Indices

Laws of indices for all rational exponents. The equivalence of $a^{\frac{m}{n}}$ *and* $\sqrt[n]{a^m}$ *should be known.*

We should already know from GCSE, the three Laws of indices :

(I)	$a^m \times a^n = a^{m+n}$	(e.g. $a^2 \times a^3 = a^5$)
(II)	$a^m \div a^n = a^{m-n}$	(e.g. $a^7 \div a^4 = a^3$)
(III)	$\left(a^m\right)^n = a^{mn}$	(e.g. $\left(a^5\right)^7 = a^{35}$)

In addition to these we need to remember the following:

REMEMBER $\sqrt{a} = a^{\frac{1}{2}}$, $\sqrt[n]{a} = a^{\frac{1}{n}}$, $\frac{1}{a} = a^{-1}$, $a^{-n} = \frac{1}{a^n}$ etc.

When dealing with negative indices we must note that, for example that

$$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$$

and

$$\frac{1}{\sqrt[3]{a^4}} = \frac{1}{a^{\frac{4}{3}}} = a^{-\frac{4}{3}}$$

When dealing with fractional indices we must note that:

$$a^{\frac{1}{n}} = \sqrt[n]{a} \qquad e.g.\, 125^{\frac{1}{3}} = \sqrt[3]{125} = 5$$

and also

$$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$$

So, for example,

$$27^{\frac{4}{3}} = \left(27^{\frac{1}{3}}\right)^4 = 3^4 = 81$$

REMEMBER $a^0 = 1$ and $a^1 = a$ for all values of a.

Example 1

(a) Given that $8 = 2^k$, write down the value of k.

$k = 3$

(b) Given that $4^x = 8^{2-x}$, find the value of x.

$4^x = 8^{2-x}$

$\Rightarrow \left(2^2\right)^x = \left(2^3\right)^{2-x}$

$\Rightarrow 2^{2x} = 2^{6-3x}$

$\Rightarrow 2x = 6 - 3x$

$\Rightarrow 5x = 6$

$\Rightarrow x = \dfrac{6}{5}$

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Surds

Use and manipulation of surds. Candidates should be able to rationalise denominators.

The square roots of certain numbers are integers (e.g. $\sqrt{9}=3$) but when this is not the case it is often easier to leave the square roots sign in the expression (e.g. it is simpler to write $\sqrt{3}$ than it is to write the value our calculator gives, i.e. 1.7320508…). Numbers of the form $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ etc. are called **surds**. We need to be able to simplify expressions involving surds.

It is important to realise the following : $\sqrt{ab}=\sqrt{a\times b}=\sqrt{a}\times\sqrt{b}$ *e.g.* $\sqrt{6}=\sqrt{2\times 3}=\sqrt{2}\times\sqrt{3}$.

Now we see from the above that we can sometimes simplify surds.

For example

$\sqrt{75}$

$=\sqrt{25\times 3}$

$=\sqrt{25}\times\sqrt{3}$

$=5\times\sqrt{3}$

$=5\sqrt{3}$

or $\sqrt{50}=\sqrt{25\times 2}=5\sqrt{2}$

and $\dfrac{21}{\sqrt{3}}=\dfrac{21\sqrt{3}}{3}=7\sqrt{3}$

We can only simplify an expression of the form $\sqrt{a}$ if *a* has a factor which is a perfect square (as the number 25 was in the above example).

Multiplying surds: Multiply out brackets in usual way (using FOIL or a similar method). Then collect similar terms.

e.g. $\left(5-\sqrt{3}\right)\left(7+\sqrt{3}\right)=35+5\sqrt{3}-7\sqrt{3}-3=32-2\sqrt{3}$

$\left(3-\sqrt{5}\right)\left(3+\sqrt{5}\right)=9+3\sqrt{5}-3\sqrt{5}-5=4$

Dividing surds: For example simplify $\dfrac{\left(14+2\sqrt{5}\right)}{\left(3-\sqrt{5}\right)}$.

The denominator of this fraction is $\left(3-\sqrt{5}\right)$. This is irrational and we need to be able to express this in a form in which the denominator is rational. To do this we must multiply top and bottom of the fraction by an expression that will "rationalise the denominator".

We saw from above that $\left(3-\sqrt{5}\right)\left(3+\sqrt{5}\right)=4$ so we multiply top and bottom by $\left(3+\sqrt{5}\right)$.

So we have the following

$$\frac{\left(14+2\sqrt{5}\right)}{\left(3-\sqrt{5}\right)}=\frac{\left(14+2\sqrt{5}\right)\left(3+\sqrt{5}\right)}{\left(3-\sqrt{5}\right)\left(3+\sqrt{5}\right)}$$

$$=\frac{52+20\sqrt{5}}{4}$$

$$=13+4\sqrt{5}$$

Rationalising the denominator:

If the denominator is $a+\sqrt{b}$ then multiply top and bottom by $a-\sqrt{b}$.

If the denominator is $a-\sqrt{b}$ then multiply top and bottom by $a+\sqrt{b}$.

If the denominator is $\sqrt{a}+\sqrt{b}$ then multiply top and bottom by $\sqrt{a}-\sqrt{b}$.

If the denominator is $\sqrt{a}-\sqrt{b}$ then multiply top and bottom by $\sqrt{a}+\sqrt{b}$.

Example 1

Given that $(2 + \sqrt{7})(4 - \sqrt{7}) = a + b\sqrt{7}$, where a and b are integers,

(a) find the value of a and the value of b.

$$\left(2+\sqrt{7}\right)\left(4-\sqrt{7}\right)=8+4\sqrt{7}-2\sqrt{7}-7=1+2\sqrt{7} \text{ so } a=1,\quad b=2.$$

Given that $\dfrac{2+\sqrt{7}}{4+\sqrt{7}} = c + d\sqrt{7}$ where c and d are rational numbers,

(b) find the value of c and the value of d.

$$\frac{2+\sqrt{7}}{4+\sqrt{7}}=\frac{\left(2+\sqrt{7}\right)\left(4-\sqrt{7}\right)}{\left(4+\sqrt{7}\right)\left(4-\sqrt{7}\right)}=\frac{1+2\sqrt{7}}{16-7}=\frac{1}{16}-\frac{1}{8}\sqrt{7}$$

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Example 2

Given that $2^x = \dfrac{1}{\sqrt{2}}$ and $2^y = 4\sqrt{2}$,

(a) find the exact value of x and the exact value of y,

$$\sqrt{2}=2^{\frac{1}{2}} \text{ so } \frac{1}{\sqrt{2}}=2^{-\frac{1}{2}}. \text{ Hence } x=-\frac{1}{2}$$

$$4\sqrt{2}=2^2\times2^{\frac{1}{2}}=2^{\frac{5}{2}}. \text{ Hence } y=\frac{5}{2}$$

(b) calculate the exact value of 2^{y-x}.

$$2^{y-x}=2^{\frac{5}{2}-\frac{1}{2}}=2^3=8$$

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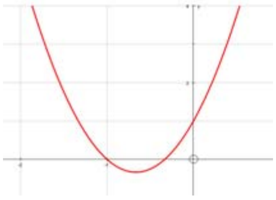
Quadratics

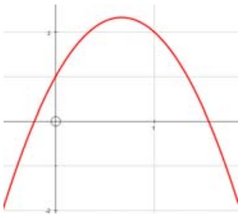
Quadratic functions and their graphs.

The graph of $y = ax^2 + bx + c$.

(i) $a > 0$



(ii) $a < 0$



The turning point can be determined by completing the square as we will see later.

The *x*-coordinates of the point(s) where the curve crosses the *x*-axis are determined by solving $ax^2 + bx + c = 0$.

The *y*-coordinates of the point where the curve crosses the *y*-axis is *c*. This is called the *y*-intercept.

The discriminant of a quadratic function.

In quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, the **Discriminant** is the name given to $b^2 - 4ac$.
The value of the discriminant determines how many real solutions (or roots) there are.

If $b^2 - 4ac < 0$ then $ax^2 + bx + c = 0$ has no real solutions.
If $b^2 - 4ac = 0$ then $ax^2 + bx + c = 0$ has one real solution.
If $b^2 - 4ac > 0$ then $ax^2 + bx + c = 0$ has two real solutions.

Completing the square

e.g. Complete the square on $x^2 + 6x + 11$. We can write $x^2 + 6x = (x+3)^2 - 9$ and so we see that $x^2 + 6x + 11 = (x+3)^2 - 9 + 11 = (x+3)^2 + 2$.

NB: The "3" is obtained by halving "6" .

So $\quad x^2 - 8x + 3 = (x-4)^2 - 16 + 3 = (x-4)^2 - 13$
and $\quad x^2 - x + 1 = (x - \tfrac{1}{2})^2 - \tfrac{1}{4} + \tfrac{4}{4} = (x - \tfrac{1}{2})^2 + \tfrac{3}{4}$.

We can use completing the square to find the turning points of quadratics. For example by completing the square on $y = x^2 + 6x + 11$ we get $y = x^2 + 6x + 11 = (x+3)^2 + 2$. Since $(x+3)^2$ can never be negative, $y = (x+3)^2 + 2$ can never be less than 2. It takes that minimum value of 2 when $(x+3)^2 = 0$, that is when $x = -3$, so the minimum point at $(-3, 2)$.

Solution of quadratic equations. Solution of quadratic equations by factorisation, use of the formula and completing the square

There are three ways of solving quadratic equations

1. Factorising

e.g. Solve $x^2 - 7x + 12 = 0$.
Factorising gives $x^2 - 7x + 12 = (x-3)(x-4)$
So we have $(x-3)(x-4) = 0$ and so $x = 3$ or $x = 4$

e.g. Factorise $2x^2 + 7x + 6$:

(i) Find two numbers which add to give 7 and which multiply to give $2 \times 6 = 12$, i.e. 4 and 3
(ii) Write $2x^2 + 7x + 6 = 2x^2 + 4x + 3x + 6$
(iii) Factorise first two terms and second two terms to give $2x(x+2) + 3(x+2)$
(iv) Hence write as $(2x+3)(x+2)$

2. Completing the square.

e.g. Solve $x^2 - 10x + 19 = 0$
Completing the square gives $x^2 - 10x + 19 = (x-5)^2 - 25 + 19 = (x-5)^2 - 6$
So we have $(x-5)^2 - 6 = 0$.
Hence we see that $(x-5)^2 = 6$ and so $x - 5 = \pm\sqrt{6}$. Thus we have solutions of $x = 5 \pm \sqrt{6}$

3. Quadratic Formula

e.g. Solve $x^2 - 10x + 19 = 0$ by the formula
We use the fact that the solutions to the equation $ax^2 + bx + c = 0$ are

$$x = \frac{-\text{b} \pm \sqrt{\text{b}^2 - 4\text{ac}}}{2\text{a}}.$$

In this case $a = 1$, $b = -10$ and $c = 19$ and so, using the formula, we have

$$x = \frac{10 \pm \sqrt{10^2 - 4 \times 1 \times 19}}{2} = \frac{10 \pm \sqrt{24}}{2}$$

We could be asked to solve a cubic, such as $x^3 + 3x^2 - 5x - 14 = 0$.
If we let $f(x) = x^3 + 3x^2 - 5x - 14$ then we see that $f(-2) = 0$ and so we have that $(x+2)$ is a factor of $f(x)$.
Factorising gives $f(x) = (x+2)(x^2 + x - 7)$. So we want to solve $(x+2)(x^2 + x - 7) = 0$.
Thus the solutions are $x = -2$ or $x = \dfrac{-1 \pm \sqrt{29}}{2}$.