) 0.0885.

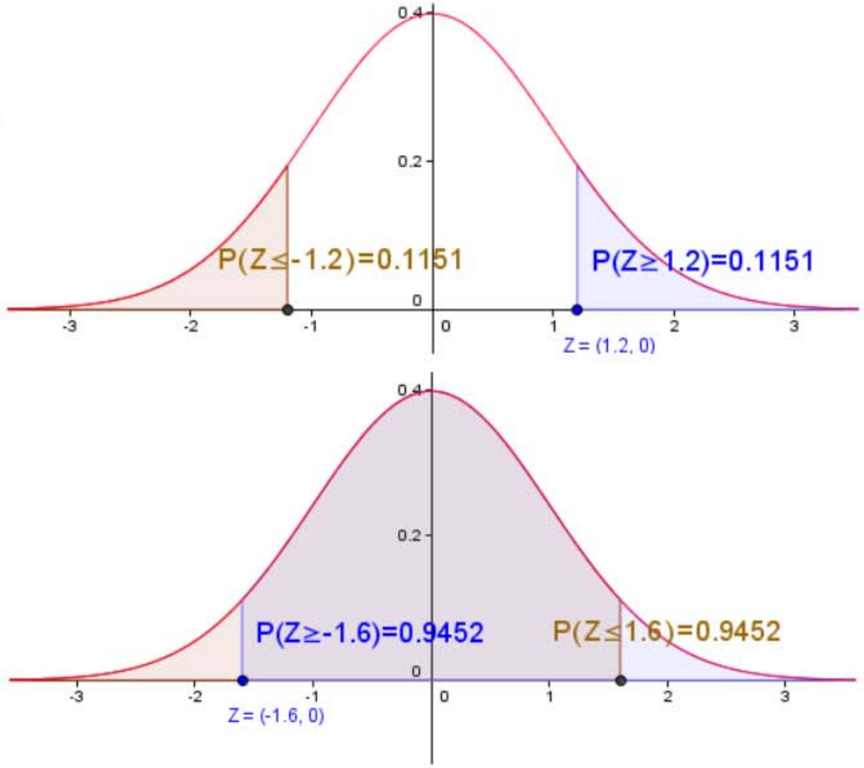
Negative values of z

Remember: the curve is symmetrical around the mean

$$P(Z \le -z) = P(Z \ge z)$$
$$P(Z \le -z) = 1 - P(Z \le z)$$

and

$$P(Z \ge -z) = P(Z \le z)$$



EXERCISE 5C

1 Find the probability that an observation from a standard normal distribution will be:

(a) less than -1.39,

(b) less than -0.58,

(c) more than -1.09,

(d) more than -0.47,

(e) less than or equal to -0.45,

(f) greater than or equal to -0.32,

(g) less than -0.64,

(h) -0.851 or greater,

(i) more than -0.747,

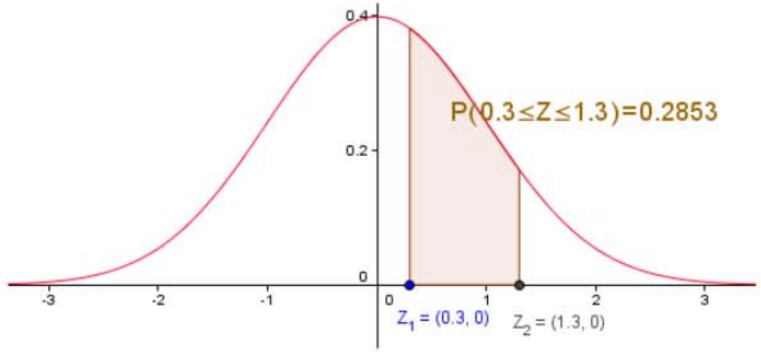
(j) less than -0.4398.

EXERCISE 5C

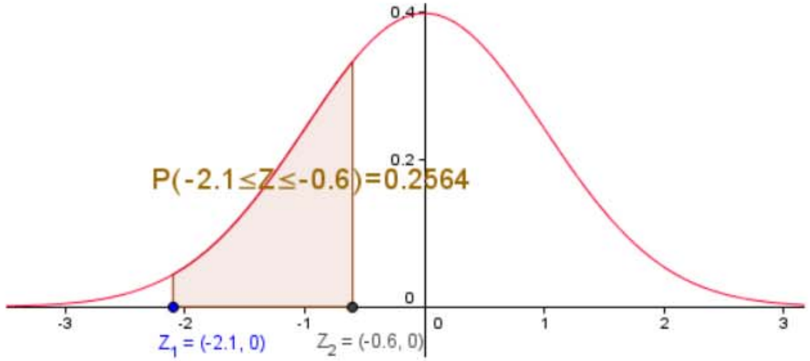
1 **(a)** 0.0823; **(b)** 0.281; **(c)** 0.862; **(d)** 0.681; **(e)** 0.326; **(f)** 0.626; **(g)** 0.261; **(h)** 0.802; **(i)** 0.773; **(j)** 0.330.

Probability between z-values

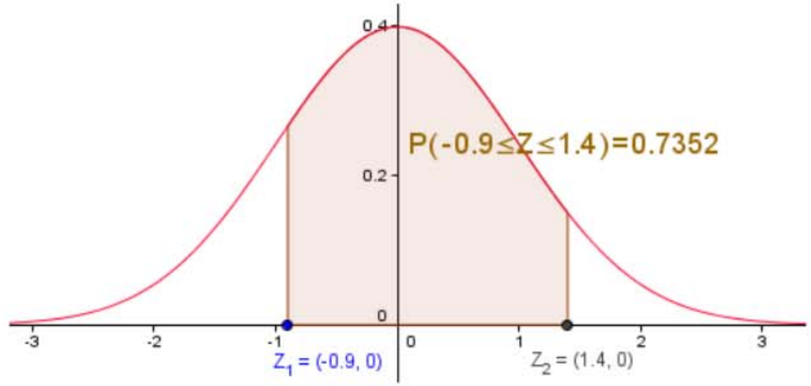
- $P(0.3 \leq Z \leq 1.3)$ =



- $P(-2.1 \leq Z \leq 0.6)$ =



- $P(-0.9 \leq Z \leq 1.4)$ =



EXERCISE

1 Find the probability that an observation from a standard normal distribution will be between:

(a) 0.2 and 0.8,

(b) −1.25 and −0.84,

(c) −0.7 and 0.7,

(d) −1.2 and 2.4,

(e) 0.76 and 1.22,

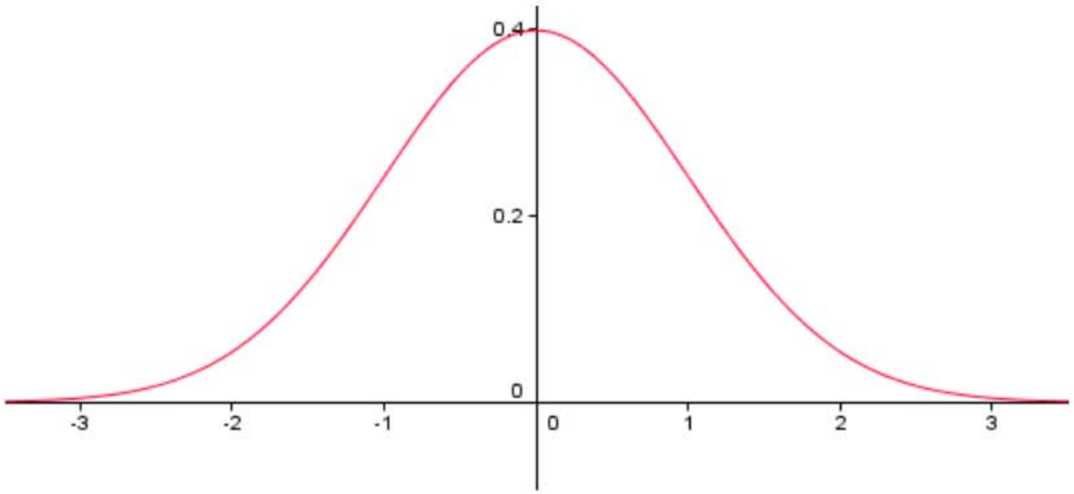
(f) −3 and −2,

(g) −1.27 and 2.33,

(h) 0.44 and 0.45,

(i) −1.2379 and −0.8888,

(j) −2.3476 and 1.9987.



EXERCISE 5D

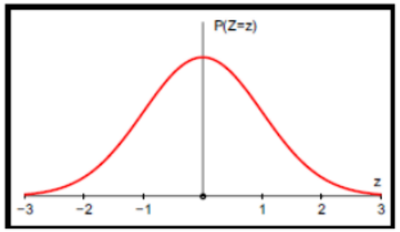
1 (a) 0.209; (b) 0.0948; (c) 0.516; (d) 0.877; (e) 0.112; (f) 0.0214; (g) 0.888; (h) 0.003 61; (i) 0.0792; (j) 0.968.

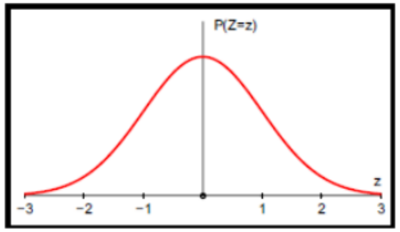
Match a blue card to the correct yellow answer.
Represent the area presented by the probabilities on the graphs

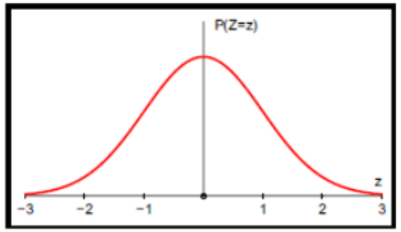
$P(-2.23 < Z < -1.18)$
$P(Z < -1.54)$
$P(Z > 0.47)$
$P(Z \geq -2.23)$
$P(2 < Z < 3)$
$P(Z < 1.7)$

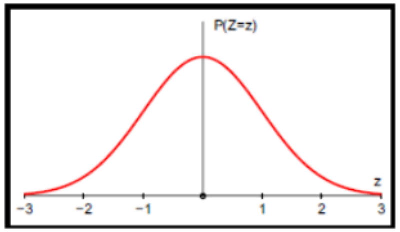
0	**0.0618**
0.955	**0.0215**
0.985	**0.118**
0.319	**0.880**
0.987	**0.619**
0.0129	**0.106**

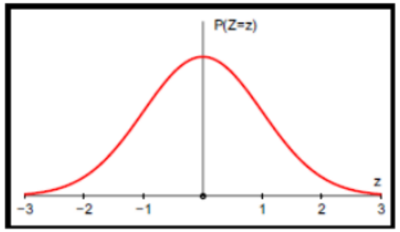
$P(Z < -2.23)$
$P(Z = 2)$
$P(1.18 \leq Z \leq 2.97)$
$P(-1.18 < Z < 2.97)$
$P(Z \leq 2.18)$
$P(-1.54 < Z < 0.47)$

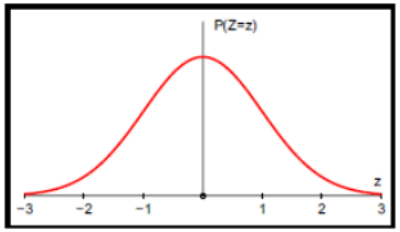


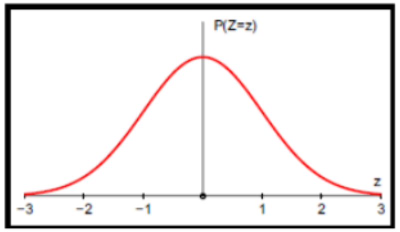


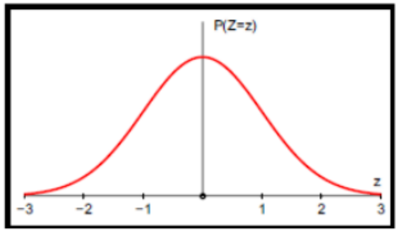


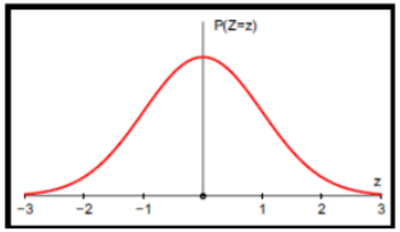


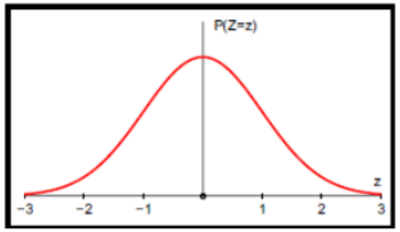












More practice:

$Z \sim N(0,1)$

Use tables of the normal distribution to find the following.

1 **a** $P(Z < 2.12)$ **b** $P(Z < 1.36)$

c $P(Z > 0.84)$ **d** $P(Z < -0.38)$

2 **a** $P(Z > 1.25)$ **b** $P(Z > -1.68)$

c $P(Z < -1.52)$ **d** $P(Z < 3.15)$

3 **a** $P(Z > -2.24)$ **b** $P(0 < Z < 1.42)$

c $P(-2.30 < Z < 0)$ **d** $P(Z < -1.63)$

4 **a** $P(1.25 < Z < 2.16)$ **b** $P(-1.67 < Z < 2.38)$

c $P(-2.16 < Z < -0.85)$ **d** $P(-1.57 < Z < 1.57)$

1 **a** 0.9830 **b** 0.9131 **c** 0.2005 **d** 0.3520
2 **a** 0.1056 **b** 0.9535 **c** 0.0643 **d** 0.9992
3 **a** 0.9875 **b** 0.4222 **c** 0.4893 **d** 0.0516
4 **a** 0.0902 **b** 0.9438 **c** 0.1823 **d** 0.8836