Factor and Remainder Theorem

Simple algebraic division; use of the Factor Theorem and the Remainder Theorem. Only division by (x + a) or (x – a) will be required.
Candidates should know that if f(x) = 0 when x = a, then (x – a) is a factor of f(x).
Candidates may be required to factorise cubic expressions such as x^3+3x^2-4 *and* $6x^3+11x^2-x-6$. *Candidates should be familiar with the terms 'quotient' and 'remainder' and be able to determine the remainder when the polynomial f(x) is divided by (ax + b).*

Factor Theorem

We need to be able to find a, b and c in the following:
$2x^3+x^2+5x-8=(x-1)(ax^2+bx+c)$.

We do this by comparing coefficients.

When we multiply out $(x-1)(ax^2+bx+c)$ we can see that the x^3 coefficient is *a*. Hence $a=2$.

When we multiply out $(x-1)(2x^2+bx+c)$ we can see that the constant coefficient is -*c*. Hence $c=8$.

When we multiply out $(x-1)(2x^2+bx+8)$ we can see that the x^2 coefficient is $b-2$. Hence $b-2=1$ and so $b=3$.

Hence we see that $2x^3+x^2+5x-8=(x-1)(2x^2+3x+8)$ (note that the x coefficient of $(x-1)(2x^2+3x+8)$ is $8-3=5$ as required).

When we write $2x^3+x^2+5x-8=(x-1)(2x^2+3x+8)$ we see that $(x-1)$ is a factor of $2x^3+x^2+5x-8$. If we used the notation $\mathrm{f}(x)\equiv 2x^3+x^2+5x-8$ we could then write $\mathrm{f}(x)\equiv(x-1)(2x^2+3x+8)$.

It follows from this that $\mathrm{f}(1)=(1-1)(2\times1^2+3\times1+8)=0\times13=0$. So we see that if $(x-1)$ is a factor of $\mathrm{f}(x)$ then $\mathrm{f}(1)=0$. In fact the following is true:

$(x-a)$ is a factor of $\mathrm{f}(x)$ if and only if $\mathrm{f}(a)=0$

We can use this to find factors of polynomials. We simply calculate $\mathrm{f}(1)$, $\mathrm{f}(2)$ etc. until we get a value of zero.

For example if we were asked to factorise $\mathrm{f}(x)=x^3+6x^2+11x+6$ we could calculate $\mathrm{f}(-1)=-1+6-11+6=0$ and so we deduce, from the Factor Theorem, that $(x+1)$ is a factor of $\mathrm{f}(x)$. We then use the above method to write $\mathrm{f}(x)=(x+1)(x^2+5x+6)$. We then need to factorise the quadratic (if possible). In this case it does factorise to give $\mathrm{f}(x)=(x+1)(x+2)(x+3)$.

Example
Find a if $(x-2)$ is a factor of $\mathrm{f}(x)=x^3+ax^2+5x+6$.

$(x-2)$ is a factor so $\mathrm{f}(2)=0$ (by Factor Theorem)
So $\mathrm{f}(2)=8+4a+10+6=4a+24=0$, hence $a=-6$.

Remainder Theorem

When the function $\mathrm{f}(x)$ is divided by $(x+a)$ the remainder is $\mathrm{f}(-a)$.

When the function $\mathrm{f}(x)$ is divided by $(ax+b)$ the remainder is $\mathrm{f}\left(-\frac{b}{a}\right)$.

NB: If $(ax+b)$ is a factor of $\mathrm{f}(x)$ then the remainder is zero, that is, from the above result, $\mathrm{f}\left(-\frac{b}{a}\right)=0$. We know this result already from the factor theorem.

When we write $\mathrm{f}(x)$ in the form $\mathrm{f}(x)\equiv(x-a)g(x)+r$ we say that $\mathrm{g}(x)$ is the **quotient** and r is the **remainder**.

e.g. Find the remainder when $2x^3-3x^2-12x+2$ is divided by $2x-7$.

If we let $\mathrm{f}(x)=2x^3-3x^2-12x+2$ then the remainder we want is $\mathrm{f}\left(\frac{7}{2}\right)$.

We should proceed as follows:

Type in the following:

7	÷	2	SHIFT	STO	)

OR

7	÷	2	STO →	X, T

You should now see the following display: $7\div2\rightarrow X$ or $7/2\rightarrow X$

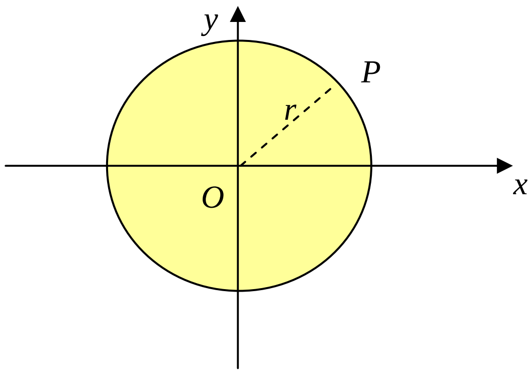
Then type in $2x^3-3x^2-12x+2$ to get out a value of 9.

So we see that $\mathrm{f}\left(\frac{7}{2}\right)=2\left(\frac{7}{2}\right)^3-3\left(\frac{7}{2}\right)^2-12\left(\frac{7}{2}\right)+2=9$.

So the remainder when $2x^3-3x^2-12x+2$ is divided by $2x-7$ is 9.
We can also see that $2x^3-3x^2-12x+2=(2x-7)(x^2+2x+1)+9$

Coordinate geometry in the (x, y) plane

Coordinate geometry of the circle using the equation of a circle in the form $(x-a)^2+(y-b)^2=r^2$… Candidates should be able to find the radius and the coordinates of the centre of the circle given the equation of the circle, and vice versa.



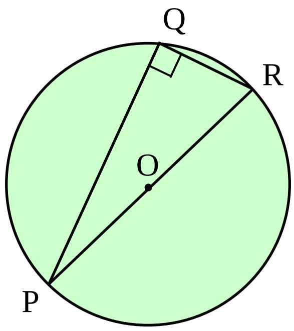
Consider the circle shown above with radius *r*. If we look at the point P we see that, from Pythagoras $x^2+y^2=r^2$. This is the equation of a circle centre the origin, radius *r*.

If we moved the centre of the circle to (a, b) then the equation would be $(x-a)^2+(y-b)^2=r^2$.
For example, find the centre and radius of the circle given by $x^2+y^2-4x+6y-3=0$

$$x^2+y^2-4x+6y-3=0$$
$$\Rightarrow (x-2)^2-4+(y+3)^2-9-3=0$$
$$\Rightarrow (x-2)^2+(y+3)^2=16$$

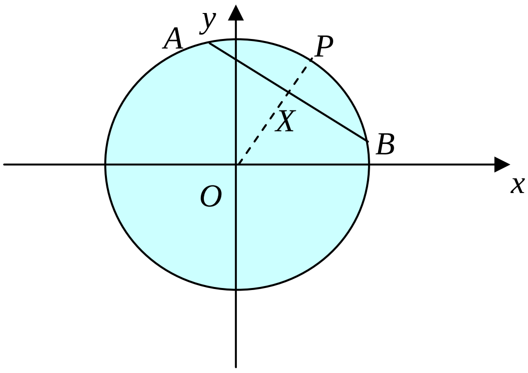
So the centre is (2, -3) and the radius is 4.

…and including use of the following circle properties: (i) the angle in a semicircle is a right angle;



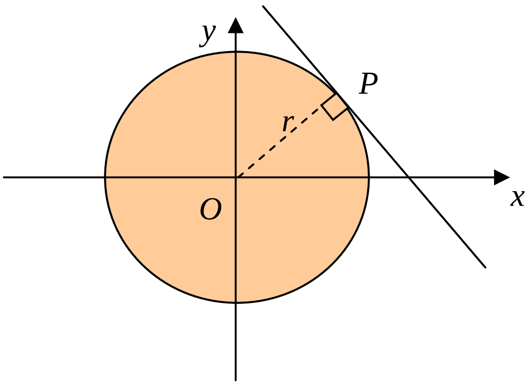
(ii) the perpendicular from the centre to a chord bisects the chord

If any chord is drawn on the circle then the perpendicular from the centre of the circle to the chord bisects the chord



This means that if we draw the chord AB on the circle and then draw the line through O perpendicular to AB, such that it hits AB at X. It follows that AX=XB.

(iii) the perpendicularity of radius and tangent.



In the diagram the tangent to the circle at P has been drawn. The line OP is at right angles to this tangent.

Geometric Series

The sum of a finite geometric series; the sum to infinity of a convergent geometric series, including the use of $|r|<1$ *.Geometric series. The general term and the sum to n terms are required.*

We have seen that an *arithmetic sequence* is a sequence in which the difference between consecutive terms is constant. A *geometric sequence* is a sequence in which the *ratio* between consecutive terms is constant. (e.g. 3, 6, 12, 24, 48, …).

As before the first term is represented by *a* and the common ratio is represented by *r*.

In the example 3, 6, 12, 24, 48, … we could write it as $3, 3\times 2, 3\times 2^2, 3\times 2^3, 3\times 2^4....$.
In fact we could write it as $3\times 2^0, 3\times 2^1, 3\times 2^2, 3\times 2^3, 3\times 2^4....$

When we write it in this was we see that the *n*th term is $u_n = 3\times 2^{n-1}$.

In general if the first term is *a* and the common ratio is *r* then the first few terms of the sequence will be $a, ar, ar^2, ar^3....$ We can see from this that the *n*th term is $u_n = ar^{n-1}$.

Suppose that *S* is the sum of the first *n* terms of the sequence whose *n*th term is $u_n = ar^{n-1}$.

We can write $S_n = a + ar + ar^2 + ... + ar^{n-1}$

If we multiply both sides by *r* we get $rS_n = ar + ar^2 + ar^3 + ... + ar^n$

If we now subtract rS_n from *S* we get

$$S_n - rS_n = \left(a + ar + ar^2 + ... + ar^{n-1}\right) - \left(ar + ar^2 + ar^3 + ... + ar^n\right)$$
$$= a - ar^n$$

Hence we see that $S_n(1-r) = a(1-r^n)$ and so $S_n = \frac{a(1-r^n)}{(1-r)}$.

The sum of the first *n* terms is $S_n = \frac{a(1-r^n)}{(1-r)}$. Alternatively we could have got $S_n = \frac{a(r^n-1)}{(r-1)}$

Example 1

Find the sum of the first 10 terms of the sequence 5, 15, 45, 135,….

The first term, *a*, is 5 and the common ratio, *r*, is 3.

So, using the formula $S = \frac{a(1-r^n)}{(1-r)}$ we have $S = \frac{5(1-3^{10})}{(1-3)} = 147620$

Example 2

Find the sum of the first 10 terms of the sequence 40, 20, 10, 5,….

The first term, *a*, is 40 and the common ratio, *r*, is ½ .

So, using the formula $S = \frac{a\left(1-r^n\right)}{\left(1-r\right)}$ we have $S = \frac{40\left(1-\left(\frac{1}{2}\right)^{10}\right)}{\left(1-\frac{1}{2}\right)} = 80$ (to 2sf).

What would have happened if we had looked at the first 100 terms instead of the first 10 terms in the two examples we have just looked at?

In example 1 the value of *S* would have been very large.
In example 2 the value of *S* would have got much closer to 80.

The sum to infinity of a convergent geometric series.

In example 1 we could not have found the sum of an infinite number of terms since this would itself have been infinite. However in example 2 we could have found the sum of an infinite number of terms and the answer would have been 80.

What determines whether or not the sum of an infinite number of terms can be found or not? Look again at the formula $S = \frac{a\left(1-r^n\right)}{\left(1-r\right)}$.

The key part of this formula when we are considering an infinite number of terms is r^n.

If $r = 3$ (as in example 1) then r^n gets very large as *n* gets large.

If $r = \frac{1}{2}$ (as in example 2) then r^n tends towards zero as *n* gets large.

The rule is as follows..

If $-1 < r < 1$ then $r^n \to 0$ as $n \to \infty$, so the sum to infinity of the sequence is simply $S_\infty = \frac{a}{1-r}$.

If $r > 1$ or $r < -1$ then the sum to infinity of the sequence cannot be found.

Example 3

Find the infinite sum of 108, 36, 12, 4,….

The first term, *a*, is 108 and the common ratio, *r*, is ⅓ .

So, using the formula $S_\infty = \frac{a}{1-r}$ we have $S = \frac{108}{\left(1-\frac{1}{3}\right)} = 162$.