SMP 16-19 Mathematics – Revision Notes
Unit 5 – Calculus Methods

Parameters

1. The equations $x = 2t$ and $y = t$ are called parametric equations, and the time *t*, which determines the *x*- and *y*-coordinates, is a parameter.
2. For a circles and ellipses:

	Circle	**Ellipse**
Cartesian	$x^2 + y^2 = r^2$	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
Parametric	$x = r\cos\theta \quad y = r\sin\theta$	$x = a\cos\theta \quad y = b\sin\theta$
Area	πr^2	πab

3. The reciprocals of trigonometric functions are represented as:

$$\sec\theta = \frac{1}{\cos\theta} \quad \operatorname{cosec}\theta = \frac{1}{\sin\theta} \quad \cot\theta = \frac{1}{\tan\theta}$$

4. The following trigonometric identities are true, for any value of θ:

$$\cos^2\theta + \sin^2\theta = 1$$
$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \cot^2\theta = \operatorname{cosec}^2\theta$$

5. Parametric equations can be converted into cartesian equations by expressing both equations equal to the same thing, and then substituting.
6. Parametric equations can be differentiated using the chain rule, whereby $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$.
7. If a moving particle has position vector $\mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix}$ then its velocity vector $\mathbf{v} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$. The gradient of the velocity vector will be given by $\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}$.

Product Rule

1. For general functions *u* and *v*, then the product rule gives:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

2. For general functions *u* and *v*, then the quotient rule gives:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

3. For implicit differentiation, differentiate each term independently, then rearrange to give $\frac{dy}{dx}$:
   a. For a term $f(y)$, then with the chain rule $\frac{d}{dx}\big(f(y)\big) = \frac{d}{dy}\big(f(y)\big) \times \frac{dy}{dx}$.
   b. For a term, $f(y) \times f(x)$, then use the product rule, with $u = f(y)$ and $v = f(x)$.

Volume

1. If $A(h)$ is the cross-sectional area of a solid at height h, the volume is given by $V = \int A(h)\, dh$.
2. For volumes of revolution of $y = f(x)$ between two points:
   a. About the x-axis, $V = \pi \int_a^b y^2\, dx$.
   b. About the y-axis, $V = \pi \int_c^d x^2\, dy$.
3. Integration by inspection can be used to integrate, by differentiating the 'main part' of the function to begin with. Compare this to what you are trying to integrate, and modify your answer accordingly.
4. To integrate trigonometric functions, the sum and difference trigonometric identities can be used:

$$2\cos A \cos B = \cos(A+B) + \cos(A-B)$$
$$2\sin A \sin B = -\cos(A+B) + \cos(A-B)$$
$$2\sin A \cos B = \sin(A+B) + \sin(A-B)$$

Integration Techniques

1. For integrating some products of functions (where one is x or x^2), integration by parts can be used:

$$\int u \frac{dv}{dx}\, dx = uv - \int v \frac{du}{dx}\, dx$$

2. For integrating some functions of functions, integration by substitution can be used. This means changing a function $f(x)$ to u, and changing the integral by replacing dx with $\frac{dx}{du}\, du$.
3. If the graph of $f(x)$ is continuous between a and b then $\int_a^b \frac{f'(x)}{f(x)}\, dx = \left[\ln|f(x)|\right]_a^b$.
4. For more complex fractions, they can be separated into simpler fractions using the technique of partial fractions:
   a. For $\frac{A}{f(x)} + \frac{B}{g(x)}$, first cross-multiply, then eliminate A and B in turn to find the other.
   b. For $A + \frac{B}{f(x)}$, separate the numerator to give a fraction that can be evaluated to give A.

Polynomial Approximations And First Principles

1. Taylor's first approximation gives a linear approximation to a graph at a specific point, by finding the equation of the tangent to the graph at that point.
2. For small x: $\sin x \approx x$, $\cos x \approx 1$ and $\tan x \approx x$.
3. The Newton-Raphson method gives a sequence of numbers that converges to a zero of $f(x)$:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

4. The notation $f^{(n)}(x)$ means that $f(x)$ is differentiated n times, i.e. it is $\frac{d^n y}{dx^n}$.
5. Maclaurin's series gives a polynomial approximation to a function close to $x = 0$:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \ldots + \frac{f^{(n)}(0)x^n}{n!}$$

6. The notation $\lim_{x \to 0}(f(x)) = a$ is used to say that $f(x)$ tends towards a when x tends towards 0.
7. For any function, first principles states that $f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$.