Matrix transformation

From the specifications

Transformations of points in the $x-y$ plane represented by 2×2 matrices.

Transformations will be restricted to rotations about the origin, reflections in a line through the origin, stretches parallel to the $x-$ and $y-$axes, and enlargements with centre the origin.
Use of the standard transformation matrices given in the formulae booklet.
Combinations of these transformations

$$\text{e.g. } \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

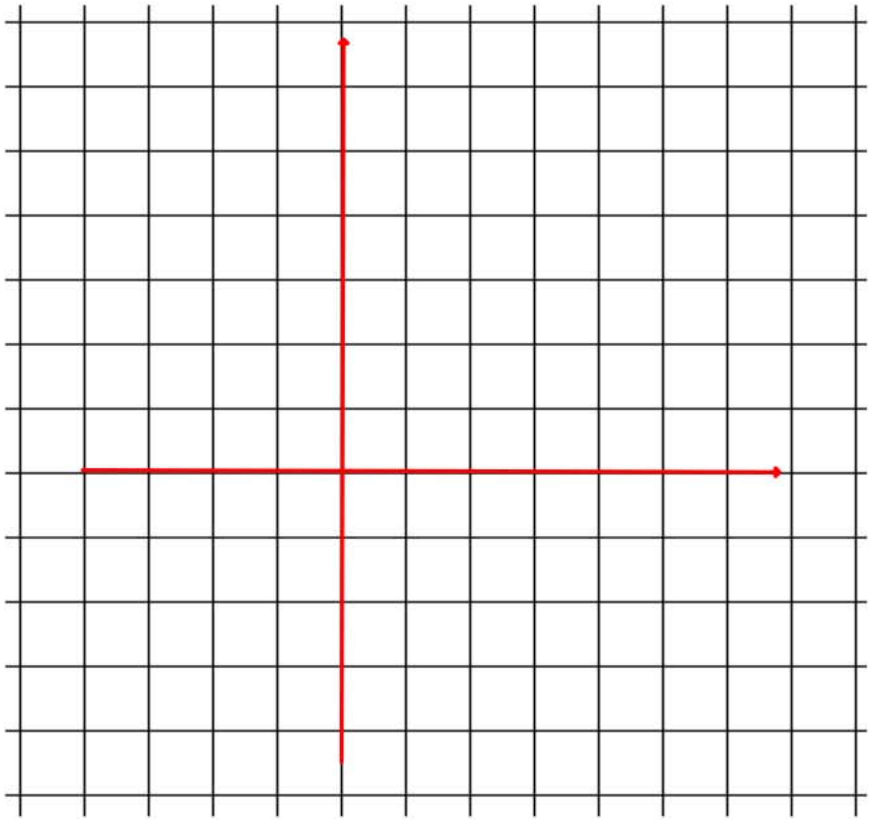
From the formulae booklet:

Matrix transformations

Anticlockwise rotation through θ about O: $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Reflection in the line $y = (\tan\theta)x$: $\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$

The principle of the transformation



A matrix 2x1 can be seen as a position vector (position of a point)

$$A = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \leftrightarrow A(x, y)$$

Consider a matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$A' = M \times A$ is the image of A through the matrix transformation

Example:

Consider the triangle ABC with A(1, 2), B(6,4) and C(2,6)

i) Work out the image of ABC through the matrix transformation $M = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{-1}{2} & 0 \end{bmatrix}$

ii) Work out the image of ABC through the matrix transformation $T = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

1. Plot the object and image for each of the following on the same diagram and describe each as a single transformation

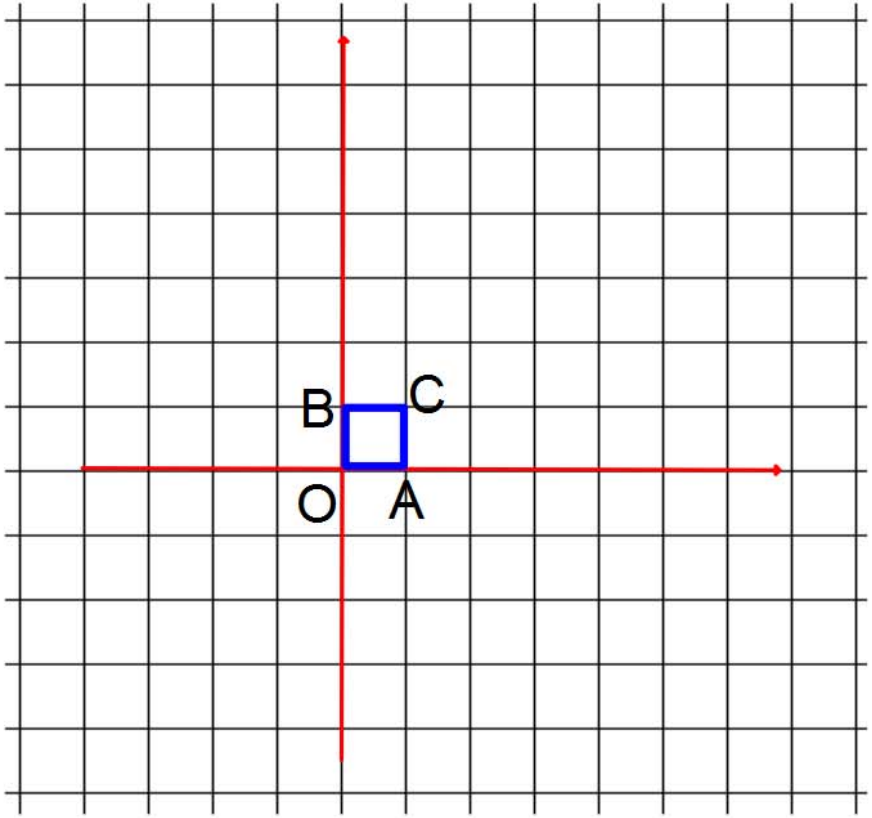
	Object	Matrix
a)	P(4,2) Q(4,4) R(0,4)	$\begin{pmatrix} -0.5 & 0 \\ 0 & -0.5 \end{pmatrix}$
b)	P(-6,8) Q(-2,8) R(-2,6)	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

2. Find the images of A (3, 1), B (3, 3), C(6, 3), D(6, 1) under the transformation $\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$. Draw a diagram to show the shape ABCD and its image.

3. A square has vertices at (0, 0), (1, 1), (0, 2), (-1, 1).

   Find the image of the square under the transformation $\begin{pmatrix} 4 & 3 \\ -3 & -2 \end{pmatrix}$.

Transforming the unit square



The unit square OACB is such that
O(0,0), A(1,0) , B(0,1) , C(1,1)

Consider a matrix $M=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

The image of O is O

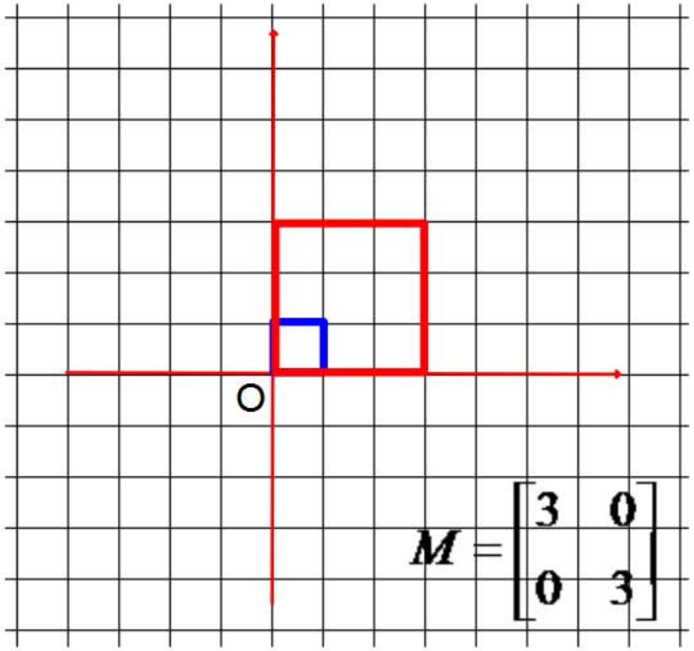
The image of A is $A'=\begin{pmatrix} a \\ c \end{pmatrix}$

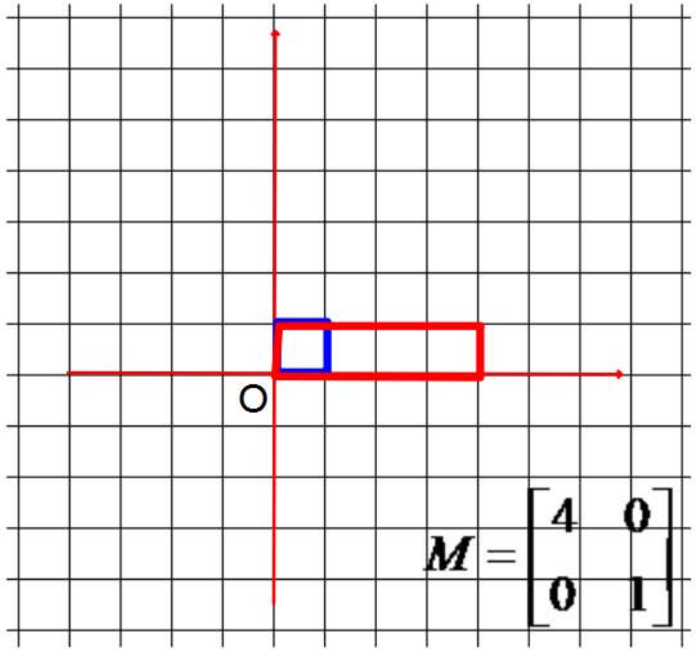
The image of B is $B'=\begin{pmatrix} b \\ d \end{pmatrix}$

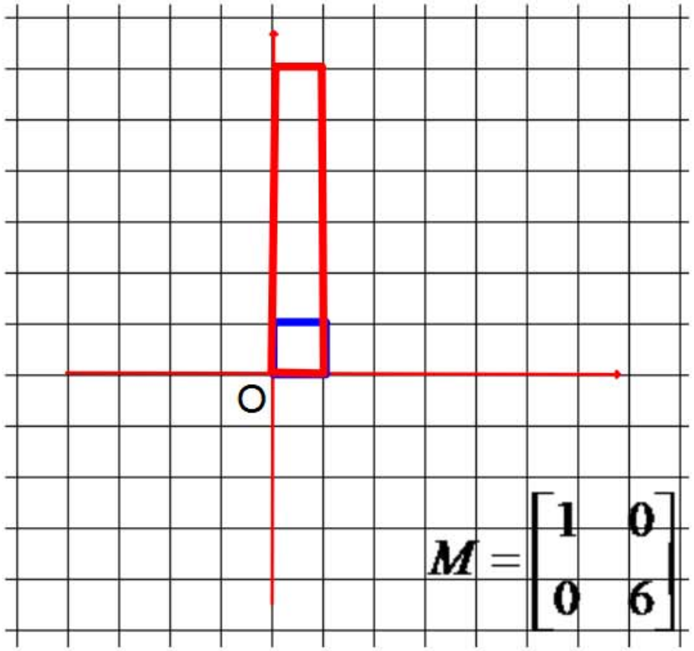
The image of C is $C'=\begin{pmatrix} a+b \\ c+d \end{pmatrix}$

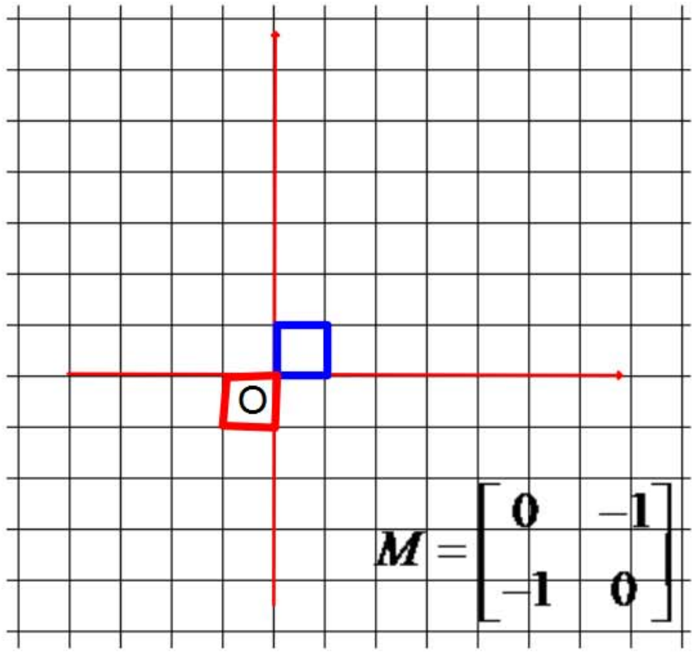
We often use the unit square to work out
what kind of transformation is a given matrix

Transform the unit square to describe the matrix transformation



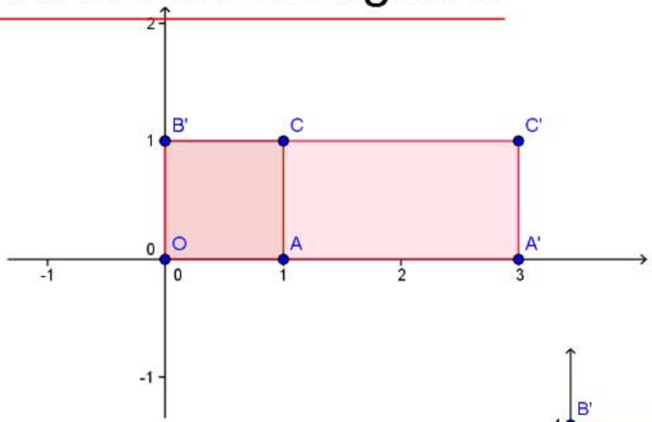




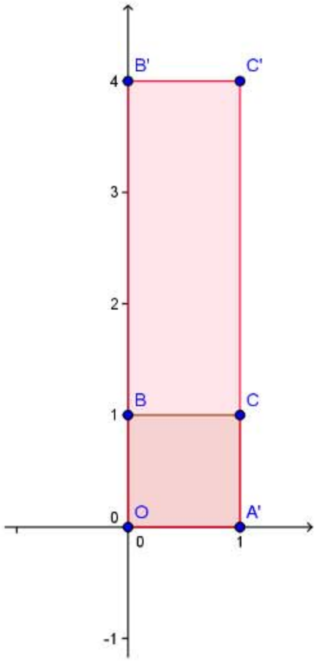


Transformation you should recognise:

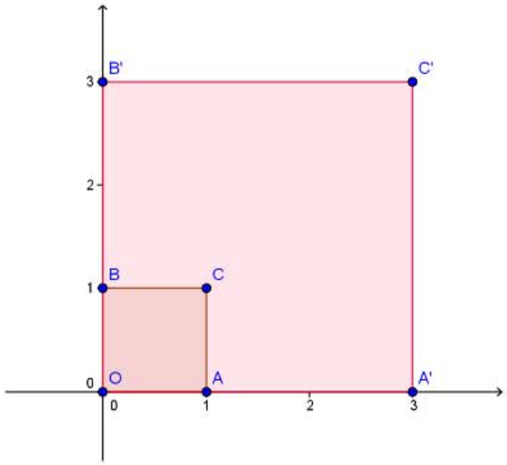
$$M = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$



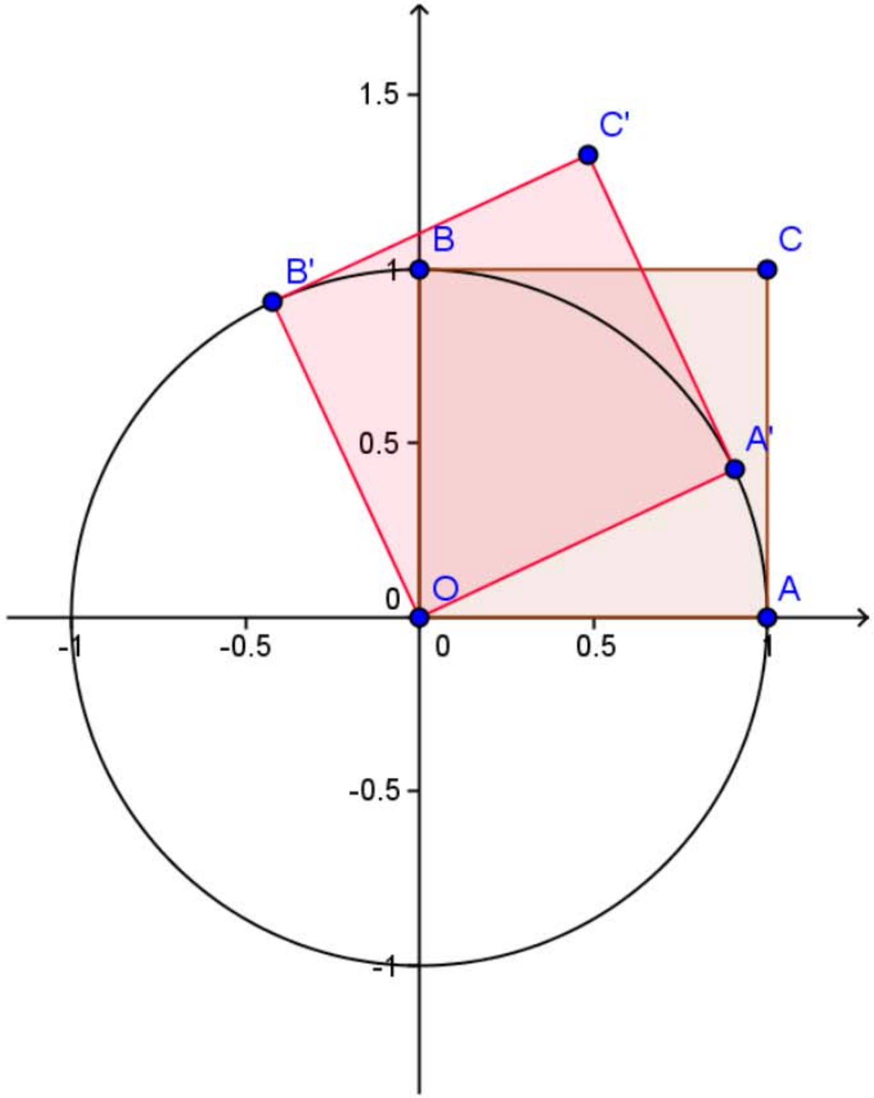
$$M = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$



$$M = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$



Rotations



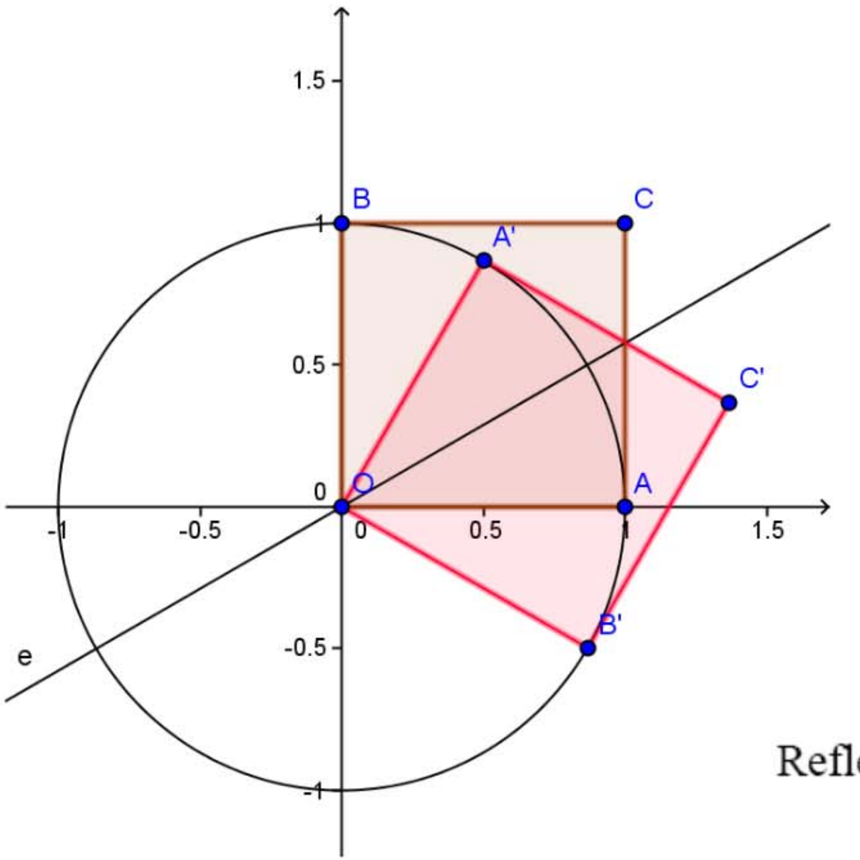
Consider a rotation of the unit square by an angle θº anticlockwise about O

To work out the element of the transformation matrix,
we need to work out the coordinates of A' and B'

θ	0°	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined

Anticlockwise rotation through θ about O: $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

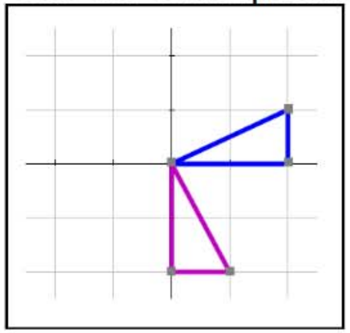
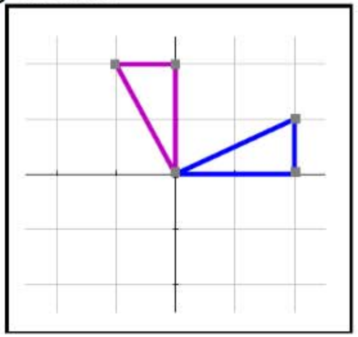
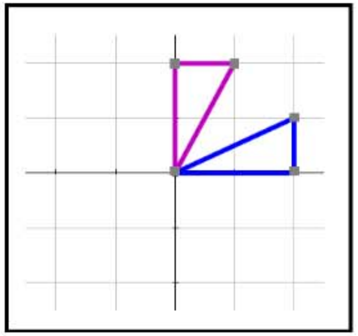
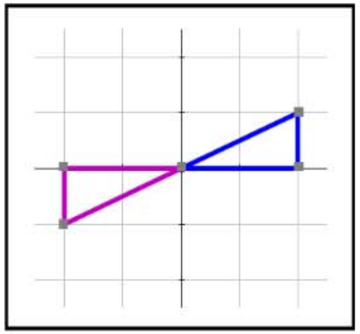
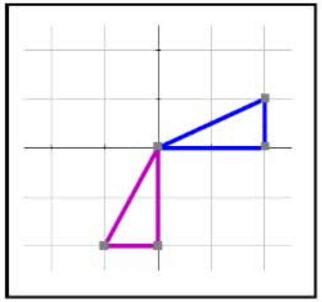
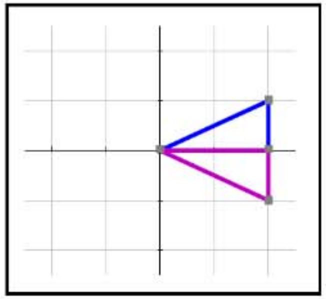
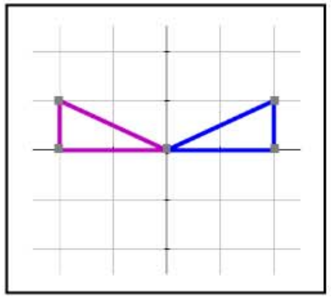
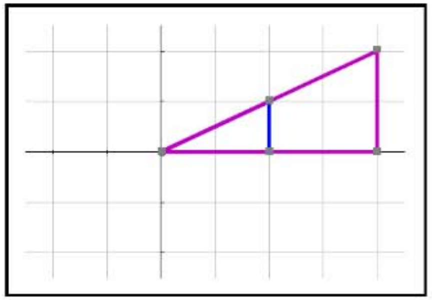
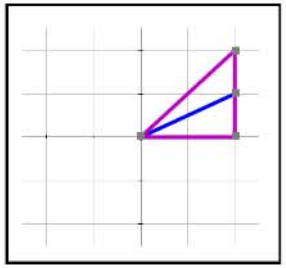
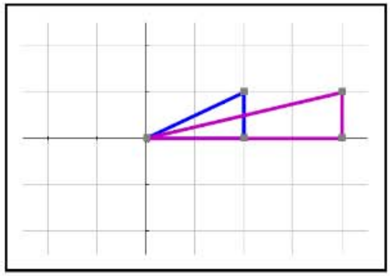
Reflections (through a line y = tan(θ)x)



Reflection in the line $y=(\tan\theta)x$: $\begin{bmatrix}\cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta\end{bmatrix}$

Describe the tranformation which map the blue triangle into the pink one.
Give the corresponding matrix

Match them up

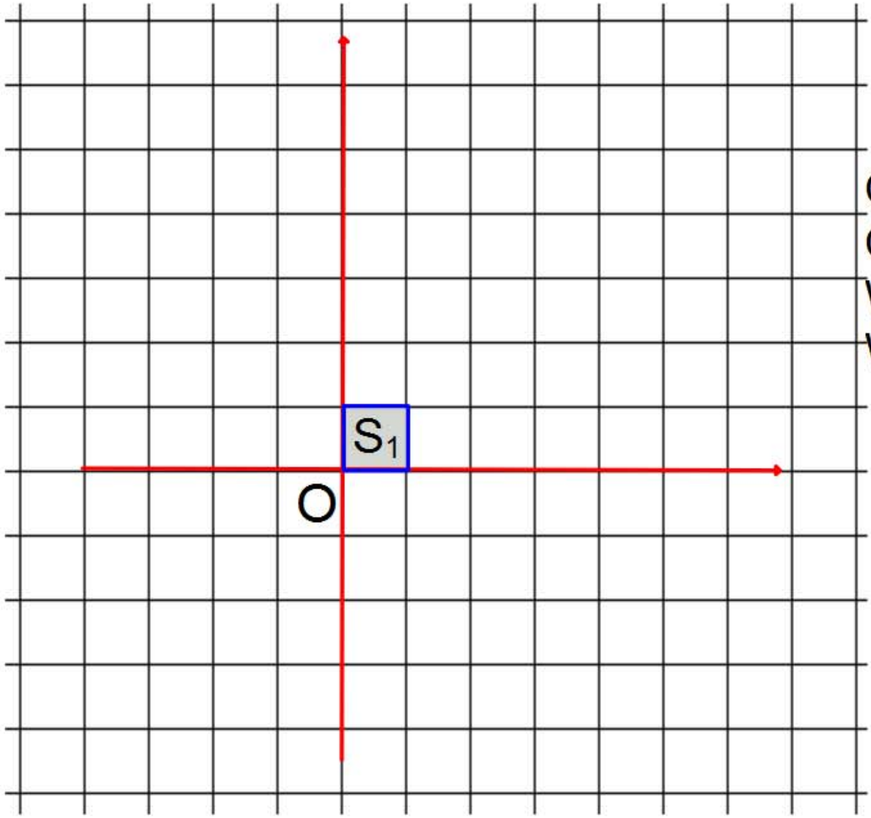
$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

Enlargement with scale factor 2, centre the origin	Reflection in the line $y=-x$	Reflection in the line $y=x$	
Stretch with scale factor 2, in the direction of the y-axis	Rotation by 90° anticlockwise	Rotation by 90° clockwise	
Reflection in y-axis	Reflection in x-axis	Stretch with scale factor 2, in the direction of the x-axis	Rotation by 180°

Composing transformations



Consider the matrices $M = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ *and* $T = \begin{bmatrix} -1 & 2 \\ 3 & -2 \end{bmatrix}$

Construct S_2 the image of S_1 through the M transformation.
Construct S_3 the image of S_2 through the T transformation.
Work out the matrix which map S_1 into S_3.
Work out M×T and T×M. What do you notice?

Consider two matrices A and B.

The product **BA** represent the transformation equivalent to applying **A THEN B**

EXERCISE 6C

1 The composite transformation **T** is formed by applying a rotation of 30° anticlockwise about the origin followed by a stretch in the x-direction of factor 2. Find the matrix that represents **T**.

2 The composite transformation **C** is formed by first reflecting in the line $y = x$ followed by an enlargement with centre the origin and scale factor 5. Find the matrix that represents **C**.

3 The composite transformation **M** is formed by applying a rotation of 90° anticlockwise about the origin followed by a reflection in the line $y = -x$.

(a) Find the matrix that represents **M**.

(b) Give a full description of transformation **M**.

4 The matrix **A** represents a rotation of 60° anticlockwise about the origin.

(a) Find matrix **A**.

(b) Give a full geometric description of the transformation represented by the matrix $\mathbf{A}^2$.

5 The transformation **A** is a reflection in the line $y = x$ and the transformation **B** is a reflection in the line $y = -x$. The composite transformation **T** is formed by applying **B** followed by **A**. Give a complete description of **T**.

6 Prove that a composite transformation formed by two successive reflections in any two straight lines through the origin that are perpendicular is equivalent to performing a half-turn about the origin.

7 The composite transformation **M** is defined by **M** = **ABC** where **A**, **B** and **C** are the transformations:

A a reflection in the line through the origin at 30° to the positive x-axis;

B a stretch in the y-direction of factor 2;

C a rotation of 45° anticlockwise about the origin.

Find the matrix that represents **M**.

8 **A**, **B** and **C** are the transformations:

A a rotation of 30° anticlockwise about the origin;

B a reflection in the line $y = \sqrt{3}x$;

C a rotation of 150° anticlockwise about the origin.

Transformation **T** is formed by applying **C** then **B** then **A**. Describe fully the composite transformation **T**.

9 Give a full geometrical description of the transformation represented by the matrix $\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$.

Answers

1) $\begin{bmatrix} \sqrt{3} & -1 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$ **2)** $\begin{bmatrix} 0 & 5 \\ 5 & 0 \end{bmatrix}$ **3)** *a)* $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ *b)* **Stretch in the x-direction SF -1 (or reflection in y-axis)**

4)a) $\begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ *b) rotation* **120°** ***anticlockwise about O*** **5) Enlargement centre O, SF -1**

7) $\begin{bmatrix} \frac{1+2\sqrt{3}}{2\sqrt{2}} & \frac{2\sqrt{3}-1}{2\sqrt{2}} \\ \frac{\sqrt{3}-2}{2\sqrt{2}} & \frac{-\sqrt{3}-2}{2\sqrt{2}} \end{bmatrix}$ **8) Stretch in the y-direction of scale factor -1**

9) Anticlockwise rotation of 53.1° about O followed by an enlargement , centre O, SF 5.