Wednesday 25 January 2012 – Afternoon

A2 GCE MATHEMATICS (MEI)

4758/01 Differential Equations

QUESTION PAPER

4733210112

Candidates answer on the Printed Answer Book.

OCR supplied materials:
- Printed Answer Book 4758/01
- MEI Examination Formulae and Tables (MF2)

Other materials required:
- Scientific or graphical calculator

Duration: 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

These instructions are the same on the Printed Answer Book and the Question Paper.

- The Question Paper will be found in the centre of the Printed Answer Book.
- Write your name, centre number and candidate number in the spaces provided on the Printed Answer Book. Please write clearly and in capital letters.
- **Write your answer to each question in the space provided in the Printed Answer Book**. Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Answer any **three** questions.
- Do **not** write in the bar codes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The acceleration due to gravity is denoted by $g\,\mathrm{m\,s^{-2}}$. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION FOR CANDIDATES

This information is the same on the Printed Answer Book and the Question Paper.

- The number of marks is given in brackets **[ ]** at the end of each question or part question on the Question Paper.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is **72**.
- The Printed Answer Book consists of **16** pages. The Question Paper consists of **4** pages. Any blank pages are indicated.

INSTRUCTION TO EXAMS OFFICER / INVIGILATOR

- Do not send this Question Paper for marking; it should be retained in the centre or recycled. Please contact OCR Copyright should you wish to re-use this document.

1 Fig. 1 shows a particle of mass 0.5 kg hanging from a light vertical spring. At time t seconds its displacement is x m below its equilibrium level and its velocity is $v\,\mathrm{m\,s^{-1}}$ vertically downwards. The forces on the particle are

- its weight, $0.5g$ N,
- the tension in the spring, $2.5(x + 0.2g)$ N,
- the resistance to motion, kv N, where k is a positive constant.

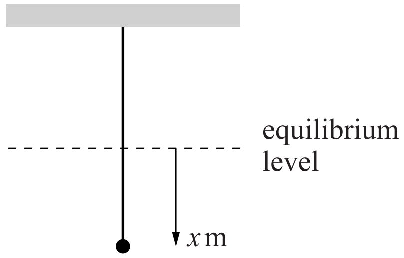

Fig. 1

(i) Use Newton's second law to write down the equation of motion for the particle, justifying the signs of the terms. Hence show that the displacement is described by the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2k\frac{\mathrm{d}x}{\mathrm{d}t} + 5x = 0.$$ **[4]**

The particle is initially at rest with $x = 0.1$.

(ii) Find the set of values of k for which the system is

(A) over-damped, (B) under-damped, (C) critically damped.

In each of the cases (A) and (B), sketch a possible displacement-time graph of the motion. **[7]**

(iii) Sketch a displacement-time graph of the motion of the particle in the case $k = 0$. **[1]**

A subsequent motion of the particle is modelled by the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\frac{\mathrm{d}x}{\mathrm{d}t} + 5x = \sin 4t.$$

(iv) Find the particular solution subject to the conditions that the particle is initially at rest with $x = 0$. **[12]**

2 A population of bacteria grows from an initial size of 1000. After t hours the size of the population is P. After 10 hours the size of the population is 4000.

At first the rate of growth is modelled as being proportional to the size of the population.

(i) Write down a differential equation modelling the population growth and solve it for P in terms of t. **[4]**

To allow for constraints on the population growth, the model is revised to give

$$\frac{dP}{dt} = kP(5000 - P),$$

where k is a constant.

(ii) Solve this differential equation to find t in terms of P, subject to the given conditions. **[9]**

(iii) Find the time it takes for the population to reach 4900, giving your answer in hours, correct to two decimal places. **[1]**

The model is further refined to give

$$\frac{dP}{dt} = 10^{-15}P^{\alpha}(5000 - P),$$

where α is a constant, and it is observed that the maximum *rate* of growth occurs when $P = 4000$.

(iv) Show that $\alpha = 4$. **[5]**

Starting from $t = 10$, $P = 4000$, Euler's method is used with a step length of 0.2 to solve this differential equation. The algorithm is given by $t_{r+1} = t_r + h$, $P_{r+1} = P_r + h\dot{P}_r$.

(v) Continue the algorithm for two steps to estimate the size of the population when $t = 10.4$. **[5]**

3 Consider the differential equation

$$\frac{dy}{dx} - y = x.$$

(i) Sketch the isoclines $\frac{dy}{dx} = m$ for $m = 0, \pm 1, \pm 2$. Hence draw a sketch of the tangent field. **[3]**

(ii) State which of the isoclines is an asymptote to any solution curve. **[1]**

(iii) Sketch on your tangent field the solution curves through $(2, 0)$ and $(0, -2)$. **[3]**

(iv) Use the integrating factor method to solve the differential equation for y in terms of x, subject to the condition $y = 3$ when $x = 0$. **[7]**

Now consider the differential equation

$$\frac{dy}{dx} - y = \sin x.$$

(v) Find the complementary function and a particular integral. Hence state the general solution. **[6]**

(vi) Find the solution subject to the condition $y = 3$ when $x = 0$ and sketch the solution curve. **[4]**

4 The simultaneous differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -x + 2y$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -x - 4y + \mathrm{e}^{-2t}$$

are to be solved.

(i) Eliminate y to obtain a second order differential equation for x in terms of t. Hence find the general solution for x. **[14]**

(ii) Find the corresponding general solution for y. **[3]**

Initially $x = 5$ and $y = 0$.

(iii) Find the particular solutions. **[4]**

(iv) Show that $\frac{y}{x} \to -\frac{1}{2}$ as $t \to \infty$. Show also that there is no value of t for which $\frac{y}{x} = -\frac{1}{2}$. **[3]**