Calculus

Specifications

Calculus

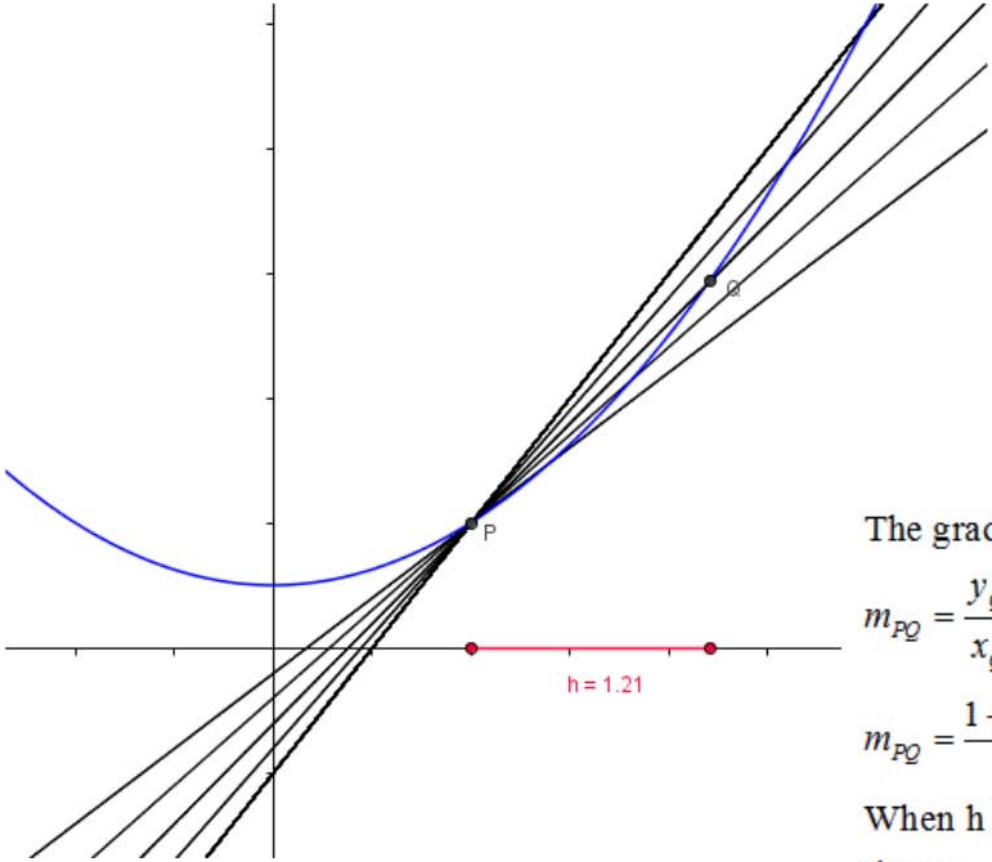
Finding the gradient of the tangent to a curve at a point, by taking the limit as h tends to zero of the gradient of a chord joining two points whose x-coordinates differ by h.

The equation will be given as $y = \mathrm{f}(x)$, where $\mathrm{f}(x)$ is a simple polynomial such as $x^2 - 2x$ or $x^4 + 3$.

Evaluation of simple improper integrals.

E.g. $\int_0^4 \frac{1}{\sqrt{x}}\,\mathrm{d}x$, $\int_4^\infty x^{-\frac{3}{2}}\,\mathrm{d}x$.

From chord to tangent



$f(x)=x^2+1$

The point P(1,2) is on the curve.

Consider the point Q on the curve with coordinates $(1+h, (1+h)^2+1)$

The gradient of the line PQ is

$$m_{PQ} = \frac{y_Q - y_P}{x_Q - x_P} = \frac{(1+h)^2+1-2}{1+h-1}$$

$$m_{PQ} = \frac{1+2h+h^2-1}{h} = \frac{2h+h^2}{h} = 2+h$$

When h tends to 0, meaning that Q gets closer and closer to P

then m_{PQ} tends to 2. The chord PQ tends to the tangent to the curve at P:

The gradient to the tangent at P is 2

or we also say "the gradient to the curve at P is 2".

The gradient of the tangent at P
= the gradient of the curve at P = gradient of PQ when h tends to 0

Another example

1 A curve has equation $y = x^2$. The points P and Q on the curve have coordinates (3, 9) and $(3 + h, (3 + h)^2)$, respectively.

(a) Find the gradient of the chord PQ in as simplified a form as possible.

(b) Hence find the gradient of the curve at the point P.

your go...

2 A curve has equation $y = 3x^2 + 2$. The points S and T on the curve have coordinates (1, 5) and $(1 + h, 3(1 + h)^2 + 2)$, respectively.

(a) Show that the gradient of the chord ST is $6 + 3h$.

(b) Hence find the gradient of the curve at the point S.

3 The points P and Q on the curve with equation $y = x^2 - 2x + 3$ have coordinates (2, 3) and $(2 + h, (2 + h)^2 - 2(2 + h) + 3)$, respectively.

(a) Show that the y-coordinate of Q is $h^2 + 2h + 3$.

(b) Find the gradient of the chord PQ in as simplified a form as possible.

(c) Hence find the gradient of the curve at the point P.

More complicated functions: using the binomial expansion

(core 2 topic)

Expand:

$$(a+b)^2 =$$

$$(a+b)^3 =$$

$$(a+b)^4 =$$

Exercises

1 A curve has equation $y = x^3$. The points P and Q on the curve have coordinates $(2, 8)$ and $(2 + h, (2 + h)^3)$, respectively.

(a) Find the gradient of the chord PQ in as simplified a form as possible.

(b) Hence find the gradient of the curve at the point P.

2 The points C and D on the curve with equation $y = 4x^3 + 7$ have coordinates $(1, 11)$ and $(1 + h, 4(1 + h)^3 + 7)$, respectively.

(a) Show that the gradient of the chord CD is $12 + 12h + 4h^2$.

(b) Hence find the gradient of the curve at the point C.

3 The points P and Q on the curve with equation $y = x^4 - 5x$ have coordinates $(2, 6)$ and $(2 + h, (2 + h)^4 - 5(2 + h))$, respectively.

(a) Show that the y-coordinate of Q is $h^4 + 8h^3 + 24h^2 + 27h + 6$.

(b) Find the gradient of the chord PQ in as simplified a form as possible.

(c) Hence find the gradient of the curve at the point P.

4 A curve has equation $y = 7x^2 - x^3$. The points A and B on the curve have x-coordinates 3 and $3 + h$, respectively.

(a) Find the gradient of the chord AB in as simplified a form as possible.

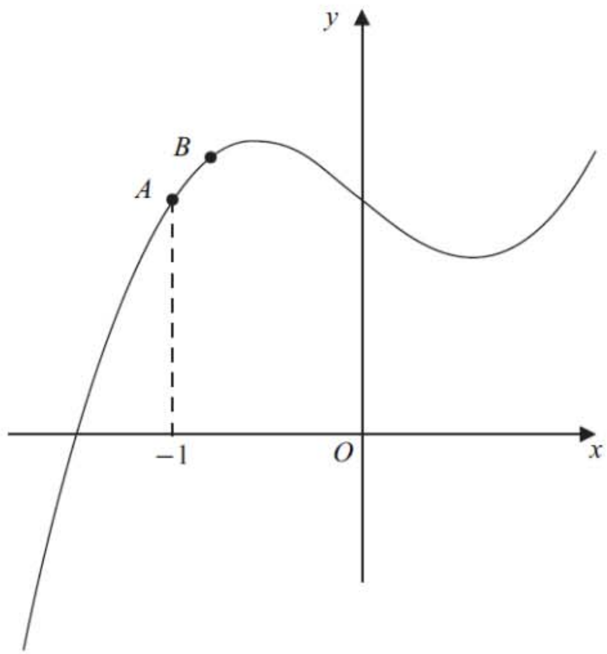
(b) Hence find the gradient of the curve at the point A.

1 **(a)** $h^2 + 6h + 12$; **(b)** 12.
2 **(b)** 12.
3 **(b)** $h^3 + 8h^2 + 24h + 27$; **(c)** 27.
4 **(a)** $-h^2 + 2h + 15$; **(b)** 15.

Exam question

The diagram shows the curve

$$y = x^3 - x + 1$$



The points A and B on the curve have x-coordinates -1 and $-1+h$ respectively.

(a) (i) Show that the y-coordinate of the point B is

$$1 + 2h - 3h^2 + h^3$$

(3 marks)

(ii) Find the gradient of the chord AB in the form

$$p + qh + rh^2$$

where p, q and r are integers. *(3 marks)*

(iii) Explain how your answer to part (a)(ii) can be used to find the gradient of the tangent to the curve at A. State the value of this gradient. *(2 marks)*

Exam question

The quadratic function g is defined for all real numbers by

$$g(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}x - \frac{1}{4}x^2$$

(ii) A chord joins the points on the curve $y = g(x)$ for which $x = 0$ and $x = h$.
Find an expression in terms of h for the gradient of this chord. *(2 marks)*

(iii) Using your answer to part (b)(ii), find the value of $g'(0)$. *(1 mark)*

Exam question

A curve has equation

$$y = x^2 - 6x + 5$$

The points A and B on the curve have x-coordinates 2 and $2 + h$ respectively.

(a) Find, in terms of h, the gradient of the line AB, giving your answer in its simplest form. *(5 marks)*

(b) Explain how the result of part (a) can be used to find the gradient of the curve at A.
State the value of this gradient. *(3 marks)*

Improper integrals with limits involving infinity

<u>Reminder:</u> $$\int x^n dx = \frac{1}{n+1}x^{n+1} + c \qquad for\ all\ \ n \neq -1$$

Find each of the following:

1 $\int 12x^{\frac{1}{2}}\,dx$

2 $\int 20x^{\frac{3}{2}}\,dx$

3 $\int 2x^{-3}\,dx$

4 $\int x^{-7}(4x+6)\,dx$

5 $\int 88x^{-12}\,dx$

6 $\int 20x^{-4}\,dx$

7 $\int 3x^{-2}\,dx$

8 $\int x^{-2}(4+6x^{7})\,dx$

9 $\int \frac{6}{x^4}\,dx$

10 $\int 3\sqrt{x}\,dx$

11 $\int \frac{1}{x^3}\,dx$

12 $\int \frac{4}{x^2}+\frac{6}{x^3}\,dx$

13 $\int \sqrt{x}(3x+5)\,dx$

14 $\int \frac{(6x-5)}{\sqrt{x}}\,dx$

15 $\int x^2\sqrt{x}\,dx$

16 $\int \frac{x^2-1}{\sqrt{x}}\,dx$

17 $\int \frac{(3+2x)^2}{x^4}\,dx$

18 $\int \frac{(2-3x)^2}{\sqrt{x}}\,dx$

19 $\int (\sqrt{x}+3)^2\,dx$

20 $\int (\sqrt{x}-1)^2\,dx$

21 $\int \sqrt{x}(3-x)(x+3)\,dx$

22 $\int (x^3-1)\sqrt[3]{x}\,dx$

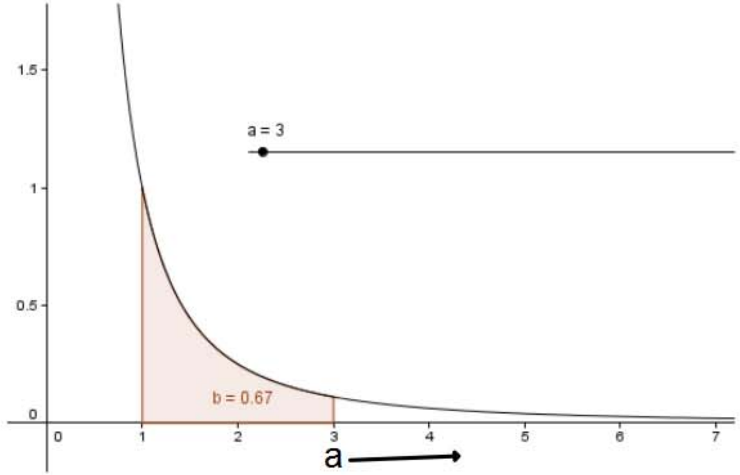
Answers are on the last page

Consider $\int_1^3 \frac{1}{x^2}\,dx$

$$\int_1^3 \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^3 = -\frac{1}{3} + 1 = \frac{2}{3}$$

Now, instead of 3, consider a variable "a".

Let's work out $\int_1^a \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^a = 1 - \frac{1}{a}$



What happens when a becomes "bigger and bigger"

or in more mathematical terms,

what happens when a tends to infinity?

Answer: $\frac{1}{a}$ then tends to 0 and the integral tends to 1.

This integral is called $\boxed{\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{a\to\infty} \int_1^a \frac{1}{x^2}\,dx}$

When f(x) is defined for $x \geqslant a$, we define the improper integral $\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$, provided the limit exists.

When f(x) is defined for $x \leqslant b$, we define the improper integral $\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$, provided the limit exists.

Examples

(a) Find each of the following integrals:

(i) $\int_2^a x^{-2}\,dx$, **(ii)** $\int_1^b x^{-4}\,dx$, **(iii)** $\int_{\frac{1}{2}}^c x^{-2}\,dx$, **(iv)** $\int_1^d x\,dx$.

(b) Hence determine whether each of the following improper integrals exists. If it exists, find its value.

(i) $\int_2^\infty x^{-2}\,dx$, **(ii)** $\int_1^\infty x^{-4}\,dx$, **(iii)** $\int_{\frac{1}{2}}^\infty x^{-2}\,dx$, **(iv)** $\int_1^\infty x\,dx$.

2 **(a)** Find each of the following integrals:

(i) $\int_a^{-3} x^{-2}\,dx$, **(ii)** $\int_b^{-1} x^{-5}\,dx$,

(iii) $\int_c^{-1} x^3\,dx$, **(iv)** $\int_d^{-0.5} x^{-6}\,dx$.

(b) Hence determine whether each of the following improper integrals exists. If it exists, find its value.

(i) $\int_{-\infty}^{-3} x^{-2}\,dx$, **(ii)** $\int_{-\infty}^{-1} x^{-5}\,dx$,

(iii) $\int_{-\infty}^{-1} x^3\,dx$, **(iv)** $\int_{-\infty}^{-0.5} x^{-6}\,dx$.

3 **(a)** Find each of the following integrals:

(i) $\int_a^{9} \frac{1}{\sqrt{x}}\,dx$, **(ii)** $\int_b^{2} x^{-2}\,dx$,

(iii) $\int_c^{8} x^{-\frac{1}{3}}\,dx$, **(iv)** $\int_d^{16} x^{-\frac{1}{4}}\,dx$.

(b) Hence determine whether each of the following improper integrals exists. If it exists, find its value.

(i) $\int_0^{9} \frac{1}{\sqrt{x}}\,dx$, **(ii)** $\int_0^{2} x^{-2}\,dx$,

(iii) $\int_0^{8} x^{-\frac{1}{3}}\,dx$, **(iv)** $\int_0^{16} x^{-\frac{1}{4}}\,dx$.

4 Explain why $\int_0^{81} \frac{1}{\sqrt{x}}\,dx$ is an improper integral and find its value.

5 A student evaluates $\int_{-2}^{a} x^{-2}\,dx$ as $-\frac{1}{a} - \frac{1}{2}$ and concludes that $\int_{-2}^{\infty} x^{-2}\,dx$ is equal to $-\frac{1}{2}$. Explain why she is incorrect.

2 **(a)** **(i)** $\frac{1}{a} + \frac{1}{3}$, **(ii)** $\frac{1}{4b^4} - \frac{1}{4}$, **(iii)** $\frac{1}{4} - \frac{c^4}{4}$, **(iv)** $\frac{1}{5d^5} + \frac{32}{5}$;

(b) **(i)** $\frac{1}{3}$, **(ii)** $\frac{1}{4}$, **(iii)** does not exist, **(iv)** $\frac{32}{5}$.

3 **(a)** **(i)** $6 - 2\sqrt{a}$, **(ii)** $\frac{1}{b} - \frac{1}{2}$, **(iii)** $6 - \frac{3c^{\frac{2}{3}}}{2}$, **(iv)** $\frac{32}{3} - \frac{4d^{\frac{3}{4}}}{3}$;

(b) **(i)** 6, **(ii)** does not exist, **(iii)** 6, **(iv)** $\frac{32}{3}$.

4 $\frac{1}{\sqrt{x}}$ is not defined at lower limit when $x = 0$, 18.

5 $\frac{1}{x^2}$ is not defined when $x = 0$ which is part of the interval of integration.

Key point summary

1 The gradient of the chord PQ is $\dfrac{y_Q - y_P}{x_Q - x_P}$.

2 When the x-coordinate of P is a, and the x-coordinate of Q is $a + h$, the gradient of the chord PQ can be simplified to an expression involving h.
The gradient of the curve at the point P is obtained by letting h tend to zero.

3 When $f(x)$ is defined for $x \geqslant a$, we define the improper integral $\int_a^\infty f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx$, provided the limit exists.

4 When $f(x)$ is defined for $x \leqslant b$, we define the improper integral $\int_{-\infty}^a f(x)\,dx = \lim_{a \to -\infty} \int_a^b f(x)\,dx$, provided the limit exists.

5 When $f(x)$ is defined for $p < x < q$, but $f(x)$ is not defined when $x = p$, then the improper integral $\int_p^q f(x)\,dx = \lim_{a \to p+} \int_a^q f(x)\,dx$, provided the limit exists.

Answers p.9

1 $8x^{\frac{3}{2}} + c.$

2 $8x^{\frac{5}{2}} + c.$

3 $\dfrac{-1}{x^2} + c.$

4 $-\dfrac{4}{5x^5} - \dfrac{1}{x^6} + c.$

5 $-\dfrac{8}{x^{11}} + c.$

6 $\dfrac{-20}{3x^3} + c.$

7 $-3x^{-1} + c.$

8 $-4x^{-1} + x^6 + c.$

9 $-\dfrac{2}{x^3} + c.$

10 $2x^{\frac{3}{2}} + c.$

11 $\dfrac{-1}{2x^2} + c.$

12 $-\dfrac{4}{x} - \dfrac{3}{x^2} + c.$

13 $\dfrac{6}{5}x^{\frac{5}{2}} + \dfrac{10}{3}x^{\frac{3}{2}} + c.$

14 $4x^{\frac{3}{2}} - 10x^{\frac{1}{2}} + c.$

15 $\dfrac{2}{7}x^{\frac{7}{2}} + c.$

16 $\dfrac{2}{5}x^{\frac{5}{2}} - 2x^{\frac{1}{2}} + c.$

17 $-3x^{-3} - 6x^{-2} - 4x^{-1} + c.$

18 $8x^{\frac{1}{2}} - 8x^{\frac{3}{2}} + \dfrac{18}{5}x^{\frac{5}{2}} + c.$

19 $\dfrac{x^2}{2} + 4x\sqrt{x} + 9x + c.$

20 $\dfrac{x^2}{2} - \dfrac{4}{3}x^{\frac{3}{2}} + x + c.$

21 $6x^{\frac{3}{2}} - \dfrac{2}{7}x^{\frac{7}{2}} + c.$

22 $\dfrac{3}{13}x^{\frac{13}{3}} - \dfrac{3}{4}x^{\frac{4}{3}} + c.$