Matrices

Specifications:

Matrices and Transformations

2×2 and 2×1 matrices;
addition and subtraction,
multiplication by a scalar.

Multiplying a 2×2 matrix by a 2×2 matrix or by a 2×1 matrix.

The identity matrix $\mathbf{I}$ for a 2×2 matrix.

Introduction:

A matrix is a rectangular **ARRAY** of numbers.
Each entry is called an **ELEMENT**

Examples:

Order of a matrix

A matrix with m rows and n columns is an $m \times n$ matrix (read "m by n")
This is called the **ORDER** of the matrix

In this module, we only work with 2x2 and 2x1 matrices

Special matrices

- Square matrices
- Identity matrix
- The Zero matrix

Operations with matrices

Adding / subtracting

You can add/subtract matrices provided that they have the same order

Rule: $$if\ A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\ and\ B=\begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

$$then\ A+B=\begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}\ and\ A-B=\begin{bmatrix} a-e & b-f \\ c-g & d-h \end{bmatrix}$$

Examples/exercise:

The matrices **A**, **B**, and **C** are given by:

$$\mathbf{A}=\begin{bmatrix} 6 & -3 \\ 9 & 17 \end{bmatrix},\quad \mathbf{B}=\begin{bmatrix} -3 & 9 \\ 0 & -4 \end{bmatrix} \text{ and } \mathbf{C}=\begin{bmatrix} 10 & -5 \\ 1 & 4 \end{bmatrix}.$$

Find:

(a) $\mathbf{A}+\mathbf{B}$, **(b)** $\mathbf{B}+\mathbf{A}$, **(c)** $\mathbf{C}-\mathbf{A}$,

(d) $\mathbf{B}+\mathbf{C}$, **(e)** $\mathbf{A}-\mathbf{B}$, **(f)** $\mathbf{A}+\mathbf{B}+\mathbf{C}$,

(g) $\mathbf{B}+\mathbf{C}-\mathbf{A}$, **(h)** $\mathbf{A}-\mathbf{C}+\mathbf{B}$.

Multiplying matrices by a scalar

If $\mathbf{A}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *and* λ is a scalar

then $\lambda\mathbf{A}=\begin{bmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{bmatrix}$

Multiplying matrices

- *Consider the matrices* $\begin{bmatrix} 2 & 4 & 7 \end{bmatrix}$ *and* $\begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}$

  Let's multiply them:

- General rule

$$\text{if } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

$$\text{then } A \times B = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

Note: you can only multiply the matrices A and B if the order of a is m×n and the order of B is n×p

1 Find the following matrix products:

(a) $\begin{bmatrix} 6 & 2 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 7 \end{bmatrix}$ **(b)** $\begin{bmatrix} 9 & 0 \\ 4 & 7 \end{bmatrix}\begin{bmatrix} 5 \\ 10 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & 7 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix}$ **(d)** $\begin{bmatrix} 5 & -9 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix}$

(e) $\begin{bmatrix} 11 & 4 \\ -3 & -10 \end{bmatrix}\begin{bmatrix} 4 \\ -2 \end{bmatrix}$ **(f)** $\begin{bmatrix} 4 & -8 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ 12 \end{bmatrix}$

2 Find each of the following:

(a) $\begin{bmatrix} 4 & 7 \\ 11 & 2 \end{bmatrix}\begin{bmatrix} 7 & 1 \\ 2 & 2 \end{bmatrix}$ **(b)** $\begin{bmatrix} 3 & 9 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$

(c) $\begin{bmatrix} -5 & 2 \\ 8 & -1 \end{bmatrix}\begin{bmatrix} -2 & 6 \\ 4 & 2 \end{bmatrix}$ **(d)** $\begin{bmatrix} -2 & 10 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 7 & -1 \\ -5 & 3 \end{bmatrix}$

3 Given that matrix $\mathbf{H} = \begin{bmatrix} 3 & 1 \\ -2 & -4 \end{bmatrix}$ find: **(a)** $\mathbf{H}^2$, **(b)** $\mathbf{H}^3$.

4 Find the values of x and y in the following cases:

(a) $\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ -5 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ **(b)** $\begin{bmatrix} -4 & 5 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} -6 \\ 8 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

(c) $\begin{bmatrix} 3 & 6 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -6 \end{bmatrix}$ **(d)** $\begin{bmatrix} -2 & 1 \\ -5 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \end{bmatrix}$

(e) $\begin{bmatrix} x & 0 \\ y & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 16 \\ -21 \end{bmatrix}$ **(f)** $\begin{bmatrix} 8 & x \\ y & 2 \end{bmatrix}\begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -28 \\ -5 \end{bmatrix}$

Key point summary

1 A **matrix** is a rectangular array of numbers. Each entry in the matrix is called an **element**.

2 A matrix with m rows and n columns is an $m \times n$ matrix. This is called the **order** of the matrix.

3 We can add or subtract matrices provided they have the **same order**.

4 To add/subtract matrices you add/subtract **corresponding elements**.

5 To multiply a matrix by a constant, simply multiply each element of the matrix by the constant.

6 We can multiply two matrices **A** and **B** only if the number of columns of **A** equals the number of rows of **B**.

7 If **A** is an $(a \times b)$ matrix and **B** is a $(c \times d)$ matrix then the product matrix **AB** exists if and only if $b = c$.
The product **AB** will be of order $a \times d$.

8 In general, $\mathbf{AB} \neq \mathbf{BA}$.
Matrix multiplication is not, in general, commutative.

9 A matrix which has the same number of rows and columns is called a **square matrix**.

10 The matrix $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is called the 2×2 **identity matrix** because when you multiply any 2×2 matrix **A** by **I** you get **A** as the answer.
This means that for any 2×2 matrix **A**,

$$\mathbf{IA} = \mathbf{AI} = \mathbf{A}.$$

11 The 2×2 matrix $\mathbf{Z} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is called the **zero matrix**

since $\mathbf{Z} + \mathbf{A} = \mathbf{A} + \mathbf{Z} = \mathbf{A}$

and $\mathbf{ZA} = \mathbf{AZ} = \mathbf{Z}$ for any 2×2 matrix **A**.