Further rational functions and maximum and minimum points

Specifications:

Algebra and Graphs

Graphs of rational functions of the form

$$\frac{ax+b}{cx^2+dx+e} \text{ or } \frac{x^2+ax+b}{x^2+cx+d}.$$

Sketching the graphs.

Finding the equations of the asymptotes which will always be parallel to the coordinate axes.

Finding points of intersection with the coordinate axes or other straight lines.

Solving associated inequalities.

Using quadratic theory (not calculus) to find the possible values of the function and the coordinates of the maximum or minimum points on the graph.

E.g. for $y = \frac{x^2+2}{x^2-4x}$, $y = k \Rightarrow x^2+2 = kx^2 - 4kx$,

which has real roots if $16k^2 + 8k - 8 \geqslant 0$, i.e. if $k \leqslant -1$ or $k \geqslant \frac{1}{2}$;

stationary points are $(1, -1)$ and $(-2, \frac{1}{2})$.

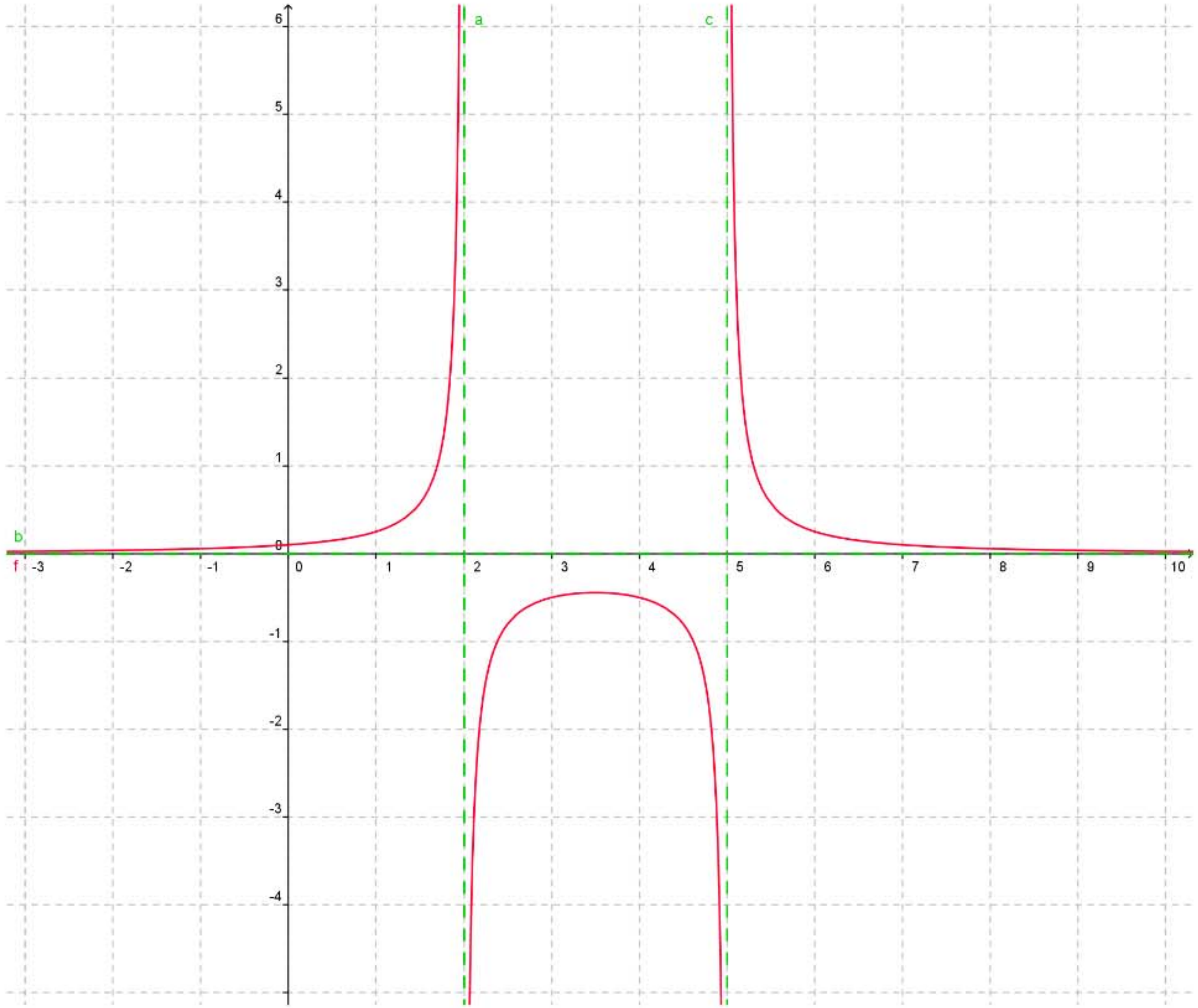
Rational functions with QUADRATIC denominators

Consider the function $f(x)=\dfrac{1}{(x-2)(x-5)}$ *and its graph.*

a) Work out the coordinates of the point(s) of intersection with the axes.

b) Work out the equation of the asymptotes and the relative position with the graph.

c) Sketch the graph of f.



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Rational functions of the form $y = \dfrac{px+q}{ax^2+bx+c}$

General remarks:

Intersection with the y-axis: Work out y(0)

Intersection with the x-axis: Solve px+q=0

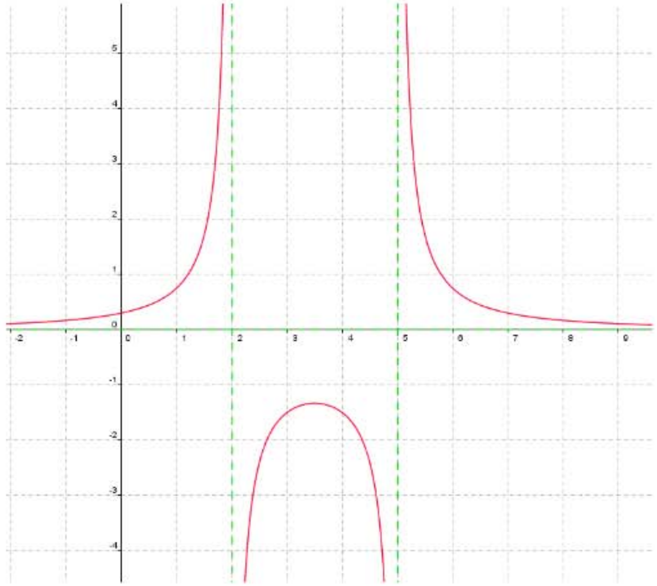
"Horizontal" asymptote: Divide the numerator and the denominator by the highest power of x

Vertical asymptotes: $y = \dfrac{px+q}{ax^2+bx+c}$

Case 1: "b^2-4ac > 0"

The denominator has two roots, x_1 *and* x_2.

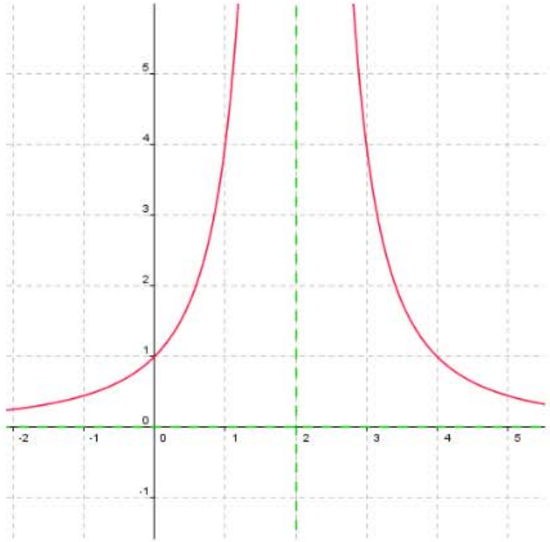
There are two asymptotes with equations $x = x_1$ *and* $x = x_2$



Case 2: "b^2-4ac=0"

The denominator has one (repeated) root, x_0.

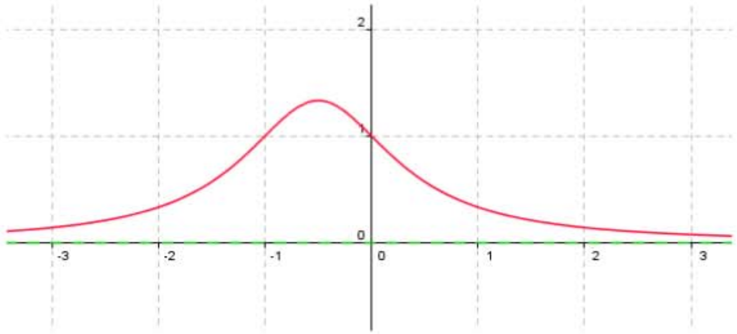
There is only one "vertical" asymptote with equation $x = x_0$.



Case 3: "b^2-4ac<0"

The denominator has no root.

There is no "vertical" asymptote.



Exercises

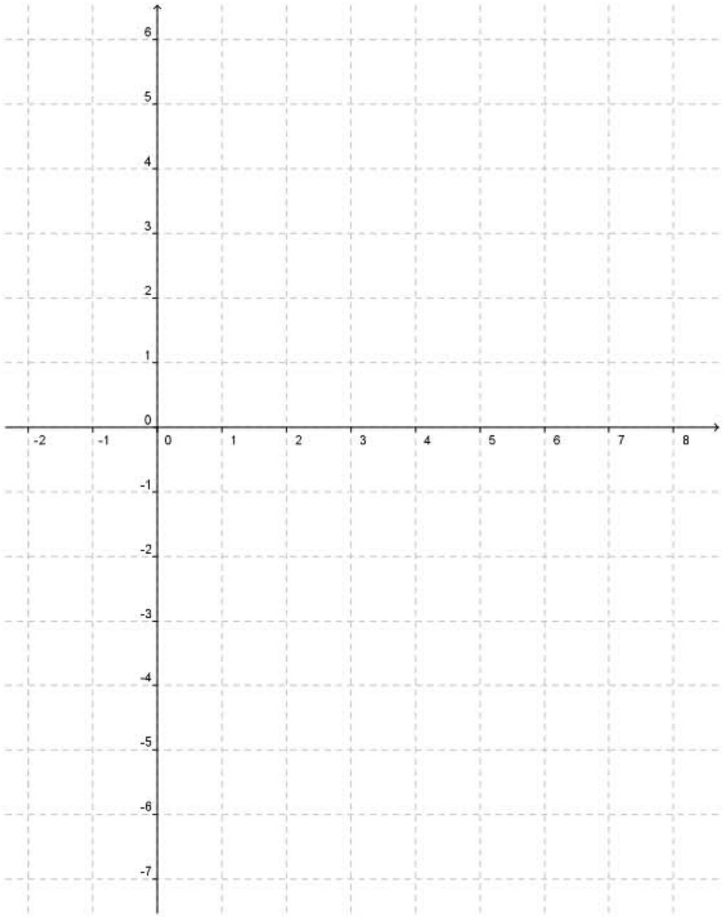
For each of the curves,

i) Find the equations of any asymptotes

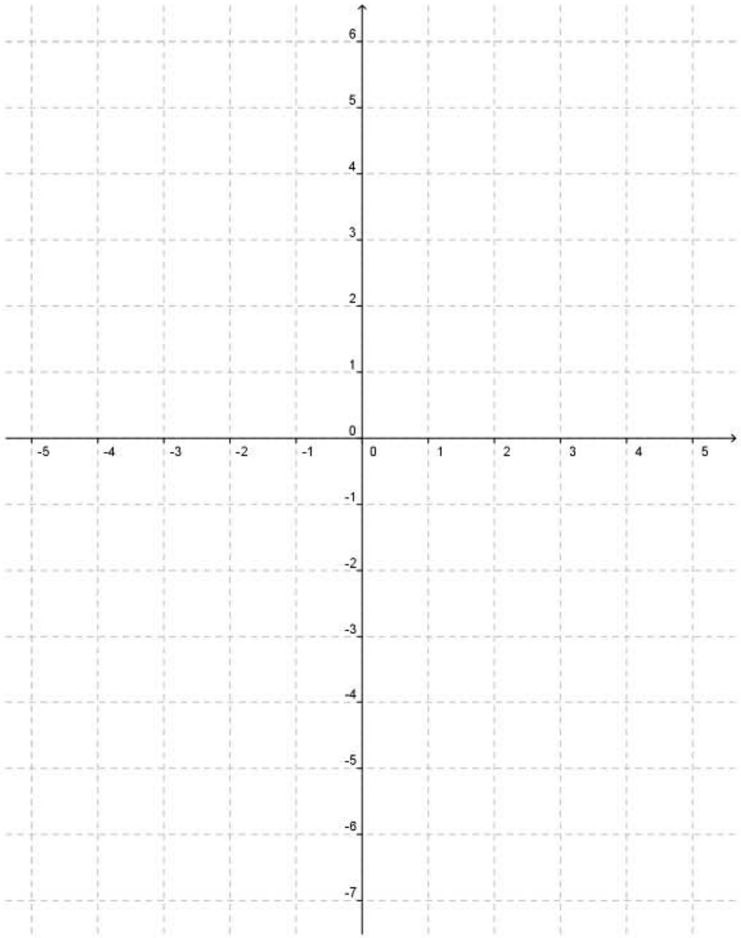
ii) Find the points of intersection with the axes

iii) Sketch the curves

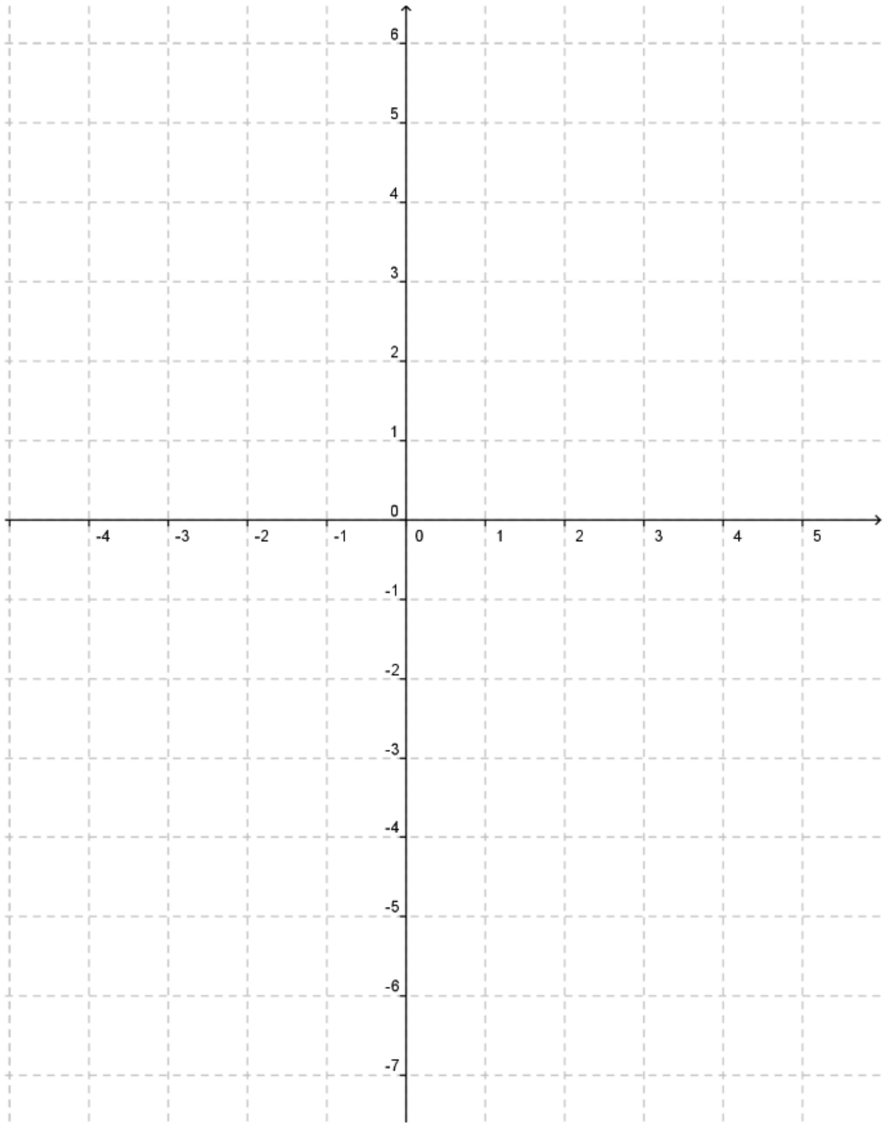
1) $y = \dfrac{1}{(x-3)(x-4)}$



2) $y = \dfrac{1}{x^2 - 4}$



3) $y = \dfrac{x-5}{x^2 - x - 6}$

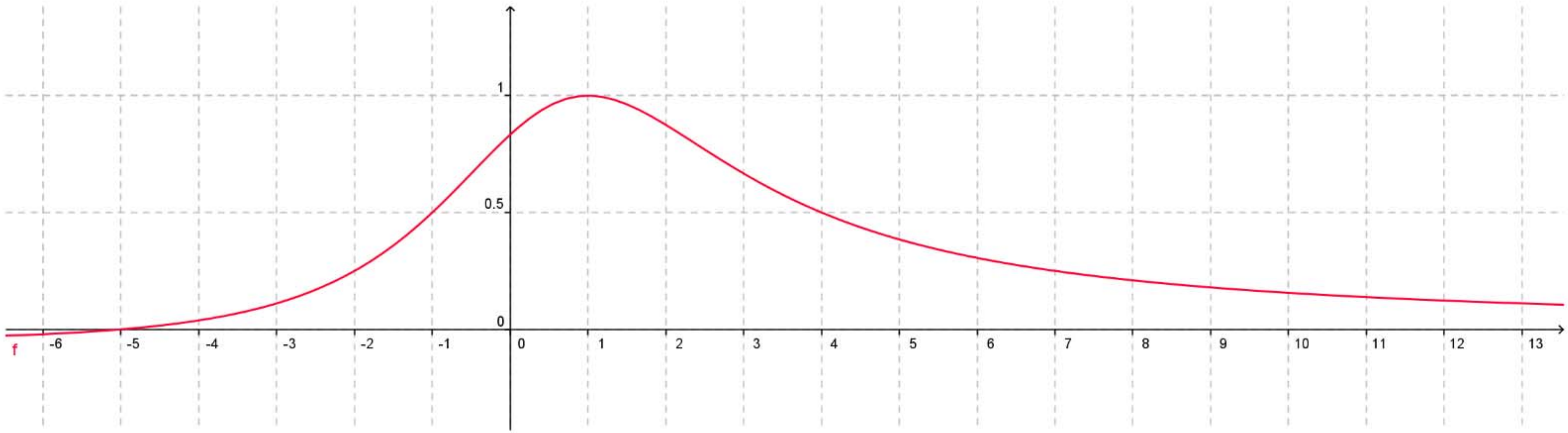


4) $y = \dfrac{x+4}{x^2 - 3x + 10}$

5) The curve C has equation $y = \dfrac{x+5}{x^2 - x + 6}$.

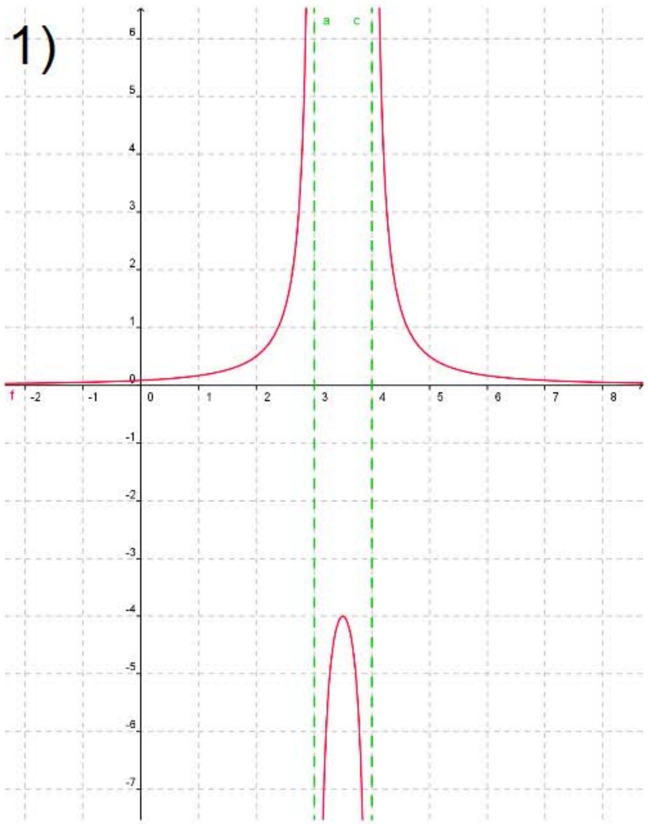
The line $y = k$ intersects the curve C. Find the points of intersection, if any, in the case when

$i)\ k = \dfrac{1}{2}$ $\qquad ii)\ k = 1$ $\qquad iii)\ k = 2$

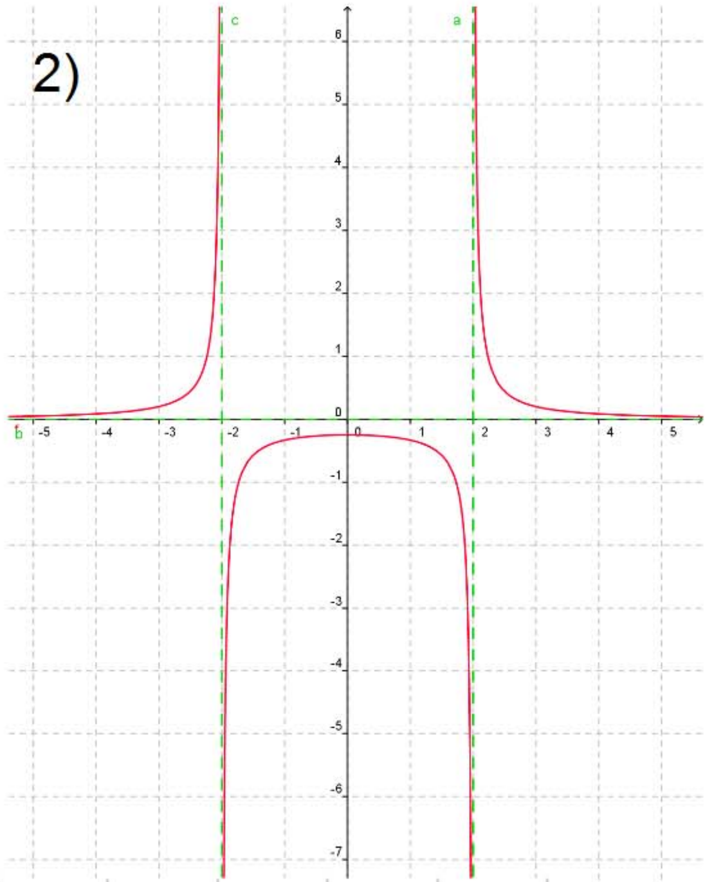


Illustrated answers

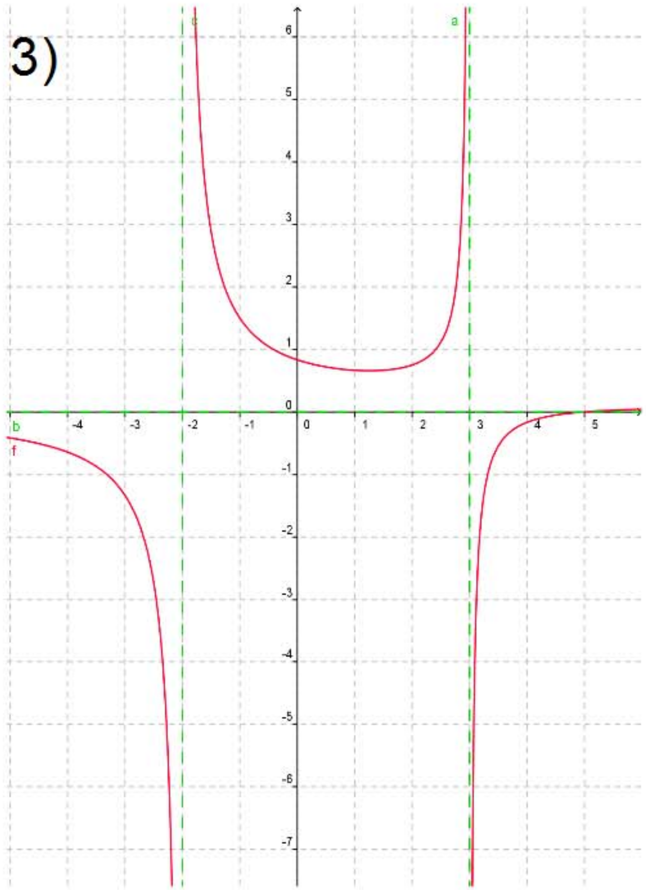
1)



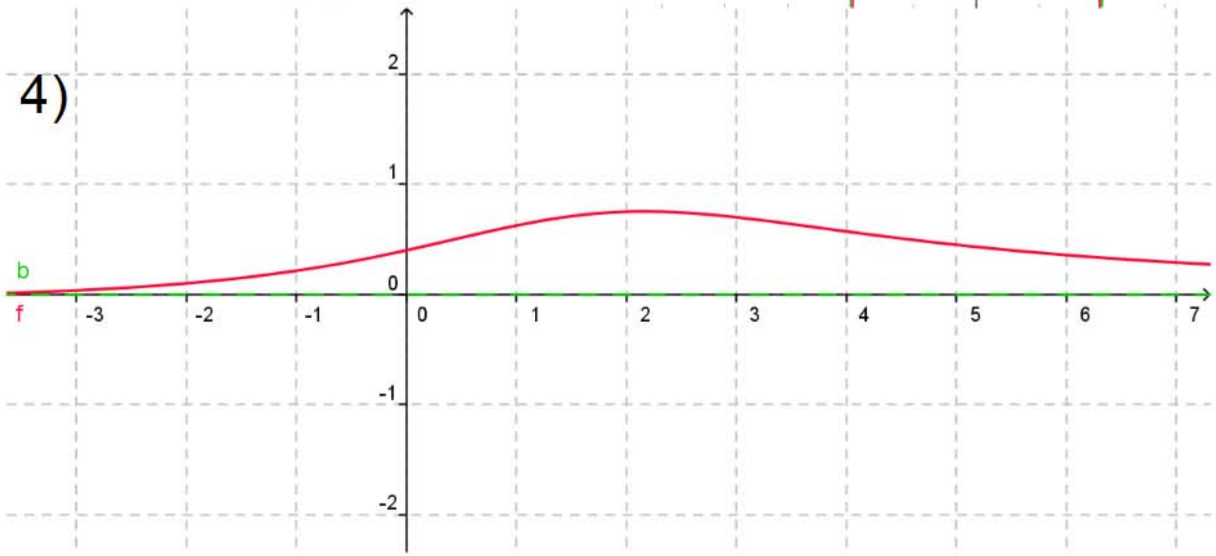
2)



3)



4)



5) i) x=-1 or x=4

ii) x=1

iii) no solution

Rational functions of the form $y = \dfrac{ax^2 + bx + c}{dx^2 + ex + f}$

General remarks:

If possible, FACTORISE the numerator and the denominator

Intersection with the x-axis: Solve $ax^2+bx+c=0$

Intersection with the y-axis: Work out y for x=0

"Horizontal" asymptote:

Divide by the "highest power of x": here, divide by x^2 both numerator and denominator:

$$y = \frac{ax^2 + bx + c}{dx^2 + ex + f} = \frac{a + \frac{b}{x} + \frac{c}{x^2}}{d + \frac{e}{x} + \frac{f}{x^2}}$$, when x tends to infinity, y tends to $\frac{a}{d}$.

"Vertical" asymptotes: see page 4

Exercise:

Study the curve C with equation $y = \dfrac{x^2 - 4}{x^2 + 4x + 6}$.

Sketch the curve.

x	-10	-7	-5	-4	-3	-2.5	-2	-1.5	-1	-0.5	0	1	2	3	5	7	10
y																	

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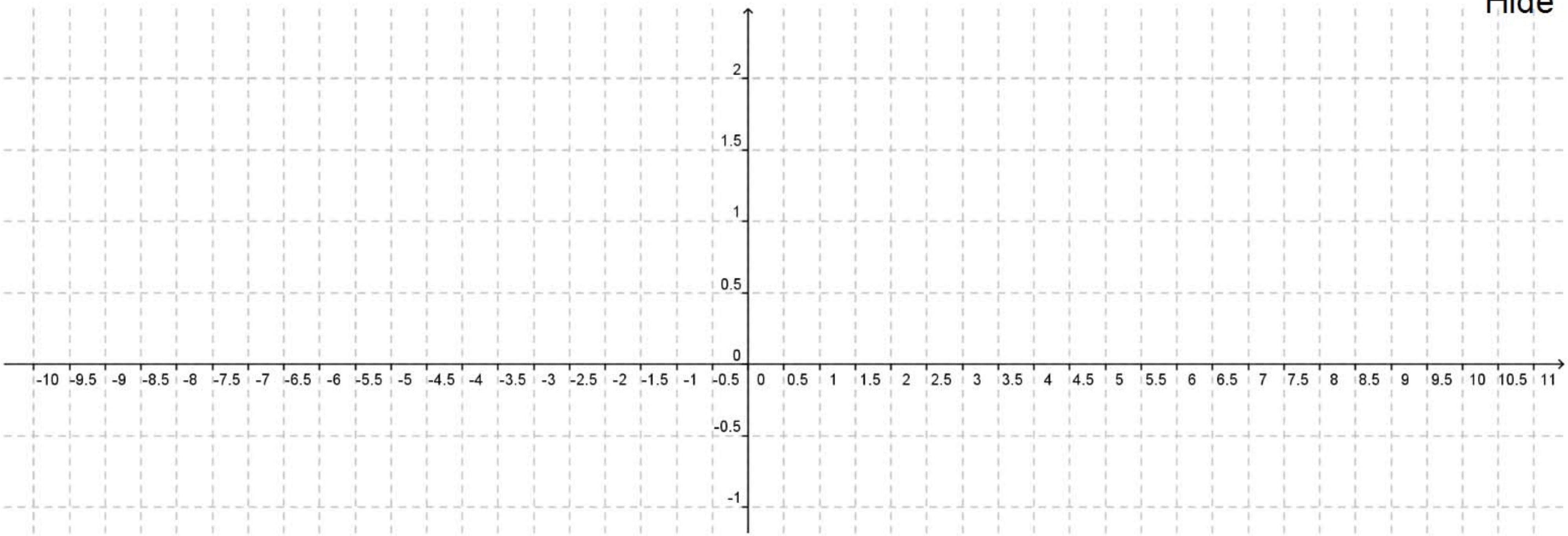


Finding the regions for which a curve is defined

We have sketched the curve with equation

$$y = \frac{x^2 - 4}{x^2 + 4x + 6}$$

Let's show that the curve exists for $-1 \leq y \leq 2$.

In order to find the set of values of y for which a curve of the form $y = \dfrac{ax^2 + bx + c}{dx^2 + ex + f}$ exists:

1 consider where $y = k$ cuts the curve by writing $k = \dfrac{ax^2 + bx + c}{dx^2 + ex + f}$;

2 multiply out to obtain a quadratic of the form $Ax^2 + Bx + C = 0$, where A, B and C will involve k;

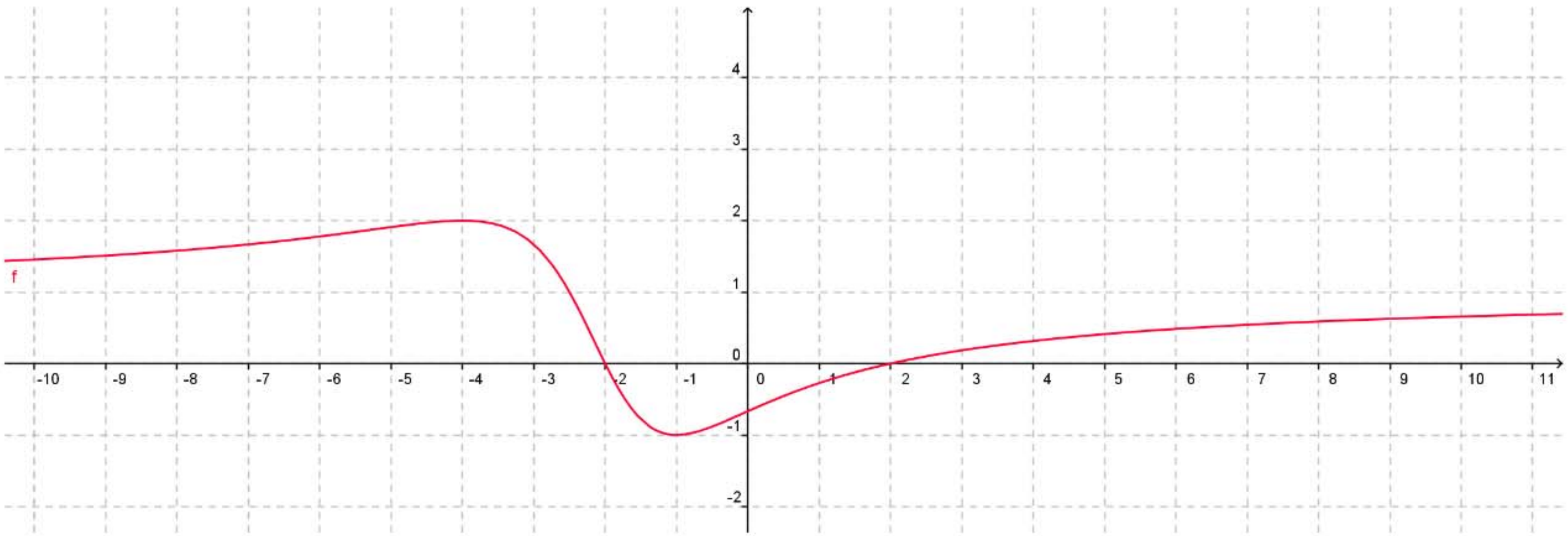
3 use the condition for real roots $B^2 - 4AC \geqslant 0$ to obtain a quadratic inequality involving k;

4 convert the solution involving k to a condition involving y.

Method: Consider the line y = k and its intersection with the curve.

To intersection exits only if the equation $\dfrac{x^2 - 4}{x^2 + 4x + 6} = k$ has solution(s).

So we are going to re-arrange this equation into a quadratic equation and work out the values of k for which the discriminant is positive.



$$\frac{x^2-4}{x^2+4x+6}=k$$

Solution:

$\frac{x^2-4}{x^2+4x+6}=k$ *becomes* $x^2-4=kx^2+4kx+6k$

$$(1-k)x^2-4kx-(4+6k)=0$$

This equation has solutions when the discriminant is positive:

$$(-4k)^2-4(1-k)(-4-6k)\geq 0$$

$$16k^2-24k^2+8k+16\geq 0$$

$$-8k^2+8k+16\geq 0$$

$$k^2-k-2\leq 0$$

$$(k-2)(k+1)\leq 0$$

$$-1\leq k\leq 2$$

The line $y=k$ intersects the curve for all values of k between -1 and 2.

Remark: This method also gives you the maximum and the minimum values of the function without using calculus.

Consequence: Coordinates of the minimum and the maximum of the curve (without using differentiation)

The equation $\dfrac{x^2-4}{x^2+4x+6}=k$ leads us to

the quadratic equation : $(1-k)x^2 - 4kx - 4-6k = 0$

We then worked out that $-1 \leq k \leq 2$

If a curve of the form $y = \dfrac{ax^2+bx+c}{dx^2+ex+f}$ is shown to exist only for $P \leqslant y \leqslant Q$, then it means that the curve must have a minimum point when $y = P$ and a maximum point when $y = Q$.

Substitute $y = P$ in order to find the x-coordinate of the minimum point. The resulting quadratic in x will always have a repeated root.

Repeat by substituting $y = Q$ to find the x-coordinate of the maximum point.

Your go...

1) Show that the curve $y = \dfrac{x+5}{x^2 - x + 6}$ exists for the values of y in the interval $-\dfrac{1}{23} \le y \le 1$.

2) Consider the curve with equation $y = \dfrac{x^2+2}{x^2 - 4x}$.

Show that NO part of the curve exists for $-1 < y < \dfrac{1}{2}$.

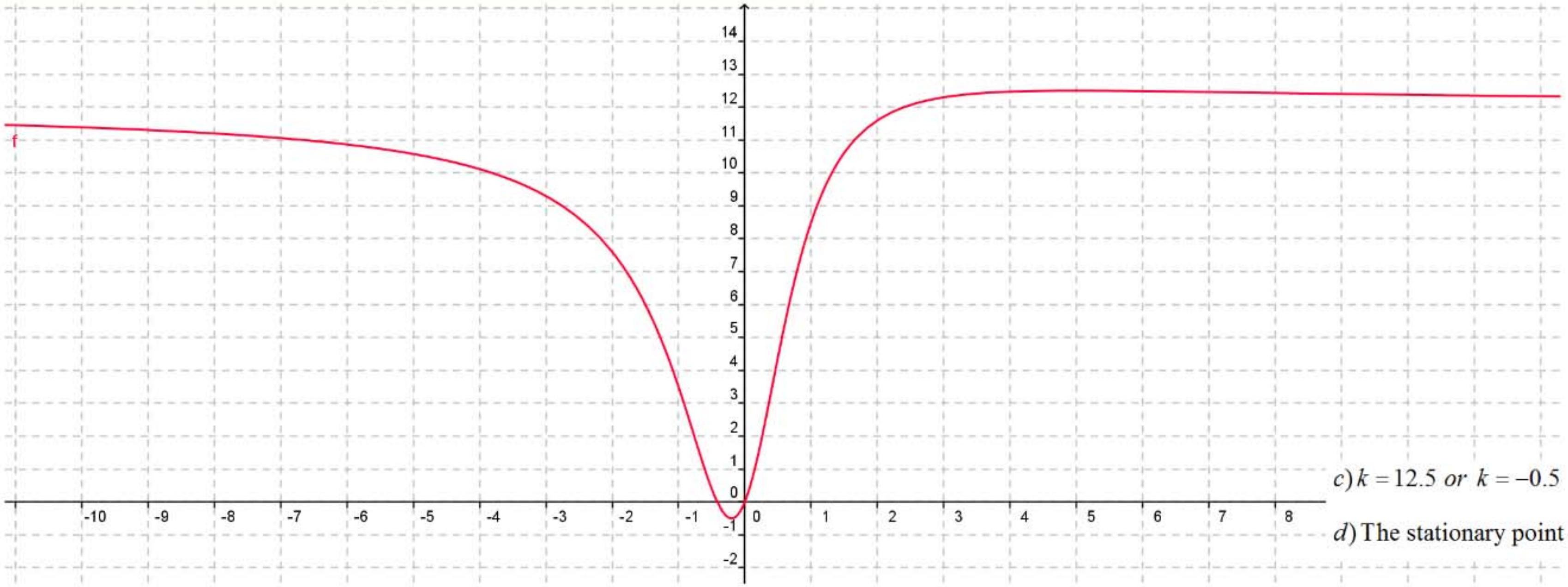
3) The function f is defined by $f(x) = \dfrac{12x^2 + 5x}{x^2 + 1}$.

(a) Write down the equation of the asymptote of the curve $y = f(x)$.

(b) Show that the equation $f(x) = k$ can be written in the form $(12 - k)x^2 + 5x - k = 0$.

(c) Hence find the values of k for which the equation $f(x) = k$ has two equal roots.

(d) Deduce the coordinates of the stationary points of the curve with equation $y = \dfrac{12x^2 + 5x}{x^2 + 1}$.



$c)\ k = 12.5 \ or \ k = -0.5$

$d)$ The stationary point are $(5, \dfrac{25}{2})$ and $(\dfrac{1}{5}, -\dfrac{1}{2})$

Exercises:

1 A curve has equation $y = \dfrac{x^2 + 1}{x(4 + 3x)}$.

(a) Write down the equations of its asymptotes.

(b) Show that the curve intersects the line $y = k$ when $(3k - 1)x^2 + 4kx - 1 = 0$.

(c) Hence show that the line $y = k$ is a tangent to the curve when $4k^2 + 3k - 1 = 0$.

(d) Hence find the coordinates of the stationary points of the curve.

2 A curve has equation $y = \dfrac{x^2 - 6}{x^2 + 4x + 5}$.

(a) Prove that the curve only exists for $-3 \leqslant y \leqslant 2$.

(b) Hence determine the coordinates of the stationary points of the curve.

3 Show that the curve $y = \dfrac{x^2 + 2x + 2}{x^2 - x - 1}$ exists only when $y \leqslant -2$ and for $y \geqslant 0.4$. Hence find the coordinates of the stationary points of the curve.

4 A curve has equation $y = \dfrac{x + 1}{x(x - 3)}$.

(a) Show that there are no values of x for which $-1 < y < -\frac{1}{9}$.

(b) Hence determine the coordinates of any stationary points of the curve.

(c) State the equations of any asymptotes of the curve.

5 A curve has equation $y = \dfrac{x^2}{x^2 + 3x + 3}$.

(a) Prove that, for all real values of x, y satisfies the inequality $0 \leqslant y \leqslant 4$.

(b) Hence find the coordinates of the stationary points of the curve.

(c) Find the equations of any asymptotes of the curve.

6 A curve has equation $y = \dfrac{x + 2}{3 - x^2}$.

(a) Find the equations of any asymptotes and state the coordinate of any points where the curve crosses the coordinate axes.

(b) Determine the possible values y can take.

(c) Hence find the coordinates of the stationary points of the curve.

7 A curve has equation $y = \dfrac{2x - 3}{2x^2 - x - 1}$.

(a) Prove algebraically that no values of y exist in the interval $\frac{2}{9} < y < 2$.

(b) Hence find the coordinates of the stationary points of the curve.

(c) Find the equations of its three asymptotes and sketch the curve.

8 A curve has equation $y = \dfrac{x^2 - 4x - 5}{x^2 + x + 2}$.

(a) Prove algebraically that, for all real values of x, y satisfies the inequality $-\frac{18}{7} \leqslant y \leqslant 2$.

(b) Hence find the coordinates of the stationary points of the curve.

(c) Find the equations of any asymptotes and sketch the curve.

Key point summary

1 It is always useful to draw any asymptotes as the first stage in sketching the graph of a rational function.

2 When a curve has a vertical asymptote at $x = a$, it is useful to check the values of y when x is a little smaller than a and when x is a little larger than a.

By considering the behaviour very close to the asymptotes, it is often possible to deduce the main shape of the graph.

3 In order to find the set of values of y for which a curve of the form $y = \dfrac{ax^2 + bx + c}{dx^2 + ex + f}$ exists:

1 consider where $y = k$ cuts the curve by writing $k = \dfrac{ax^2 + bx + c}{dx^2 + ex + f}$;

2 multiply out to obtain a quadratic of the form $Ax^2 + Bx + C = 0$, where A, B and C will involve k;

3 use the condition for real roots $B^2 - 4AC \geqslant 0$ to obtain a quadratic inequality involving k,

4 convert the solution involving k to a condition involving y.

4 If a curve of the form $y = \dfrac{ax^2 + bx + c}{dx^2 + ex + f}$ is shown to exist only for $P \leqslant y \leqslant Q$, then it means that the curve must have a minimum point when $y = P$ and a maximum point when $y = Q$.

Substitute $y = P$ in order to find the x-coordinate of the minimum point. The resulting quadratic in x will always have a repeated root.

Repeat by substituting $y = Q$ to find the x-coordinate of the maximum point.

5 If a curve of the form $y = \dfrac{ax^2 + bx + c}{dx^2 + ex + f}$ is shown to exist only for $y \leqslant M$ or $y \geqslant N$, then it means that the curve must have a maximum point when $y = M$ and a minimum point when $y = N$.

Substitute $y = M$ in order to find the x-coordinate of the maximum point. The resulting quadratic in x will always have a repeated root.

Repeat by substituting $y = N$ to find the x-coordinate of the minimum point.

Answers (previous page exercises)

1 **(a)** $x = 0, x = -\frac{4}{3}, y = \frac{1}{3}$; **(d)** $(-\frac{1}{2}, -1)$ and $(2, \frac{1}{4})$.

2 **(b)** $(-\frac{3}{2}, -3)$ and $(-4, 2)$.

3 $(0, -2)$ and $(-2, 0.4)$.

4 **(b)** $(1, -1)$ and $(-3, -\frac{1}{9})$; **(c)** $x = 0, x = 3, y = 0$.

5 **(b)** $(0, 0)$ and $(-2, 4)$; **(c)** $y = 1$, no vertical asymptotes.

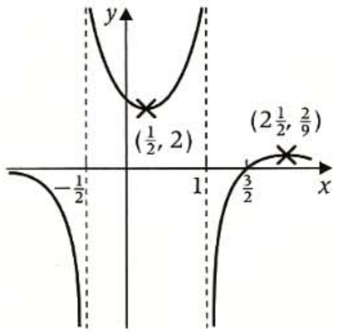
6 **(a)** $x = \pm\sqrt{3}, y = 0, (-2, 0)$ and $(0, \frac{2}{3})$;

(b) $y \leqslant \frac{1}{6}$ and $y \geqslant \frac{1}{2}$;

(c) $(-3, \frac{1}{6})$ and $(-1, \frac{1}{2})$.

7 **(b)** $(\frac{5}{2}, \frac{2}{9})$ and $(\frac{1}{2}, 2)$;

(c) $x = -\frac{1}{2}, x = 1, y = 0$.



8 **(b)** $(-3, 2)$ and $(\frac{1}{5}, -\frac{18}{7})$;

(c) $y = 1$, no vertical asymptotes