Linear laws

Specifications

Numerical Methods

Reducing a relation to a linear law.

E.g. $\dfrac{1}{x}+\dfrac{1}{y}=k$; $y^2=ax^3+b$; $y=a\,x^n$; $y=ab^x$

Use of logarithms to base 10 where appropriate.

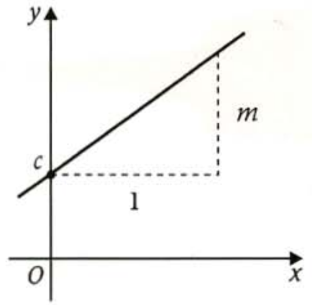
Given numerical values of (x, y), drawing a linear graph and using it to estimate the values of the unknown constants.

"y = mx+c" - Revision

$y = mx + c$ is the equation of a straight line with gradient m and y-intercept c.

You can find the value of m by taking any two points (x_1, y_1) and (x_2, y_2) **on** the line and using $m = \frac{y_2 - y_1}{x_2 - x_1}$.

You can usually find the value of c by reading the y-coordinate of the point where the line intersects the y-axis.



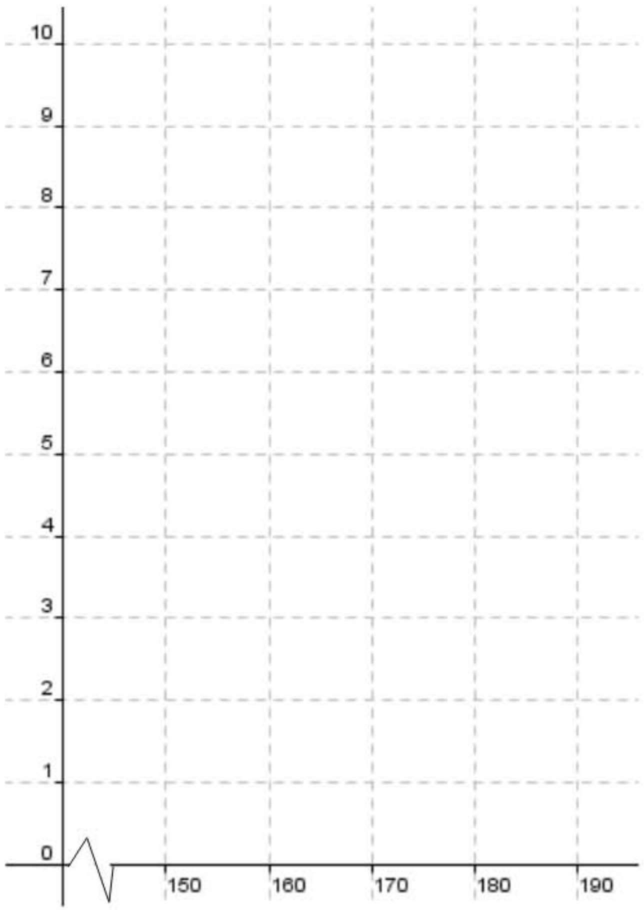
A common error is to use coordinates of points from a table which may not always lie on the line.

An example:

Two quantities x and y are measured experimentally and the following values obtained:

x	150	160	170	180	190
y	9.0	6.9	5.0	3.1	1.0

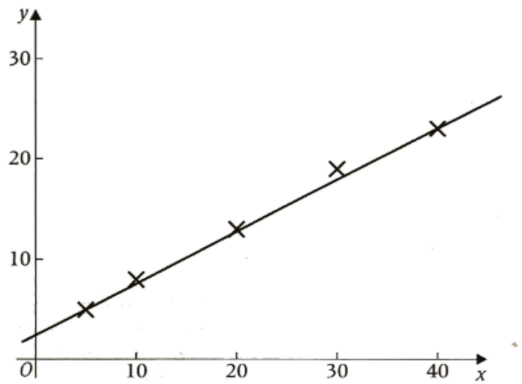
It is thought that they are connected by a law of the form $y = ax + b$. Test if this is so and, by drawing a suitable straight line graph, estimate the values of a and b.



The law connecting y and x is $y = -0.2x + 39$.

Exercises:

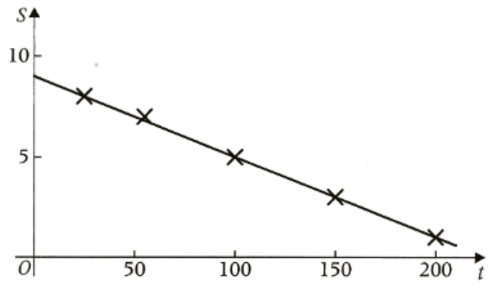
1 The scatter diagram shows the results of a scientific experiment between two variables y and x. A line of best fit has been drawn.



(a) Find the equation of the line of best fit.

(b) Use your equation to estimate the value of x when $y = 40$.

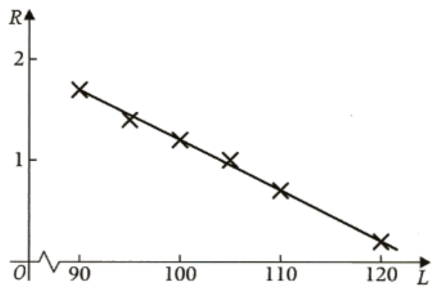
2 The scatter diagram shows the results of a scientific experiment between two variables S and t. A line of best fit has been drawn.



(a) Find the equation of the line of best fit.

(b) Use your equation to estimate the value of t when $S = 0$.

3 The scatter diagram shows the results of a scientific experiment between two variables R and L. A line of best fit has been drawn.



(a) Find the equation of the line of best fit.

(b) Use your equation to estimate the value of R when $L = 84$.

1 (a) $y = \frac{1}{2}x + 3$; **(b)** 74.

2 (a) $S = 9 - 0.04t$; **(b)** 225.

3 (a) $R = 62 - 0.05L$; **(b)** 57.8.

Reducing a relation to a linear law

Equations of the form $y = ax^2 + b$

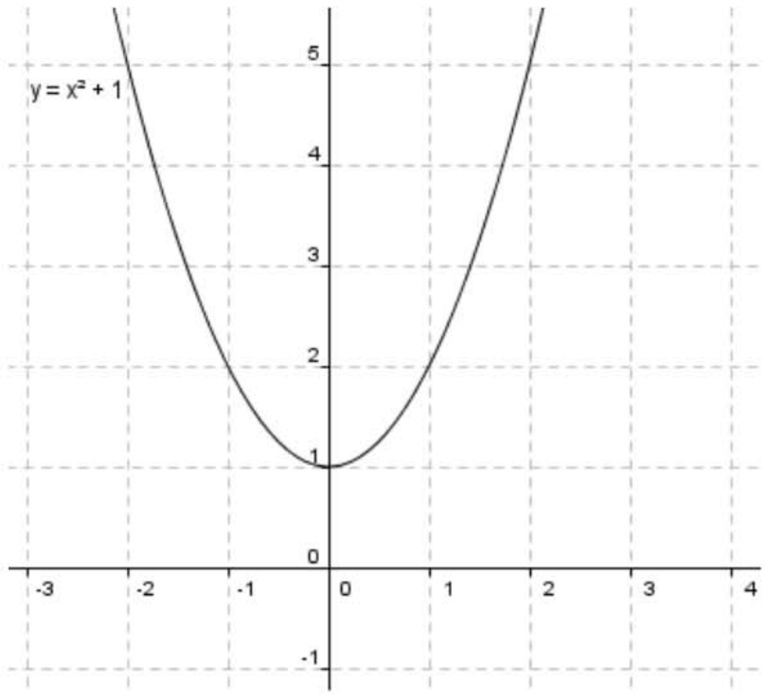
Consider the equation $y = ax^2 + b$

Numerical example

$y = x^2 + 1$

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
y	5	3.3	2	1.3	1	1.3	2	3.3	5
$X=x^2$									
$Y=y$									

Complete the table and plot the point (X,Y)



Your turn...

The table shows some experimental values of the variables L and T.

T	5	10	12	15	20
L	11	29	41	61	105

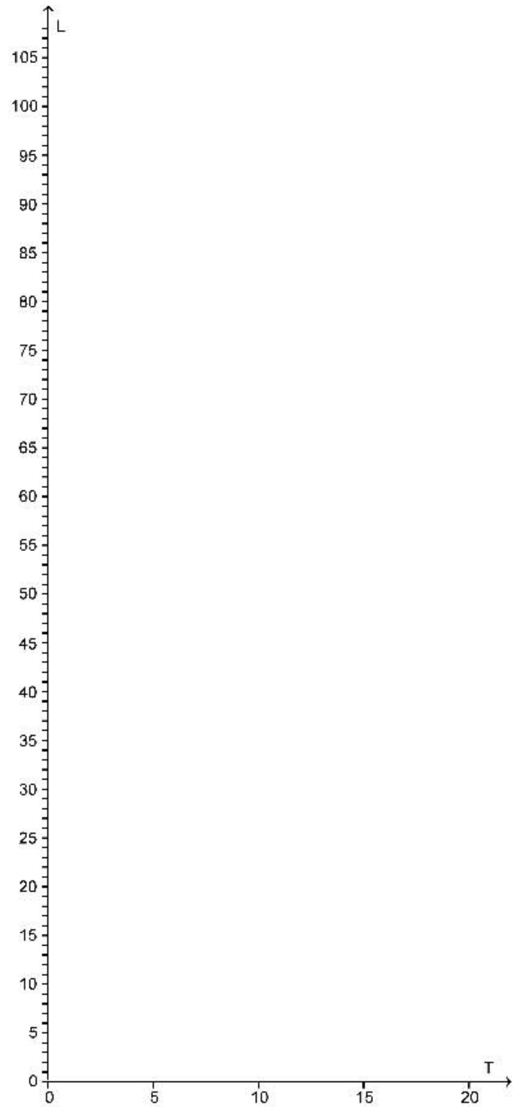
A scientist believes that the variables T and L satisfy a relation of the form $L = aT^2 + b$.

(a) By drawing an appropriate graph, explain why the scientist is correct.

(b) Use your graph to estimate the values of the constants a and b.

(c) Given that $T > 0$, use the relation to find the value of T when $L = 149$.

T^2					
L					

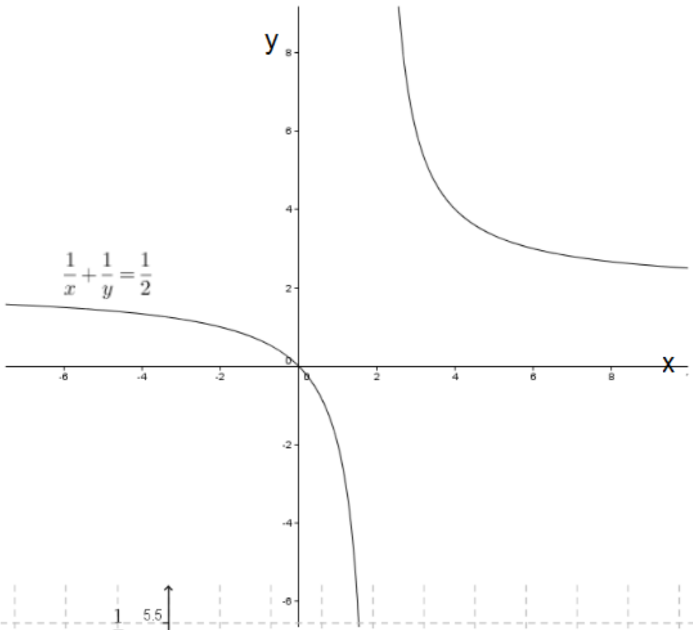


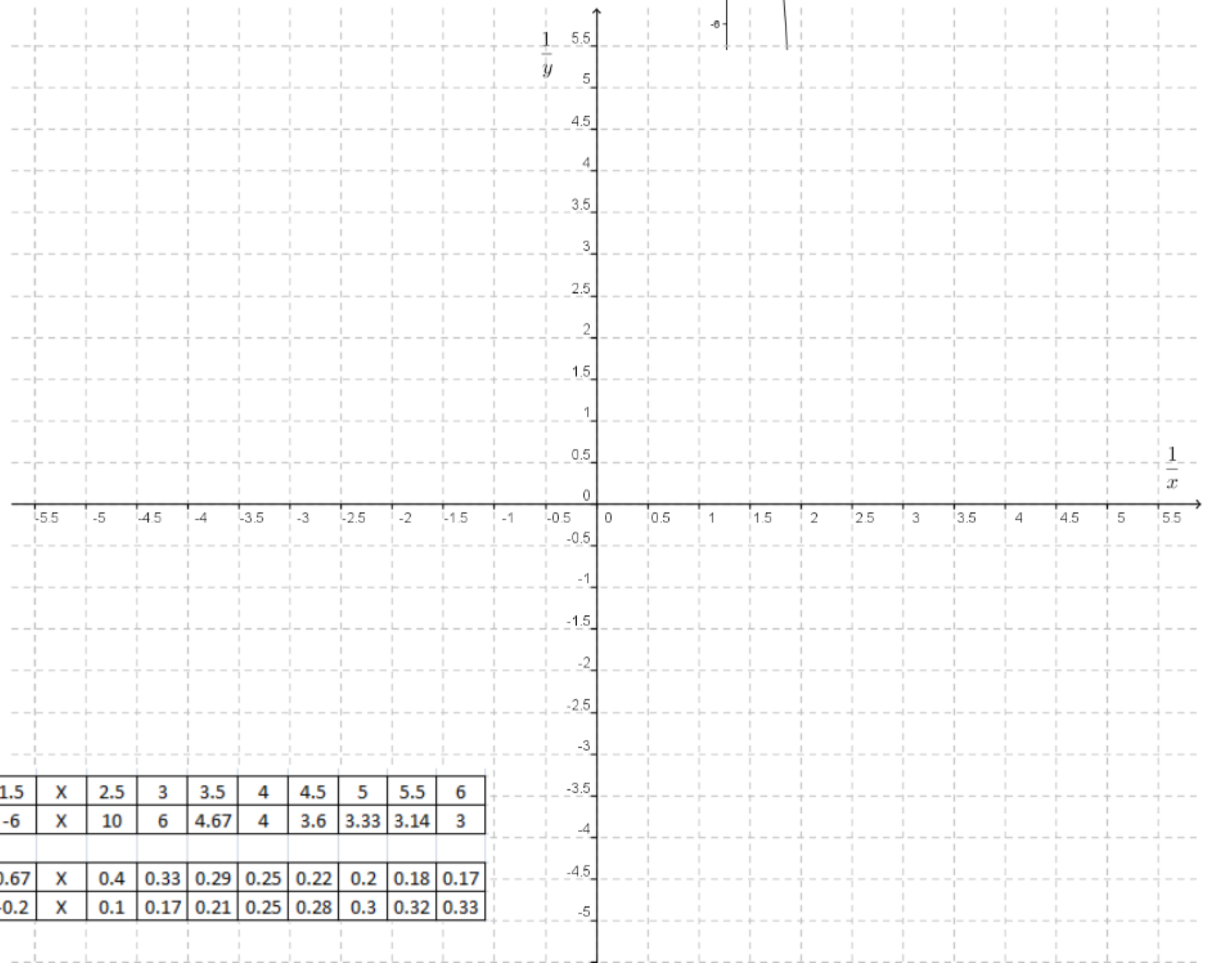
$$L = \frac{1}{4}T^2 + 5$$

If $L = 149, T = 24$

Equations of the form $\dfrac{1}{y}+\dfrac{1}{x} = a$

Consider the equation $\dfrac{1}{y}+\dfrac{1}{x} = a$





x	-2	-1.5	-1	-0.5	-0.4	-0.3	-0.2	-0.1	X	0.1	0.2	0.3	0.4	0.5	1	1.5	X	2.5	3	3.5	4	4.5	5	5.5	6
y	1	0.86	0.67	0.4	0.33	0.26	0.18	0.1	X	-0.1	-0.2	-0.4	-0.5	-0.7	-2	-6	X	10	6	4.67	4	3.6	3.33	3.14	3

1/x	-0.5	-0.7	-1	-2	-2.5	-3.3	-5	-10	X	10	5	3.33	2.5	2	1	0.67	X	0.4	0.33	0.29	0.25	0.22	0.2	0.18	0.17
1/y	1	1.17	1.5	2.5	3	3.83	5.5	10.5	X	-9.5	-4.5	-2.8	-2	-1.5	-0.5	-0.2	X	0.1	0.17	0.21	0.25	0.28	0.3	0.32	0.33

Summary

To test a belief that the relation between x and y is of the form $\frac{1}{x}+\frac{1}{y}=a$, you need to plot $\frac{1}{y}$ against $\frac{1}{x}$. If the points are roughly in a straight line with gradient -1, you can deduce that the relation between x and y is of the form $\frac{1}{x}+\frac{1}{y}=a$. The intercept on the vertical axis ($X=0$) gives an estimate for a.

if $\frac{1}{y}+\frac{1}{x}=a,$

Let $\frac{1}{y}$ *be* Y *and* $\frac{1}{x}$ *be* X

then you have $Y=-X+a$

Exercise:

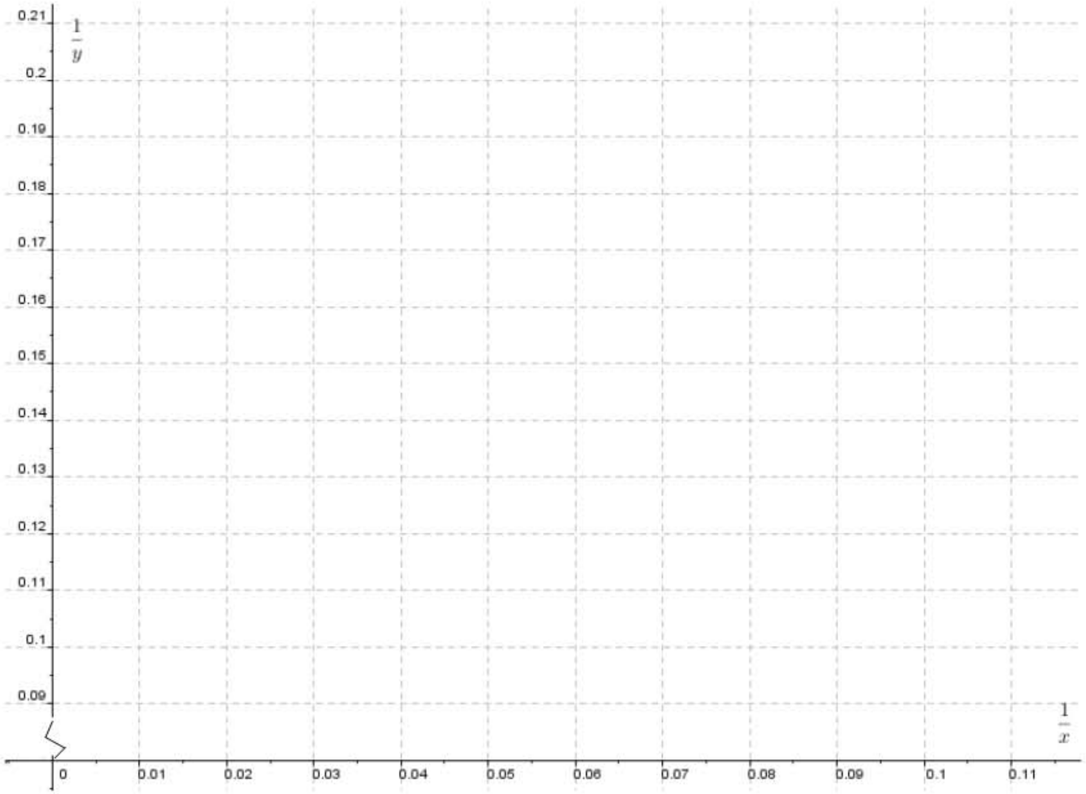
The table shows some experimental values of the variables x and y.

x	10	15	25	40	50
y	11.1	8.1	6.7	6.1	5.9

(a) By plotting $\frac{1}{y}$ against $\frac{1}{x}$ show that these values are consistent with the relation $\frac{1}{x}+\frac{1}{y}=a$, where a is a constant.

(b) Estimate the value of a, giving your answer to two decimal places.

(c) Hence estimate the value of y when $x=0.5$, giving your answer to two decimal places.

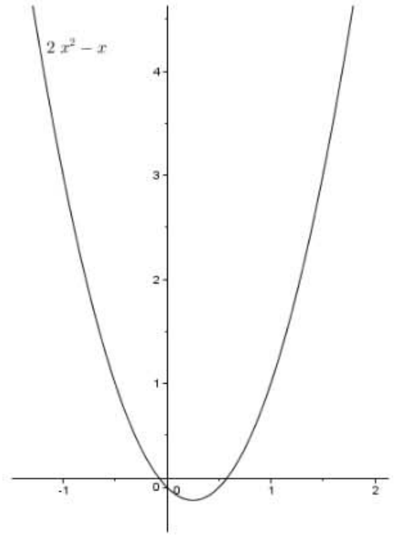


a)

1/x					
1/y					

Equations of the form y=ax²+bx

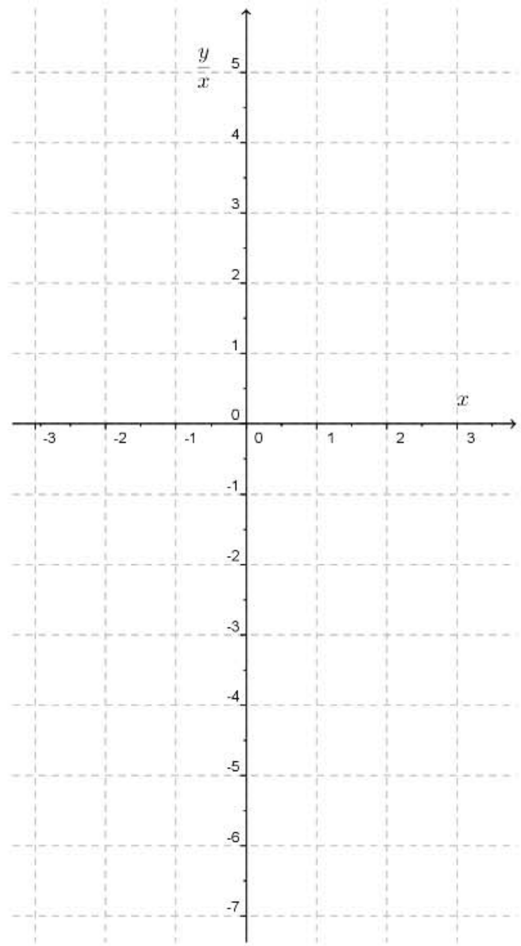
Consider the equation $y = ax^2 + bx$



Numerical example: $y = 2x^2 - x$

x	-3	-2.5	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2	2.5	3
y	21	15	10	6	3	1	0	0	1	3	6	10	15

x	-3	-2.5	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2	2.5	3
y/x													



Summary

To test a belief that the relation between x and y is of the form $y = ax^2 + bx$, you can plot $\frac{y}{x}$ against x. If the points are roughly in a straight line, you can deduce that the relation between x and y is of the form $y = ax^2 + bx$. The gradient of the line gives an estimate for a and the intercept on the vertical axis ($X = 0$) gives an estimate for b.

if $y = ax^2 + bx$,

Let $\frac{y}{x}$ *be* Y *and* x *be* X

then you have $Y = aX + b$

Exercise:

The table shows some experimental values of the variables x and the corresponding values of y.

x	2	4	5	6	8
y	14	32	42	54	80

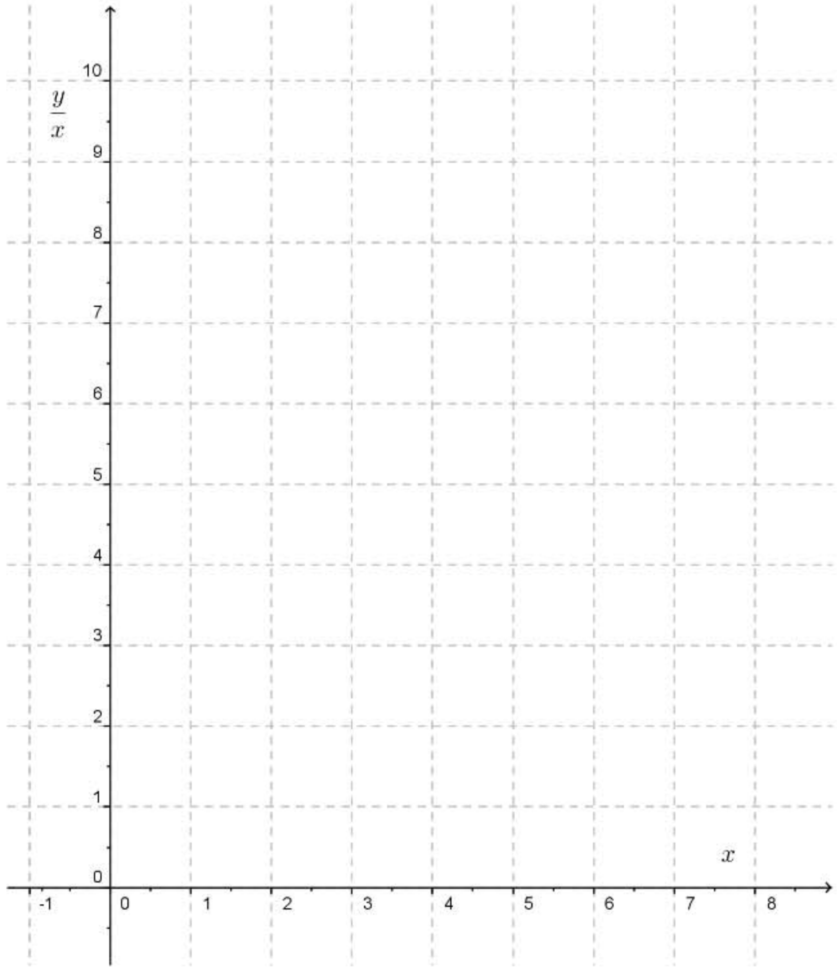
(a) By plotting $\frac{y}{x}$ against x show that these values are consistent with the relation $y = ax^2 + bx$, where a and b are constants.

(b) Draw a suitable straight line to illustrate the relation and use your line to estimate the value of y when $x = 7$.

(c) Estimate the values of a and b, giving your answers to one decimal place.

a)

x	2	4	5	6	8
y/x					



Other equations

a) $ay^2 = x - b$

b) $y^3 = ax^2 + bx$

For each of the following relations between the variables x and y, find possible variables which can be plotted to obtain a straight line graph and explain how the graph can be used to estimate the value of the constant a and the value of the constant b:

(a) $y = ax^3 + b$ **(b)** $y = a + b\sqrt{x}$ **(c)** $y^2 = ax + b$

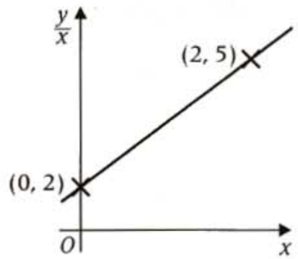
(d) $\dfrac{1}{y} = a + \dfrac{b}{\sqrt{x}}$ **(e)** $y = ax^3 + bx$ **(f)** $y = ax + by^2$

(g) $y = \dfrac{a}{x} + bx$ **(h)** $xy = ax^2 + b$ **(i)** $y^2 - ay + bx^2 = 0$

answers p12

Exercises

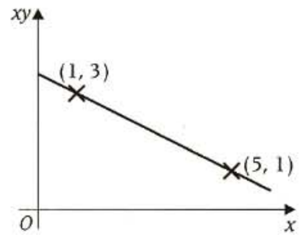
The diagram shows a straight line graph of $\frac{y}{x}$ against x passing through the points (0, 2) and (2, 5).



(a) Express y in the form $ax^2 + bx$, where a and b are constants to be found.

(b) Hence verify that $y = 47.5$ when $x = 5$.

5 The diagram shows a straight line graph of xy against x passing through the points (1, 3) and (5, 1).



(a) Express y in the form $\frac{a}{x} + b$, where a and b are constants to be found.

(b) Hence verify that $y = -0.43$ when $x = 50$.

3 It is assumed that x and y are related by a law of the form $y = a + bx^2$, where a and b are constants. Experimental measurements of x and y are taken to give the following pairs of values:

x	8	10	12	14	16
y	40	60	82	108	138

(a) By means of a straight line graph verify that the law is valid.

(b) Use your graph to estimate approximate values for a and b.

6 The variables x and y are known to satisfy an equation of the form $y = a + b\sqrt{x}$, where a and b are constants. Corresponding approximate values of x and y (each rounded to one decimal place) were obtained experimentally and are given in the following table.

x	3.2	6.8	16.0	25.2	33.6	40.4
y	4.0	5.0	6.6	8.2	9.0	9.8

By drawing a suitable linear graph, estimate the values of a and b, giving both answers to one decimal place. [A]

4 The table shows corresponding values of the variables x and y obtained in an experiment.

x	2.5	3	3.5	4	4.5
y	1.020	0.955	0.914	0.885	0.864

(a) Draw a straight line graph to verify that x and y are approximately connected by a relation of the form $\frac{1}{x} + \frac{1}{y} = a$, where a is a constant.

(b) Use your graph to estimate the value, to two decimal places, of
(i) x when $y = 0.9$, **(ii)** a.

Answers of page 10 exercise

(Note that there are alternative correct answers.)

(a) Plot y against x^3, gradient of line gives a, intercept on y-axis gives b;

(b) Plot y against $\sqrt{x}$, gradient of line gives b, intercept on y-axis gives a;

(c) Plot y^2 against x, gradient of line gives a, intercept on y^2-axis gives b;

(d) Plot $\frac{1}{y}$ against $\frac{1}{\sqrt{x}}$, gradient of line gives b, intercept on $\frac{1}{y}$-axis gives a;

(e) Plot $\frac{y}{x}$ against x^2, gradient of line gives a, intercept on $\frac{y}{x}$-axis gives b;

(f) Plot $\frac{y}{x}$ against $\frac{y^2}{x}$, gradient of line gives b, intercept on $\frac{y}{x}$-axis gives a;

(g) Plot $\frac{y}{x}$ against $\frac{1}{x^2}$, gradient of line gives a, intercept on $\frac{y}{x}$-axis gives b;

(h) Plot xy against x^2, gradient of line gives a, intercept on xy-axis gives b;

(i) Plot y against $\frac{x^2}{y}$ gradient of line gives $-b$, intercept on y-axis gives a.

2) $y = \frac{3}{2}x^2 + 2x$ 3) b) $a \approx 8$; $b \approx \frac{1}{2}$

4) b) i) ≈ 3.72 ii) ≈ 1.38 5) a) $y = \frac{3.5}{x} - 0.5$

6) $a \approx 1.7$; $b \approx 1.3$

Using the LOG function

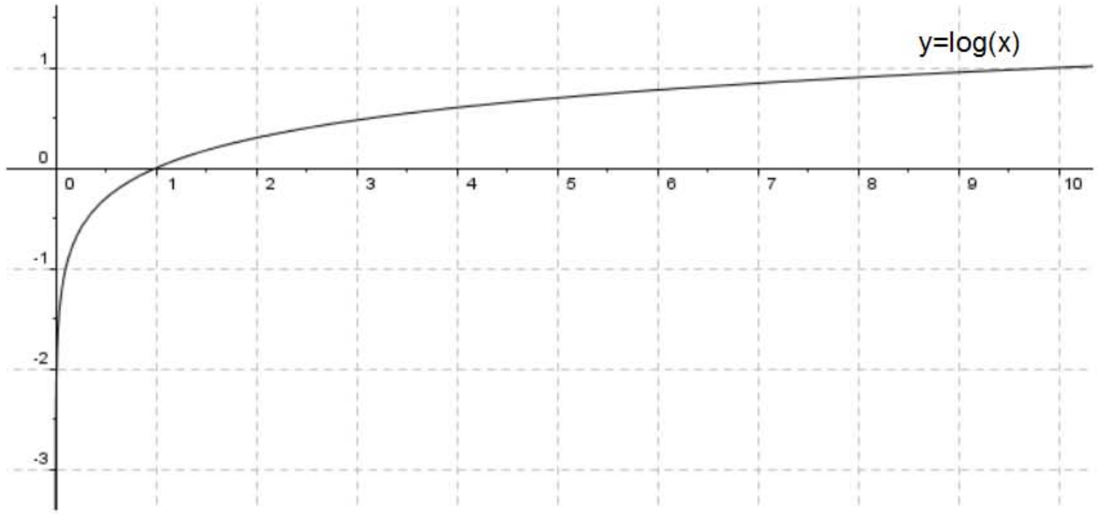
Property of the log function

- $\log(x)$ exists for $x > 0$.
- *for* $a > 0$ *and* $b > 0$,

$$\log(a \times b) = \log(a) + \log(b)$$

$$\log\left(\frac{a}{b}\right) = \log(a) - \log(b)$$

$$\log(a^b) = b \times \log(a)$$



In this chapter, we usually use the $\log_{10}$ (log *of base* 10) to change the variables.

log is a function on the calculator

Equations of the form y=ab^x

Consider the equation y = ab^x
Now take the log of both sides :

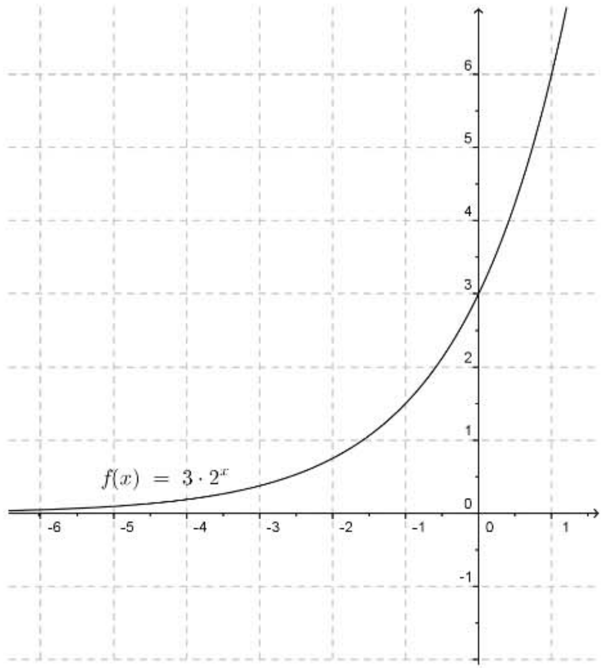
$$y = ab^x$$

Composing by log

$$\log y = \log(ab^x)$$

$$\log y = x\log(b) + \log(a)$$

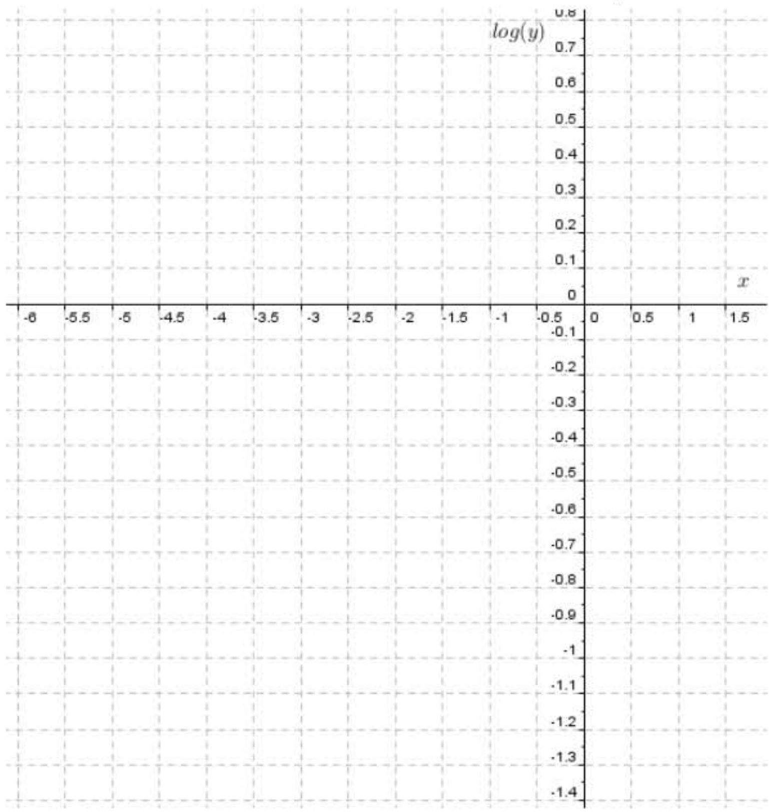
$$Y \;\; = m\,x + c \qquad with\ m = \log(b)\ and\ c = \log(a)$$



Numerical example: y = 3×2^x

x	-6	-5.5	-5	-4.5	-4	-3.5	-3	-2.5	-2	-1.5	-1	-0.5	0	0.5	1
y	0.047	0.066	0.094	0.133	0.188	0.27	0.38	0.53	0.75	1.06	1.5	2.12	3	4.24	6

x	-6	-5.5	-5	-4.5	-4	-3.5	-3	-2.5	-2	-1.5	-1	-0.5	0	0.5	1
log(y)															



Summary

Taking logarithms of both sides of
$y = ab^x \quad \Rightarrow \quad \log y = \log a + x \log b$.

To represent the relation $y = ab^x$ in a linear form you need to plot $\log y$ against x. If a straight line is obtained from the given data the relation is true. The gradient of the line is the value of $\log b$ and the intercept on the vertical axis ($X = 0$) gives the value of $\log a$. Knowing these two values, estimates for a and b can be found.

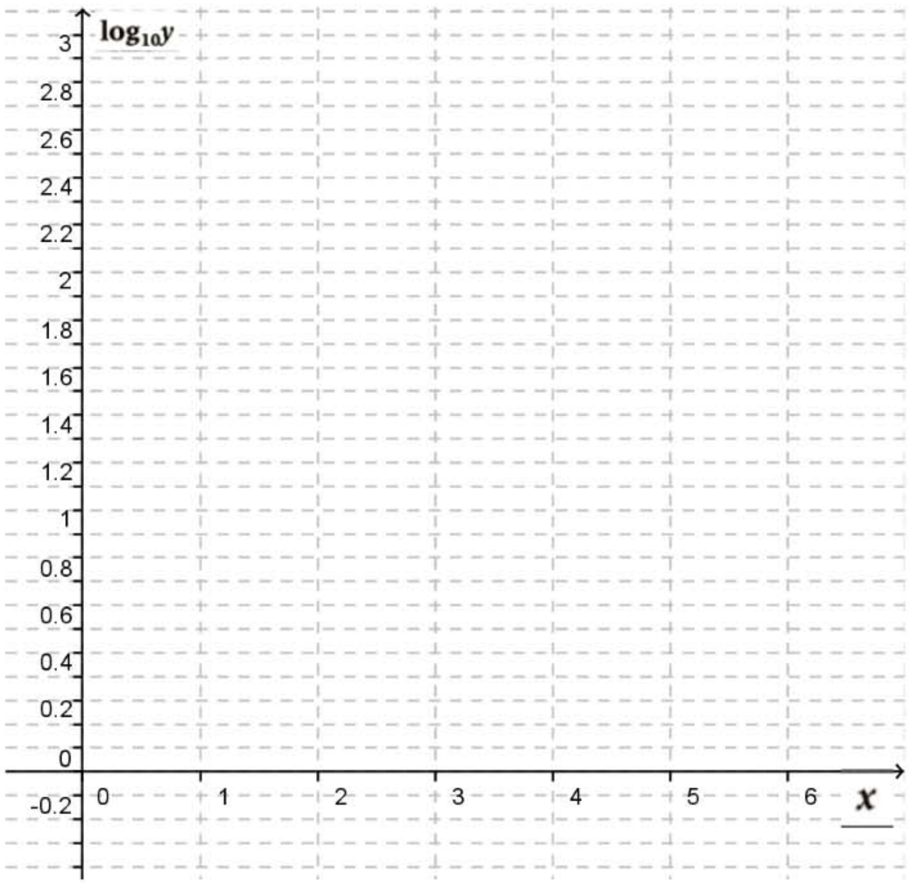
Exercise:

The data for x and y as given in the table below are related approximately by a law of the form $ky = h^x$, where h and k are constants.

x	1	2	3	4	5	6
y	17	49	110	330	810	2200

By drawing a suitable graph find estimates, to two significant figures, for h and k. [A]

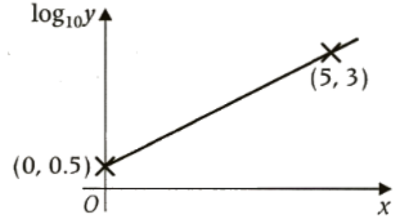
x	1	2	3	4	5	6
$\log_{10} y$						



. $h = 2.6$ and $k = 0.14$.

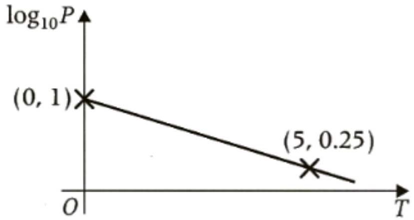
Exercises:

1 The diagram shows a straight line graph of $\log_{10}y$ against x passing through the points (0, 0.5) and (5, 3).



Express y in the form ab^x, where a and b are constants to be found.

2 The diagram shows a straight line graph of $\log_{10}P$ against T passing through the points (0, 1) and (5, 0.25).



Express P in the form ab^T, where a and b are constants to be found.

3 T is thought to relate L by a law of the form $T = ka^L$, where k and a are constants.

(a) Express $\log_{10}T$ in terms of L, $\log_{10}k$ and $\log_{10}a$.

The pairs of values of L and T are:

L	1	2	3	4	5
T	13.95	43.24	134.06	415.58	1288.31

(b) Plot $\log_{10}T$ against L on graph paper and hence draw a suitable straight line to illustrate the relation between the data.

(c) Use your line to estimate, to two significant figures:
(i) the value of T when $L = 2.5$,
(ii) the values of k and a.

4 The data for x and y, as given in the table below, are related approximately by a law of the form $y = ab^x$, where a and b are constants.

x	0.5	1	1.5	2	2.5
y	10.6	15.0	21.2	30.0	42.4

By drawing a suitable graph find estimates, to two significant figures, for a and b.

5 The data for x and y, as given in the table below, are related approximately by a law of the form $ky = h^x$, where h and k are constants.

x	1	2	3	4	5	6
y	0.96	2.30	5.53	13.27	31.85	76.44

By drawing a suitable graph find estimates, to two significant figures, for h and k.

6 The data for x and y, as given in the table below, are related approximately by a law of the form $y = pq^{-x}$, where p and q are constants.

x	1	2	3	4	5
y	15.0	9.38	5.86	3.66	2.29

By drawing a suitable graph find estimates, to two significant figures, for p and q.

1 $y \approx 3.16x^{3.16}$. **2** $P \approx 10 \times 0.708^T$.
3 (a) $\log_{10}T = \log_{10}k + L\log_{10}a$; **(c) (i)** 83, **(ii)** $a = 3.2$, $k \approx 4.7$.
4 $a \approx 7.5$, $b \approx 2.0$.
5 $h \approx 2.4$, $k \approx 2.5$.
6 $p \approx 24$, $q \approx 1.6$.

Equations of the form y=ax^n

Consider the equation y = ax^n
Now take the log of both sides :

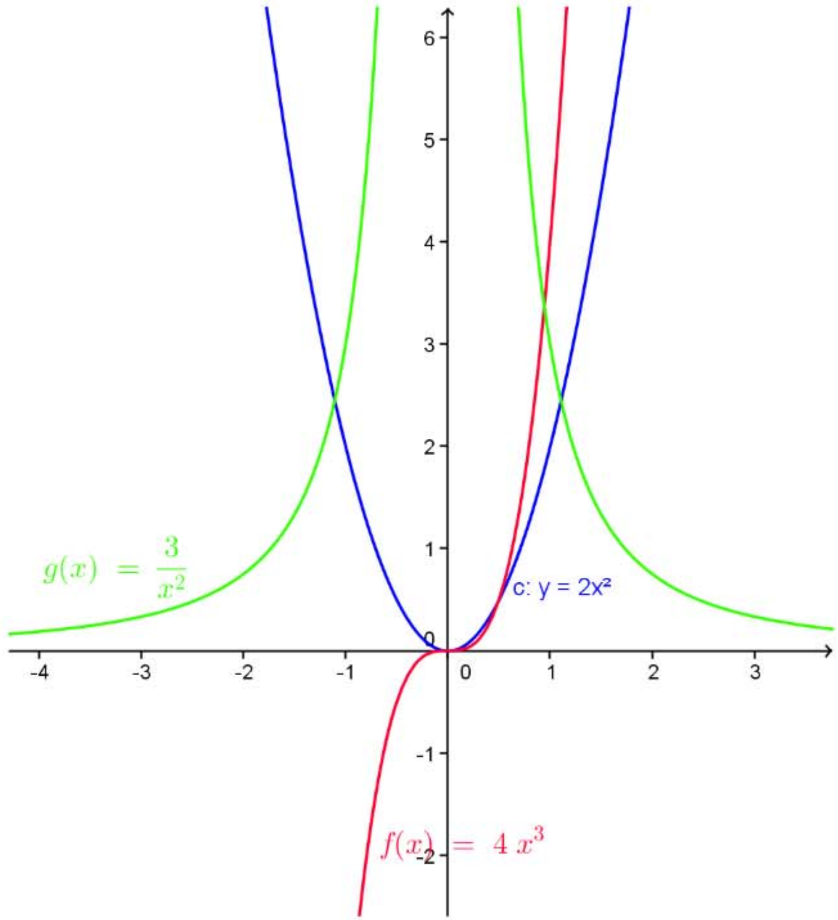
$y = ax^n$

Composing by log

$\log y = \log(ax^n)$

$\log y = n\log(x) + \log(a)$

$Y \quad = nX + c \qquad\qquad with\ c = \log(a)$



Taking logarithms of both sides of $y = ax^n \Rightarrow \log y = \log a + n \log x$.

To test a belief that the relation between x and y is of the form $y = ax^n$, you need to plot $\log y$ against $\log x$. If the points are roughly in a straight line, you can deduce that the relation between x and y is of the form $y = ax^n$. The gradient of the line gives an estimate for n and the intercept on the vertical axis ($X = 0$) gives the value of $\log a$ from which the estimate for a can be found.

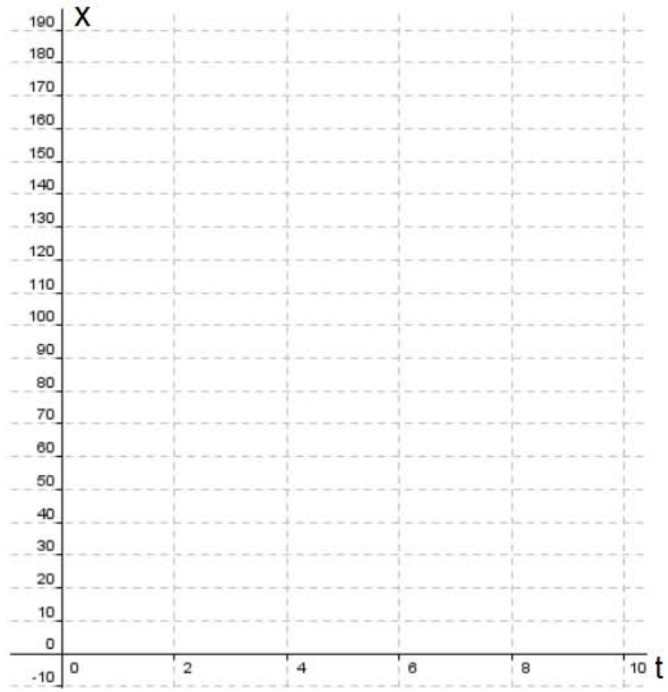
Example:

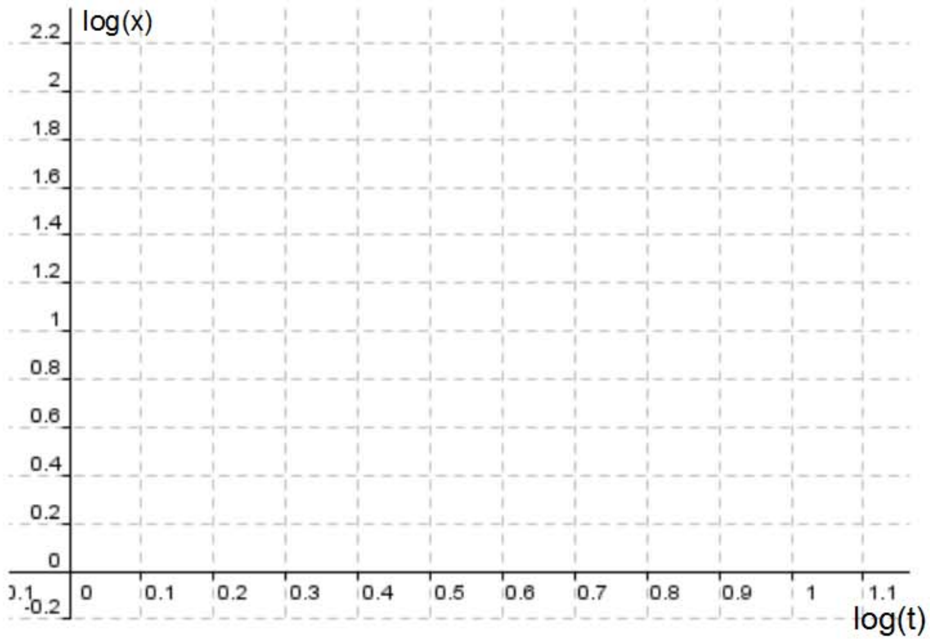
The corresponding values of two variables x and t found by experiment are:

t	2	4	6	8	10
x	3.16	17.9	49.3	101.2	176.8

By drawing a suitable linear graph, verify that the values of t and x, approximately satisfy a relation of the form $x = at^n$. Use your graph to estimate values of the constants a and n giving your answers to two significant figures.

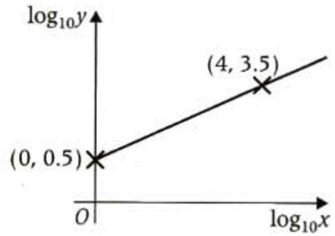
log(t)					
log(x)					





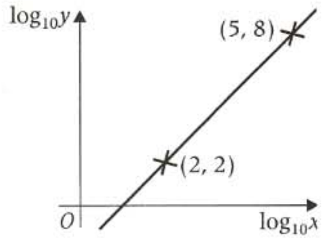
Exercises:

1 The diagram shows a straight line graph of $\log_{10}y$ against $\log_{10}x$ passing through the points (0, 0.5) and (4, 3.5).



Express y in the form ax^b, where a and b are constants to be found.

2 The diagram shows a straight line graph of $\log_{10}y$ against $\log_{10}x$ passing through the points (2, 2) and (5, 8).



Express y in the form ax^b, where a and b are constants to be found.

3 The corresponding pairs of values of two variables y and x found by experiment are:

x	2	3	4	5	6
y	9.2	14.9	21.1	27.6	34.4

By drawing a suitable linear graph, verify that the values of x and y, approximately satisfy a relation of the form $y = ax^b$. Use your graph to estimate values of the constants a and b giving your answers to two significant figures.

4 V is thought to relate T by a law of the form $V = aT^{-n}$, where a and n are constants.

(a) Express $\log_{10}V$ in terms of n, $\log_{10}T$ and $\log_{10}a$.

Pairs of values of T and V are:

T	10	12	14	16	18
V	0.980	0.895	0.829	0.775	0.731

(b) Plot $\log_{10}V$ against $\log_{10}T$ and hence draw a suitable straight line to illustrate the relation between the data.

(c) Use your line to estimate, to two significant figures:
(i) the value of V when $T = 12.6$,
(ii) the values of a and n.

5 The variables Q and x satisfy a relation of the form $Q = ax^b$, where a and b are constants.

Measurements of Q for given values of x gave the following results:

x	0.4	0.5	0.6	0.7	0.8
Q	1.72	3.02	4.74	6.98	9.73

(a) Express $\log_{10}Q$ in terms of $\log_{10}a$, b and $\log_{10}x$.

(b) **(i)** Plot $\log_{10}Q$ against $\log_{10}x$.
(ii) Draw a suitable straight line to illustrate the relation between the data.

(c) Use your line to estimate:
(i) the value of Q when $x = 0.54$, giving your answer to two significant figures,
(ii) the values of a and b, giving your answer to two significant figures. [A adapted]

1 $y = 10x^{0.75}$. **2** $y = 0.01x^2$. **3** $a \approx 4.0$, $b \approx 1.2$.
4 **(a)** $\log_{10}V = \log_{10}a - n\log_{10}T$;
(c) **(i)** 0.87, **(ii)** $a \approx 3.1$, $n \approx 0.50$.
5 **(a)** $\log_{10}Q = \log_{10}a + b\log_{10}x$;
(c) **(i)** 3.6 (or 3.7), **(ii)** $a = 16$ (or 17), $b \approx 2.5$.

Key point summary

1 Since experimental data is not exact, due to measuring errors, points plotted are unlikely to all lie exactly in line so a line of best fit is drawn.

2 $y = mx + c$ is the equation of a straight line with gradient m and y-intercept c.

3 If the variables used are not x and y, the method to find the equation of the line of best fit is exactly the same. In the general equation $y = mx + c$ you just replace y by the variable on the vertical axis and replace x by the variable on the horizontal axis.

4 If the graph does not show the y-intercept, you can find the value of the gradient m as usual and then find the value of c by substituting the coordinates of a point on the line into the equation $y = mx + c$. Alternatively, you can use the coordinates of two points on the line to form and solve a pair of simultaneous equations in m and c.

5 To test a belief that the relation between x and y is of the form $y = ax^2 + b$, you need to plot y against x^2. If the points are roughly in a straight line, you can deduce that the relation between x and y is of the form $y = ax^2 + b$. The gradient of the line gives an estimate for a and the intercept on the vertical axis ($X = 0$) gives an estimate for b.

6 To test a belief that the relation between x and y is of the form $\frac{1}{x} + \frac{1}{y} = a$, you need to plot $\frac{1}{y}$ against $\frac{1}{x}$. If the points are roughly in a straight line with gradient -1, you can deduce that the relation between x and y is of the form $\frac{1}{x} + \frac{1}{y} = a$. The intercept on the vertical axis ($X = 0$) gives an estimate for a.

7 To test a belief that the relation between x and y is of the form $y = ax^2 + bx$, you can plot $\frac{y}{x}$ against x. If the points are roughly in a straight line, you can deduce that the relation between x and y is of the form $y = ax^2 + bx$. The gradient of the line gives an estimate for a and the intercept on the vertical axis ($X = 0$) gives an estimate for b.

8 Taking logarithms of both sides of
$y = ax^n \quad \Rightarrow \quad \log y = \log a + n \log x$.

9 To test a belief that the relation between x and y is of the form $y = ax^n$, you need to plot $\log y$ against $\log x$. If the points are roughly in a straight line, you can deduce that the relation between x and y is of the form $y = ax^n$. The gradient of the line gives an estimate for n and the intercept on the vertical axis ($X = 0$) gives the value of log a from which the estimate for a can be found.

10 Taking logarithms of both sides of
$y = ab^x \quad \Rightarrow \quad \log y = \log a + x \log b$.

11 To represent the relation $y = ab^x$ in a linear form you need to plot $\log y$ against x. If a straight line is obtained from the given data the relation is true. The gradient of the line is the value of $\log b$ and the intercept on the vertical axis ($X = 0$) gives the value of $\log a$. Knowing these two values, estimates for a and b can be found.