Binomial distribution

Binomial Distribution

Discrete random variables.	Only an understanding of the concepts; not examined beyond binomial distributions.
Conditions for application of a binomial distribution.	
Calculation of probabilities using formula.	Use of $\binom{n}{x}$ notation.
Calculation of probabilities using tables.	
Mean, variance and standard deviation of a binomial distribution.	Knowledge, but not derivations, will be required.

Given in the formulae book

Standard discrete distributions:

Distribution of X	$P(X = x)$	Mean	Variance
Binomial $B(n, p)$	$\binom{n}{x} p^x (1-p)^{n-x}$	np	$np(1-p)$

Introduction

Consider a biased coin so that the probability of getting Head is 1/4.
We are going to throw it three times, recording H or T. e.g. (H,H,T) , (T,H,T), ...

We are interested in the number of Head in the outcome, let call it X.

- Work out the

  i) P(X=1) = P(1 H in the outcome)
  ii) P(X=2) = P(2 Hs in the outcome)
  iii) P(X=3)

- Same questions but this time we throw the coin 100 times!

This experience is typical to what is called a BINOMIAL DISTRIBUTION

Certain conditions are necessary for a situation to be modelled by the binomial distribution.

- A fixed number of trials, n.
- Just two possible outcomes resulting from each trial.
- The probability of each outcome is the same for each trial.
- The trials are independent of each other.

The letters n and p are the binomial parameters.

p is the probability of "success",
either outcome can be call the "success" but, by convention, it is often the less likely one.

Pascal's Triangle

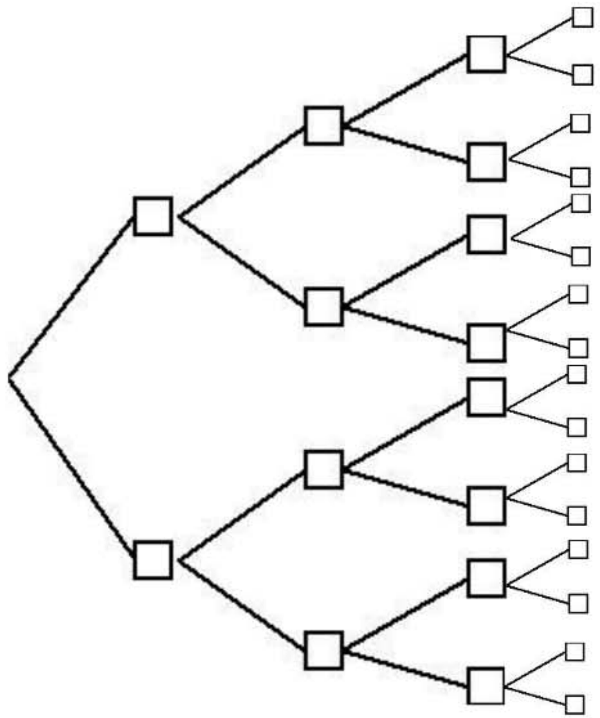
Worked example

Four fair coins are thrown. Draw a tree diagram to find the probability that:

(a) only one head is obtained,

(b) exactly two heads are obtained.

Can we do it without the tree?



Pascal's triangle

Number of trials

1 1

1 2 1

1 3 3 1 ← $n = 3$

1 4 6 4 1 ← $n = 4$

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

1 10 45 120 210 252 210 120 45 10 1 ← $n = 10$

0 success (→ 1), 1 success (→ 10), 6 successes (→ 210)

I throw the dice 9 times, what is the probability of getting exactly five 4s?

What if n is more than 10?

Use the calculator function "^nC_r"   read "n choose r"

$$^nC_r \text{ is also noted } \binom{n}{r} =$$

$$\binom{10}{6} = {}^{10}C_6 =$$

$$\binom{25}{10} = {}^{25}C_{10} =$$

Conclusion and formula

When the probability of a success is "p" and an experiment is repeated "n" times, then the probability that there are r successes is given by:

$$P(X=r)=\binom{n}{r}\times p^r\times(1-p)^{n-r}$$

The random variable X is noted

$$X \sim B(n,p)$$

EXERCISE 4B

1 Ropes produced in a factory are tested to a certain breaking strain. From past experience it is found that one-quarter of ropes break at this strain.

From a batch of four such ropes, find the probability that exactly two break.

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2 A distorted coin, where the probability of a head is $\frac{3}{5}$, is thrown five times.

Find the probability that a head shows on exactly four of these throws.

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3 A group of 10 friends plans to each buy a present for their friend who has a birthday. The probability that they will choose to buy chocolates is 0.4 and the friends all choose their present independently.

Find the probability that only three of the 10 friends decide to buy chocolates.

1) $\frac{27}{128}$ **2)** $\frac{162}{625}$ **3)** 0.215

4 A bank cash dispenser has a probability of 0.2 of being out of order on any one day chosen at random.

Find the probability that, out of the 10 of these machines which this bank owns, exactly three are out of order on any one given day.

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5 The probability that Miss Brown will make an error in entering any one set of daily sales data into a database is 0.3.

Find the probability that, during a fortnight (ten working days) she makes an error exactly four times.

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6 A restaurant takes bookings for 20 tables on Saturday night. The probability that a party does not turn up for their booking is 0.15.

Find the probability that only two of the parties who have made bookings do not turn up.

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7 A school pupil attempts a multiple-choice exam paper but has not made any effort to learn any of the information necessary. Therefore the pupil guesses the answers to all the questions.

There are five possible answers to each question and there are 30 questions on the paper. Find the probability that the pupil gets eight questions correct out of the 30 on the paper.

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8 A batch of 25 lightbulbs is sent to a small retailer. The probability that a bulb is faulty is 0.1.

Find the probability that only two of the bulbs are faulty.

4) 0.201 5) 0.200 6) 0.229 7) 0.111 8) 0.266

Cumulative binomial tables

Consider a random variable $X \sim B(8,0.5)$

Work out $P(X<4)$.

In the formulae booklet

TABLE 1 CUMULATIVE BINOMIAL DISTRIBUTION FUNCTION

The tabulated value is $P(X \leqslant x)$, where X has a binomial distribution with parameters n and p.

p	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	p
x	$n=8$																		x
0	0.9227	0.8508	0.7837	0.7214	0.6634	0.6096	0.5596	0.5132	0.4703	0.4305	0.2725	0.1678	0.1001	0.0576	0.0319	0.0168	0.0084	0.0039	0
1	0.9973	0.9897	0.9777	0.9619	0.9428	0.9208	0.8965	0.8702	0.8423	0.8131	0.6572	0.5033	0.3671	0.2553	0.1691	0.1064	0.0632	0.0352	1
2	0.9999	0.9996	0.9987	0.9969	0.9942	0.9904	0.9853	0.9789	0.9711	0.9619	0.8948	0.7969	0.6785	0.5518	0.4278	0.3154	0.2201	0.1445	2
3	1.0000	1.0000	0.9999	0.9998	0.9996	0.9993	0.9987	0.9978	0.9966	0.9950	0.9786	0.9437	0.8862	0.8059	0.7064	0.5941	0.4770	0.3633	3
4			1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9997	0.9996	0.9971	0.9896	0.9727	0.9420	0.8939	0.8263	0.7396	0.6367	4
5							1.0000	1.0000	1.0000	1.0000	0.9998	0.9988	0.9958	0.9887	0.9747	0.9502	0.9115	0.8555	5
6											1.0000	0.9999	0.9996	0.9987	0.9964	0.9915	0.9819	0.9648	6
7												1.0000	1.0000	0.9999	0.9998	0.9993	0.9983	0.9961	7
8														1.0000	1.0000	1.0000	1.0000	1.0000	8

How to use the table:

$P(X \leq x)$ direct reading

$P(X < x) = P(x \leq x\text{-}1)$

$P(X > x) = 1 - P(X \leq x)$

$P(X = x) = P(X \leq x) - P(X \leq x\text{-}1)$ or calculation using formula

$P(a \leq X \leq b) = P(X \leq b) - P(X \leq a\text{-}1)$

Your turn...

The probability that a candidate will guess the correct answer to a multiple-choice question is 0.2. In a multiple-choice test there are 50 questions. A candidate decides to guess the answers to all the questions and chooses the answers at random, each answer being independent of any other answer.

Find the probability that the candidate:

(a) gets five or fewer answers correct,

(b) gets more than 14 answers correct,

(c) gets exactly nine answers correct,

(d) gets between seven and 12 (inclusive) answers correct.

p	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**	**0.10**	**0.15**	**0.20**	**0.25**	**0.30**	**0.35**	**0.40**	**0.45**	**0.50**	p
x	$n=50$																		x
0	0.6050	0.3642	0.2181	0.1299	0.0769	0.0453	0.0266	0.0155	0.0090	0.0052	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	**0**
1	0.9106	0.7358	0.5553	0.4005	0.2794	0.1900	0.1265	0.0827	0.0532	0.0338	0.0029	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	**1**
2	0.9862	0.9216	0.8108	0.6767	0.5405	0.4162	0.3108	0.2260	0.1605	0.1117	0.0142	0.0013	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	**2**
3	0.9984	0.9822	0.9372	0.8609	0.7604	0.6473	0.5327	0.4253	0.3303	0.2503	0.0460	0.0057	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000	**3**
4	0.9999	0.9968	0.9832	0.9510	0.8964	0.8206	0.7290	0.6290	0.5277	0.4312	0.1121	0.0185	0.0021	0.0002	0.0000	0.0000	0.0000	0.0000	**4**
5	1.0000	0.9995	0.9963	0.9856	0.9622	0.9224	0.8650	0.7919	0.7072	0.6161	0.2194	0.0480	0.0070	0.0007	0.0001	0.0000	0.0000	0.0000	**5**
6		0.9999	0.9993	0.9964	0.9882	0.9711	0.9417	0.8981	0.8404	0.7702	0.3613	0.1034	0.0194	0.0025	0.0002	0.0000	0.0000	0.0000	**6**
7		1.0000	0.9999	0.9992	0.9968	0.9906	0.9780	0.9562	0.9232	0.8779	0.5188	0.1904	0.0453	0.0073	0.0008	0.0001	0.0000	0.0000	**7**
8			1.0000	0.9999	0.9992	0.9973	0.9927	0.9833	0.9672	0.9421	0.6681	0.3073	0.0916	0.0183	0.0025	0.0002	0.0000	0.0000	**8**
9				1.0000	0.9998	0.9993	0.9978	0.9944	0.9875	0.9755	0.7911	0.4437	0.1637	0.0402	0.0067	0.0008	0.0001	0.0000	**9**
10					1.0000	0.9998	0.9994	0.9983	0.9957	0.9906	0.8801	0.5836	0.2622	0.0789	0.0160	0.0022	0.0002	0.0000	**10**
11						1.0000	0.9999	0.9995	0.9987	0.9968	0.9372	0.7107	0.3816	0.1390	0.0342	0.0057	0.0006	0.0000	**11**
12							1.0000	0.9999	0.9996	0.9990	0.9699	0.8139	0.5110	0.2229	0.0661	0.0133	0.0018	0.0002	**12**
13								1.0000	0.9999	0.9997	0.9868	0.8894	0.6370	0.3279	0.1163	0.0280	0.0045	0.0005	**13**
14									1.0000	0.9999	0.9947	0.9393	0.7481	0.4468	0.1878	0.0540	0.0104	0.0013	**14**
15										1.0000	0.9981	0.9692	0.8369	0.5692	0.2801	0.0955	0.0220	0.0033	**15**

a) **0.0480**

b) **$1 - P(X \le 14) = 0.0607$**

c) **$P(X \le 9) - P(X \le 8) = 0.1364$**

d) **$P(X \le 12) - P(X \le 6) = 0.7105$**

EXERCISE

1 Components produced in a factory are tested in batches of 20. The proportion of components which are faulty is 0.2. Find the probability that a randomly chosen batch has:

(a) three or fewer faulty components,

(b) less than three faulty components,

(c) more than one faulty component.

2 A biased die, where the probability of a six showing is $\frac{2}{5}$, is thrown eight times. Find the probability that:

(a) a six shows fewer than three times,

(b) a six shows at least twice,

(c) no sixes show.

3 A group of 25 school pupils are asked to write an essay for a GCSE project. They each independently choose a subject at random from a selection of five. One of the choices available is to write a horror story. Find the probability that, out of this group,

(a) more than five write a horror story,

(b) at least six write a horror story,

(c) less than four write a horror story.

4 A cashier at a cinema has to calculate and balance the takings each evening. The probability that the cashier will make a mistake is 0.3. The manager of the cinema wishes to monitor the accuracy of the calculations over a 25-day working month. Find the probability that the cashier makes:

(a) fewer than five mistakes,

(b) no more than eight mistakes,

(c) more than three mistakes.

5 A manufacturer of wine glasses sells them in presentation boxes of 20. Random samples show that three in every hundred of these glasses are defective. Find the probability that a randomly chosen box contains:

(a) no defective glasses,

(b) at least two defective glasses,

(c) fewer than three defective glasses,

(d) exactly one defective glass.

6 The probability that a certain type of vacuum tube will shatter during a thermal shock test is 0.15. What is the probability that, if 25 such tubes are tested:

(a) four or more will shatter,

(b) no more than five will shatter,

(c) between five and ten (inclusive) will shatter?

7 A researcher calls at randomly chosen houses in a large city and asks the householder whether they will agree to answer questions on local services. The probability that a householder will refuse to answer the questions is 0.2. What is the probability that, on a day when 12 households are visited,

(a) three or fewer will refuse,

(b) exactly three will refuse,

(c) no more than one will refuse,

(d) at least ten **will agree** to answer? [A]

8 The organiser of a school fair has organised raffle tickets to be offered to all adults who attend. The probability that an adult declines to buy a ticket is 0.15. What is the probability that if 40 adults attend and are asked to buy tickets:

(a) five or fewer will decline,

(b) exactly seven will decline,

(c) between four and ten (inclusive) decline,

(d) 36 or more **will agree** to buy. [A]

9 A gardener plants beetroot seeds. The probability of a seed not germinating is 0.35, independently for each seed.

Find the probability that, in a row of 40 seeds, the number not germinating is:

(a) nine or fewer,

(b) seven or more,

(c) equal to the number germinating. [A]

10 Safety inspectors carry out checks on the fire resistance of office furniture. Twenty per cent of items of furniture checked fail the test.

(a) Three randomly selected pieces of furniture are tested. Find the probability that:
(i) none of these three items will fail the test,
(ii) exactly one of these three items will fail the test.

(b) Twenty pieces of office furniture awaiting test include exactly four which will fail. If three of these twenty pieces, selected at random, are tested, find the probability that:
(i) all three will pass the test,
(ii) exactly two will pass the test. [A]

EXERCISE Answers

Answers from tables have been given to four decimal places. However three significant figures is sufficient.

1 $n = 20, p = 0.2, X \sim B(20, 0.2)$

(a) $P(X \leq 3) = 0.4114$;

(b) $P(X < 3) = P(X \leq 2) = 0.2061$;

(c) $P(X > 1) = 1 - P(X \leq 1) = 1 - 0.0692 = 0.9308$.

2 $n = 8, p = \frac{2}{5}, X \sim B(8, 0.4)$

(a) $P(X < 3) = P(X \leq 2) = 0.3154$;

(b) $P(X \geq 2) = 1 - P(X \leq 1) = 1 - 0.1064 = 0.8936$;

(c) $P(X = 0) = 0.0168$.

3 $n = 25, p = \frac{1}{5} = 0.2, X \sim B(25, 0.2)$

(a) $P(X > 5) = 1 - P(X \leq 5) = 1 - 0.6167 = 0.3833$;

(b) $P(X \geq 6) = 1 - P(X \leq 5) = 0.3833$
at least 6 = 6 or more = more than 5 see **(a)**;

(c) $P(X < 4) = P(X \leq 3) = 0.2340$.

4 $n = 25, p = 0.3, X \sim B(25, 0.3)$

(a) $P(X < 5) = P(X \leq 4) = 0.0905$;

(b) $P(X \leq 8) = 0.6769$
no more than 8 = 8 or less;

(c) $P(X > 3) = 1 - P(X \leq 3) = 1 - 0.0332 = 0.9668$.

5 $n = 20, p = \frac{3}{100} = 0.03, X \sim B(20, 0.03)$

(a) $P(X = 0) = 0.5438$;

(b) $P(X \geq 2) = 1 - P(X \leq 1) = 1 - 0.8802 = 0.1198$;

(c) $P(X < 3) = P(X \leq 2) = 0.9790$;

(d) $P(X = 1) = P(X \leq 1) - P(X = 0) = 0.8802 - 0.5438 = 0.3364$
or $\binom{20}{1}(0.03)^1(0.97)^{19} = 0.336$ (three significant figures).

6 $n = 25, p = 0.15, X \sim B(25, 0.15)$

(a) $P(X \geqslant 4) = 1 - P(X \leq 3) = 1 - 0.4711 = 0.5289$;

(b) $P(X \leq 5) = 0.8385$
no more than 5 = 5 or less;

(c) $P(5 \leq X \leq 10) = P(X \leq 10) - P(X \leq 4) = 0.9995 - 0.6821 = 0.3174$.

7 $n = 12, p = 0.2, X \sim B(12, 0.2)$

(a) $P(X \leqslant 3) = 0.7946$;

(b) $P(X = 3) = 0.2363 = P(X \leq 3) - P(X \leq 2)$, (direct calculation gives 0.2362);

(c) $P(X \leqslant 1) = 0.2749$
no more than 1 = 1 or fewer;

(d) $P(\geqslant 10 \text{ agree}) = P(10, 11, 12 \text{ agree}) = P(0, 1, 2 \text{ refuse})$
$P(X \leq 2) = 0.5583$;

8 $n = 40, p = 0.15, X \sim B(40, 0.15)$

(a) $P(X \leqslant 5) = 0.4325$;

(b) $P(X = 7) = P(X \leq 7) - P(X \leq 6) = 0.7359 - 0.6067 = 0.1492$
(direct calculations gives 0.1493);

(c) $P(4 \leq X \leq 10) = P(X \leq 10) - P(X \leq 3) = 0.9701 - 0.1302 = 0.8399$;

(d) $P(36, 37, 38, 39, 40 \text{ agree}) = P(0, 1, 2, 3, 4 \text{ decline})$
$P(X \leq 4) = 0.2633$.

9 **(a)** 0.0644; **(b)** 0.9956; **(c)** 0.0190.

10 **(a)** **(i)** 0.512, **(ii)** 0.384;
(b) **(i)** 0.491, **(ii)** 0.421.

Mean and variance of the binomial distribution

Consider a random variable $X \sim B(n, p)$

The mean of X is given by $\mu = n \times p$

The variance of X is given by $\sigma^2 = n \times p \times (1 - p)$

Given in the formula book

Example:

A biased die is thrown 3 times, the outcomes are recorded, for example (2,3,1), ...

The probability of getting a 6 is 0.2.

The variable X is the number of sixes obtained in an outcome.

a) Work out the mean number of 6 obtained

b) Work out the variance of the number of sixes

X=	Freq/prob	
0		
1		
2		
3		
Total		

The binomial model

EXERCISE

1 A tour operator organises a trip for cricket enthusiasts to the Caribbean in March. The package includes a ticket for a one-day International in Jamaica. Places on the tour must be booked in advance. From past experience, the tour operator knows that the probability of a person who has booked a place subsequently withdrawing is 0.08 and is independent of other withdrawals.

(a) Twenty people book places. Find the probability that:

(i) none withdraw,

(ii) two or more withdraw,

(iii) exactly two withdraw.

(b) The tour operator accepts 22 bookings but has only 20 tickets available for the one-day International. What is the probability that he will be able to provide tickets for everyone who goes on tour? [A]

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2 The organiser of a fund raising event for a sports club finds that the probability of a person who is asked to buy a raffle ticket refusing is 0.15.

(a) What is the probability that, if 40 people are asked to buy raffle tickets:

(i) five or fewer will refuse,

(ii) exactly seven will refuse,

(iii) more than four will refuse.

The club also owns a fruit machine for the use of members. Inserting a 20p coin enables a member of the club to attempt to win a prize. The probability of winning a prize is a constant 0.2.

(b) Find the probability that a member, who has 25 attempts at winning on this machine, gains:

(i) three or more prizes,

(ii) no more than five prizes.

Another member of the club asks people to pay £1 to enter a game of chance. She continues to ask until 50 people have agreed to participate.

(c) X is the number of people she asks before obtaining the 50 participants. Say, giving a reason whether it is likely that X will follow a binomial distribution. [A]

Answers

Answers from tables have been given to 4 d.p. However, 3 significant figures is sufficient.

1 (a) $W \sim B(20, 0.08)$

(i) $P(W = 0) = 0.1887$,

(ii) $P(W \geqslant 2) = 1 - P(W \leq 1) = 1 - 0.5169 = 0.4831$,

(iii) $P(W = 2) = P(W \leq 2) - P(W \leq 1) = 0.7879 - 0.5169 = 0.2710$;

(b) $R \sim B(22, 0.08)$

$$P(R \geqslant 2) = 1 - P(R = 1) - P(R = 0)$$
$$= 1 - \binom{22}{1}(0.08)(0.92)^{21} - (0.92)^{22}$$
$$= 1 - 0.3055 - 0.1597 = 0.5348.$$

2 (a) $n = 40, p = 0.15, X \sim B(40, 0.15)$

(i) $P(X \leq 5) = 0.4325$,

(ii) $P(X = 7) = P(X \leq 7) - P(X \leq 6) = 0.7559 - 0.6067 = 0.1492$,

(iii) $P(X > 4) = 1 - P(X \leq 4) = 1 - 0.2633 = 0.7367$;

0, 1, 2, 3, 4, | 5, 6, ...

$X \leq 4$ $X > 4$

(b) $n = 25, p = 0.2, X \sim B(25, 0.2)$

(i) $P(X \geqslant 3) = 1 - P(X \leq 2) = 1 - 0.0982 = 0.9018$,

(ii) $P(X \leq 5) = 0.6167$;

0, 1, 2, | 3, 4 ...

$X \leq 2$ $X \geqslant 3$

(c) X is not binomial, since the total number of trials is not fixed.

Key point summary

1 The binomial distribution can only be used to model situations in which certain conditions exist. These conditions are:

- a fixed number of trials,
- two possible outcomes only at each trial,
- fixed probabilities for each outcome,
- trials independent of each other.

2 The parameters involved in describing a binomial distribution are $\boldsymbol{n}$ and $\boldsymbol{p}$, where $\boldsymbol{n}$ represents the number of trials and $\boldsymbol{p}$ represents the probability of a success.

This is written $X \sim \mathrm{B}(n, p)$.

3 The formula for evaluating a binomial probability of x successes out of n trials when the probability of a success is p is

$$\mathrm{P}(X = x) = \binom{n}{x} \times p^x \times (1-p)^{(n-x)}$$

where $\binom{n}{x}$, which can also be written as ${}^nC_x$ is found on most calculators.

It can also be found from Pascal's triangle or from the definition

$$\frac{n!}{x!(n-x)!}.$$

4 Cumulative binomial tables can be used for evaluating binomial probabilities of the type $\mathrm{P}(X \leq x)$ for certain values of n and p.

Use these tables if you can as it will save a lot of time.

Remember to choose $p \leq 0.5$

5 The mean and variance of the binomial distribution are given by

$$\text{Mean} = np$$

and

$$\text{Variance} = np(1-p).$$

Notes